*

M500
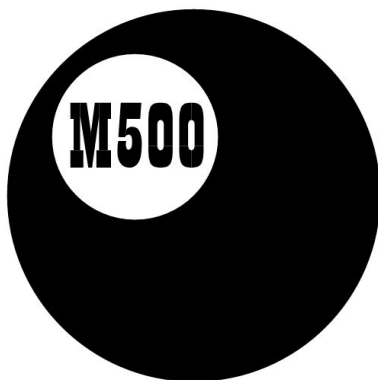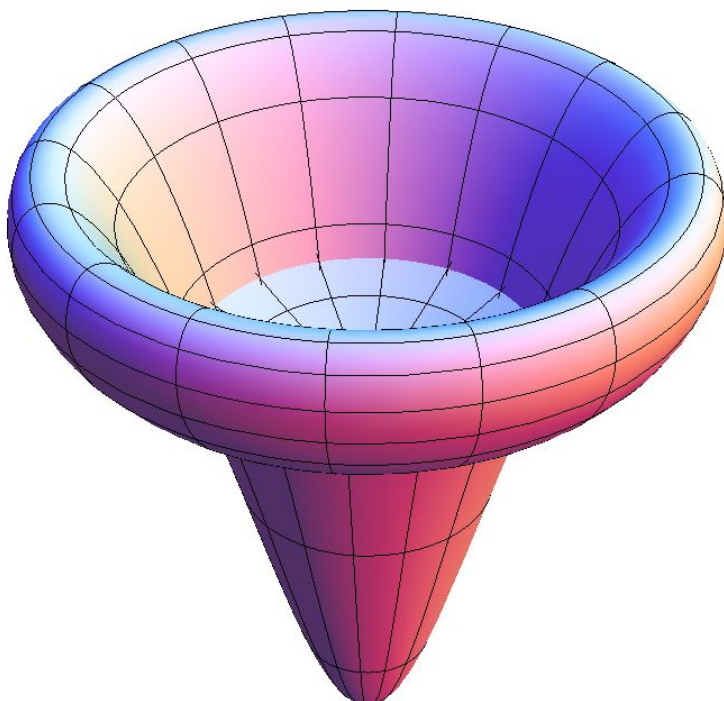
M500 172

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The M500 Special Issue, published once a year, gives students' reflections on their previous courses.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

Prime Pythagorean triangles

Tony Forbes

A *prime Pythagorean triangle* has one 90-degree angle and three integer sides, two of which are prime (not all three—that's impossible), such as (3, 4, 5), (5, 12, 13), (11, 60, 61), and so on. Harvey Dubner [1] did the analysis and concluded that for a triangle with sides a, b, c, $a < b < c$, the only possibility is: a and c prime, $c = (a^2 + 1)/2$ and $b = c - 1$.

Also in [1], Harvey announced the discovery of a sequence of six prime Pythagorean triangles, where

(i) the lengths of the hypotenuse and one leg of each triangle are primes, and

(ii) the hypotenuse is the shorter leg of the next triangle in the sequence.

This is equivalent to finding a chain of seven primes p_0, p_1, ..., p_6 linked by

$$p \rightarrow \frac{1}{2}(p^2 + 1).$$

The triangles are $(p_0, p_1 - 1, p_1)$, $(p_1, p_2 - 1, p_2)$, ..., $(p_5, p_6 - 1, p_6)$ and, as Harvey pointed out, they rapidly become very large and exceedingly thin.

Shortly afterwards, the author [2] found several new sets of seven primes and it became apparent that a search for a chain of *eight* primes (seven triangles) was a feasible proposition, except that the largest prime in the sequence might be rather difficult to verify.

Nevertheless, I started a search using the much improved sieve outlined in [2]. And on 28 September 1999, I found this chain of eight primes beginning

$$P_0 = 2185103796349763249 \quad \text{(19 digits)},$$

and continuing with

$$P_1 = \frac{1}{2}\left(P_0^2 + 1\right) \qquad \text{37 digits},$$

$$P_2 = \frac{1}{2}\left(P_1^2 + 1\right) \qquad \text{73 digits},$$

$$P_3 = \frac{1}{2}\left(P_2^2 + 1\right) \qquad \text{145 digits},$$

$$P_4 = \frac{1}{2}\left(P_3^2 + 1\right) \qquad \text{289 digits},$$

$$P_5 = \frac{1}{2}\left(P_4^2 + 1\right) \qquad \text{578 digits},$$

$$P_6 = \frac{1}{2}(P_5^2 + 1) \qquad 1155 \text{ digits,}$$

$$P_7 = \frac{1}{2}(P_6^2 + 1) \qquad 2310 \text{ digits.}$$

Jeremy Humphries and **Manfred Toplic** joined in the search and I am very grateful for their help.

I thought it would be interesting to draw a picture of the corresponding triangles, starting with the smallest, $(P_0, P_1 - 1, P_1)$, joining the next one, $(P_1, P_2 - 1, P_2)$, to it at the common edge, P_1, then attaching $(P_2, P_3 - 1, P_3)$ at P_2, then attaching $(P_3, P_4 - 1, P_4)$ at P_3, and so on. The intention was to create a sort of triangular, spiral-shaped object. However, I soon discovered that after scaling down to fit everything on the page the diagram is dominated by the hypotenuse of the big triangle $(P_6, P_7 - 1, P_7)$ and you would need a very powerful microscope to see anything other than what appears to be a single straight line.

Then I had the idea of transforming all lengths by the inverse tan function. Conveniently, $\tan^{-1} x$ transforms the interval $[-\infty, \infty]$ to $[-\pi/2, \pi/2]$ and does not have too drastic an effect on points close to zero. I got MATHEMATICA to do the work, and the result appeared (without explanation) on the front cover M500 **170**.

At the time **170** came out I was, as predicted, having difficulty validating the primality of the last element of the chain, the 2310-digit number P_7.
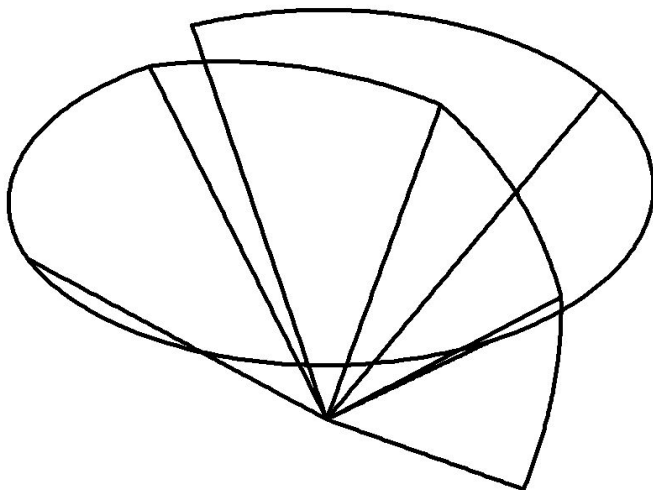
After a certain amount of deliberation I decided to create a new, general primality-prover for the PC. Eventually the programming was completed and I was at last in a position to confirm that all eight numbers P_0, P_1, ..., P_7 are true primes.

The first five, P_0, P_1, P_2, P_3 and P_4, are quite small and easily verified by the UBASIC program APRT-CLE. For the others, I was able to obtain partial factorizations of $P_k - 1$.

$$P_k - 1 \;=\; (P_0 - 1)\frac{1}{2}(P_0 + 1)\frac{1}{2}(P_1 + 1)\frac{1}{2}(P_2 + 1)\ldots\frac{1}{2}(P_{k-1} + 1),$$

$$P_0 - 1 \;=\; 2^4 \cdot 233 \cdot 586132992583091,$$

$$\tfrac{1}{2}(P_0 + 1) \;=\; 3^2 \cdot 5^3 \cdot 13 \cdot 761 \cdot 19087 \cdot 5143087,$$

$$\tfrac{1}{2}(P_1 + 1) \;=\; 7^2 \cdot 1063 \cdot 189043 \cdot 7552723 \cdot 113558719 \cdot 141341652553,$$

$$\tfrac{1}{2}(P_2 + 1) \;=\; 7058053 \cdot 584806347967357670071323 5221$$
$$\cdot\; 34520041584369005634844907730019249777,$$

$$\tfrac{1}{2}(P_3 + 1) \;=\; 2179 \cdot 1847645923 \cdot C132,$$

$$\tfrac{1}{2}(P_4 + 1) \;=\; 307 \cdot 769 \cdot 262513 \cdot P278,$$

$$\tfrac{1}{2}(P_5 + 1) \;=\; 108139 \cdot 11360649709 \cdot 5586562264501 \cdot C550,$$

$$\tfrac{1}{2}(P_6 + 1) \;=\; 4177 \cdot 1372052449 \cdot 5098721569$$
$$\cdot\; 84098816095916212867 \cdot C1113,$$

where $P278$ is a 278-digit prime:

$$
\begin{aligned}
P278 \;=\; & 66505518540598996114987486506055236521044267373138 \\
& 69473288000457727001877127498646545001634613677898 \\
& 53932112480508999228232340454335875401889420451888 \\
& 17780482079524485531037464472393979852934170207932 \\
& 02663155485302406204947222346461607409301255277393 \\
& 47884672922480556979611960 19,
\end{aligned}
$$

and $C132$, $C550$ and $C1113$ denote composite numbers of 132, 550 and 1113 digits respectively. The 28-digit factor of $P_2 + 1$ and the 20-digit factor of $P_6 + 1$ were found by **Manfred Toplic** and **Paul Zimmermann**. (Both found both.)

Thus we have a sufficient partial factorization (48.4%) of $P_5 - 1$ to construct a primality proof for P_5 by the methods of Brillhart, Lehmer and Selfridge [3]. Similarly, the primality of P_6 follows from a 33.4% factorization of $P_6 - 1$.

That leaves P_7.

I would like to thank Manfred Toplic and Paul Zimmermann for their help with the factorization of $P_7 - 1$. However, after a considerable amount of effort attempting to tease out further prime divisors of $P_7 - 1$, I decided that an easy primality proof for P_7 was not to be. I therefore resorted to a more complicated method. This was carried out by my primality-proving program, available on the Web at

www.ltkz.demon.co.uk/ar2/vfypr.zip.

The program proves that given number N is prime by trial-dividing N by all primes up to $\sqrt{N}$. The trick is to place severe restrictions on the possible divisors so that there are not too many to try. We do this in two ways: (i) with the APRCL test (this part is a more or less direct conversion of APRT-CLE from UBASIC to C); and (ii) by using prime factors of $N^2 - 1$.

For P_7, the program ran on a 400 MHz AMD/K6/2 non-stop for 19.7 days before delivering its answer.

Let d be a prime divisor of P_7. The proof that no such d exists proceeds in several stages.

1. *Pocklington's Theorem.* Gathering together all the factors found so far, let

$$
\begin{aligned}
F_1 \;=\; & 11364028773118678645863393880225035110068188490680 \\
& 74284625807644534721210969640169863192044176288720 \\
& 57382836214336492569310719940321645143241641366672 \\
& 31704620613678520580684280352992373327229897947340 \\
& 09917692032743575475918022578947700337216860293874 \\
& 96561498464943981086970289943873321681460108830000 \\
& 00131801406514260770804840415255291401064877989705 \\
& 76202962420323563098312300324091122817224414751412 \\
& 15123765209184430598589590008879997663256918503367 \\
& 07250451432160496252649191808276871593840887080642 \\
& 91103468534974000 \quad (517\ \text{digits}).
\end{aligned}
$$

Then F_1 divides $P_7 - 1$ and F_1 can be completely factorized into primes. Furthermore, I confirm that for every prime factor f of F_1, $11^{P_7-1} \equiv \pmod{P_7}$ and $\gcd(11^{(P_7-1)/f} - 1,\ P_7) = 1$. Hence the conditions of Pocklington's Theorem [3] hold and we have

$$
d \;\equiv\; 1 \pmod{F_1}. \tag{1}
$$

2. *Morrison's Theorem.* Let $F_2 = 43^2 \cdot 73 = 134977$. Then F_2 divides $P_7 + 1$. Furthermore, I confirm that for every prime factor f of F_2, I can

find a Lucas sequence U_n with discriminant 17 such that U_{P_7+1} is divisible by P_7 and $\gcd(U_{(P_7+1)/f},\ P_7) = 1$. The conditions of Morrison's Theorem [4, Theorem 16] hold and therefore we have

$$d \ \equiv \ \pm 1 \ (\mathrm{mod}\ F_2). \tag{2}$$

3. *The APRCL test.* I confirm that the conditions of Lenstra's Theorem (see, for example, Riesel [5]) are satisfied with the following collection of prime powers p^k: $\{2^5,\ 3^3,\ 5^2,\ 7,\ 11,\ 13\}$ and primes q: $\{11,\ 17,\ 19,\ 23,$ 29, 31, 37, 41, 53, 61, 67, 71, 79, 89, 97, 101, 109, 113, 127, 131, 151, 157, 181, 199, 211, 241, 271, 281, 313, 331, 337, 353, 379, 397, 401, 421, 433, 463, 521, 541, 547, 601, 617, 631, 661, 673, 701, 757, 859, 881, 911, 937, 991, 1009, 1051, 1093, 1171, 1201, 1249, 1301, 1321, 1801, 1873, 1951, 2003, 2017, 2081, 2161, 2311, 2341, 2377, 2521, 2731, 2801, 2861, 2971, 3121, 3169, 3301, 3361, 3433, 3511, 3697, 3851, 4159, 4201, 4621, 4951, 5281, 5851, 6007, 6301, 6553, 7151, 7393, 7561, 7723, 8009, 8191, 8317, 8581, 8737, 9241, 9829, 9901, 11551, 11701, 12601, 13729, 14561, 14851, 15121, 15401, 15601, 16381, 16633, 17551, 18481, 19801, 20021, 20593, 21601, 21841, 23761, 24571, 25741, 26209, 28081, 30241, 34651, 36037, 38611, 39313, 42901, 47521, 48049, 50051, 51481, 54601, 55441, 65521, 66529, 70201, 72073, 79201, 81901, 92401, 93601, 96097, 103951, 108109, 109201, 110881, 118801, 120121, 123553, 131041, 140401, 150151, 151201, 180181, 193051, 196561, 200201, 216217, 218401, 257401, 270271, 300301, 332641, 393121, 415801, 432433, 450451$\}$.

Let T be the product of the p^ks and let S be the product of the qs, multiplied by 11, the only q that divides T. Then $T = 21621600$ and

$\begin{aligned} S \ = \ &8980800822251132178222234817202160073886868223302\\ &44539522938364666709284347969136134439971184078953\\ &43448926110251734618866677160265385666702172524877\\ &14880656314884111327138509681893187202637968286542\\ &83293015406063187900321990063760228444306508719210\\ &87374318335464280164671112699177078381998108933220\\ &08595569363494379320612699391931488477880423743996\\ &04982905147776427399328285077729661659607325950022\\ &44652659557331648190118182727751315532905547543602\\ &46016541006719320534163395247117363557783380631924\\ &67962869074581133724035035597332553162773278185621\\ &96526723803946778487744027151656295879855418736649\\ &74668639216504414583262504360944184 9 \ \ (\text{636 digits}). \end{aligned}$

The result of the APRCL test is that

$$d \;\equiv\; P_7^i \;(\text{mod } S) \text{ for some } i = 1,\, 2,\, \ldots,\, T-1. \tag{3}$$

4. *The Chinese Remainder Theorem.* Let $G = F_1 F_2 S$. Since F_1, F_2 and S are pairwise coprime we can combine (1), (2) and (3) by the CRT to obtain

$$d \;\equiv\; \Delta(e, i) \;(\text{mod } G),$$

where

$$\Delta(e, i) \;\equiv\; \left(\frac{1}{F_2 S} \text{ mod } F_1 \right) F_2 S + \left(\frac{e}{F_1 S} \text{ mod } F_2 \right) F_1 S$$
$$+ \left(\frac{P_7^i}{F_1 F_2} \text{ mod } S \right) F_1 F_2 \;(\text{mod } G),$$

for some $e = +-1$ and $i = 1,\, 2,\, \ldots,\, T-1$.

5. *Trial division.* For $e = -1$ and 1, and for each i from 1 to $T-1$, we compute the value of $\Delta(e, i)$ that lies between 0 and $G-1$ and confirm that it does not divide P_7. Hence $d \geq G$.

6. At this point we have proved that P_7 has no prime divisors less than G. But $G > \sqrt{P_7}$. Therefore P_7 is prime.

References

[1] Harvey Dubner, 'Pythagorean triangles with prime leg and hypotenuse', NMBRTHRY, listserv.nodak.edu/archives/nmbrthry.html, 4 July 1999.

[2] Tony Forbes, 'Re: Pythagorean triangles with prime leg and hypotenuse', NMBRTHRY, listserv.nodak.edu/archives/nmbrthry.html, 15 July 1999.

[3] H. C. Pocklington, 'The determination of the prime or composite nature of large numbers by Fermat's theorem', *Proc. Cambridge Philos. Soc.*, **18** (1914-16), 29–30.

[4] John Brillhart, D. H. Lehmer & J. L. Selfridge, 'New primality criteria and factorizations of $2^m \pm 1$', *Mathematics of Computation*, **29** (1975), 620-647.

[5] Hans Riesel, *Prime Numbers and Computer Methods for Factorization*, 2nd edition, Birkhäuser, Boston, 1994.

Problem 172.1 – 345 triangle
Dave Ellis

I have read with admiration the solutions to Problem 168.2 – 345 square. I published a version of this myself in a computer magazine some years ago along the lines of a monastery quadrangle with a water fountain in it at the defined point. Readers were invited to solve the problem by devising a computer program.

I also published a version some time later that may be of further interest to M500.

> An equilateral triangle encloses a point. The point is 30 metres from one corner, 40 metres from another corner, and 50 metres from the remaining corner. What is the length of the triangle's side?

Once again, I solved this with a simple computer program. It would be interesting to see what M500 readers do with it.
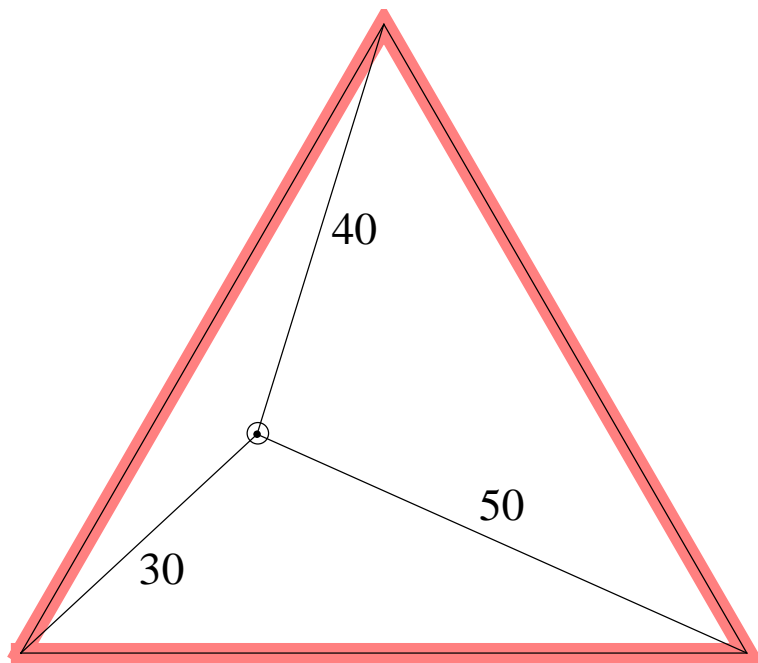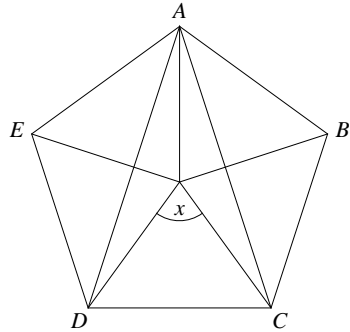
Solution 169.2 – Chords

In a regular pentagon inscribed in a circle with unit radius, show that the product of the chords from any vertex to each of the others is equal to 5;

$(AB)(AC)(AD)(AE) = 5.$

Generalize.

John Bull

From the figure it can be seen that $AE = AB = 2\sin(x/2)$; $AD = AC = 2\sin x$. Therefore $AE \cdot AD \cdot AC \cdot AB = 2^4(\sin^2 x/2)(\sin^2 x)$, where $x = 2\pi/5$.

This can be generalized to any polygon, where we are then required to show that:

Sides $= 4$, $x = 2\pi/4$: $2^3 \sin^2 x/2 = 4$,

Sides $= 5$, $x = 2\pi/5$: $2^4(\sin^2 x/2)(\sin^2 x) = 5$,

Sides $= 6$, $x = 2\pi/6$: $2^5(\sin^2 x/2)(\sin^2 x) = 6$,

Sides $= 7$, $x = 2\pi/7$: $2^6(\sin^2 x/2)(\sin^2 x)(\sin^2 3x/2) = 7$,

Sides $= 8$, $x = 2\pi/8$: $2^7(\sin^2 x/2)(\sin^2 x)(\sin^2 3x/2) = 8$,

Sides $= 9$, $x = 2\pi/9$: $2^8(\sin^2 x/2)(\sin^2 x)(\sin^2 3x/2)(\sin^2 2x) = 9$,

Sides $= 10$, $x = 2\pi/10$: $2^9(\sin^2 x/2)(\sin^2 x)(\sin^2 3x/2)(\sin^2 2x) = 10$,

Sides $= 11$, $x = \dfrac{2\pi}{11}$: $2^{10}(\sin^2 \dfrac{x}{2})(\sin^2 x)(\sin^2 \dfrac{3x}{2})(\sin^2 2x)(\sin^2 \dfrac{5x}{2}) = 11$,

Sides $= 12$, $x = \dfrac{2\pi}{12}$: $2^{11}(\sin^2 \dfrac{x}{2})(\sin^2 x)(\sin^2 \dfrac{3x}{2})(\sin^2 2x)(\sin^2 \dfrac{5x}{2}) = 12$,

etc., etc.

For a $(2n + 1)$-gon and a $(2n + 2)$-gon, the problem is equivalent to trigonometrical results

$$\prod_{k=1}^{n} \sin^2 \frac{k\pi}{2n + 1} = \frac{2n + 1}{2^{2n}} \quad \text{and} \quad \prod_{k=1}^{n} \sin^2 \frac{k\pi}{2n + 2} = \frac{2n + 2}{2^{2n+1}}$$

respectively. This can be solved using complex-variable methods. Solutions as worked examples can be found in *From Erdős to Kiev—Problems of Olympiad Caliber*, by Ross Honsberger (Mathematical Association of America, Dolciana Mathematical Expositions, Vol. 17, 1996) and *Theory and*

Problems of Complex Variables, by Murray R. Spiegel (Schaum's Outline Series, July 1964).

However, there is a reasonably easy way to see how these results arise using real variables. Return to the particular case of the pentagon, and consider the multiple angle identity for $5x$:

$$\sin 5x \;=\; (16\sin^4 x - 20\sin^2 x + 5)(\sin x).$$

Put $x = 0$, $\pi/5$, $2\pi/5$, $3\pi/5$, $4\pi/5$ in the LHS and it can easily be seen that the identity equates to zero. Thus $\pi/5$, $2\pi/5$, $3\pi/5$, $4\pi/5$ are solutions of the quartic $16\sin^4 x - 20\sin^2 x + 5 = 0$. However, $\sin \pi/5 = \sin 4\pi/5$ and $\sin 2\pi/5 = \sin 3\pi/5$ so we can take $\sin \pi/5$ and $\sin 2\pi/5$ as repeated roots. Thus we have roots $\sin^2 x/2$ and $\sin^2 x$, where $x = 2\pi/5$. The product of the roots of any polynomial equated to zero is the constant term divided by the highest order coefficient, in this case $5/16$. Hence $(\sin^2 x/2)(\sin^2 x) = 5/16$, which is the required result.

This can be generalized to other cases using multiple angle identities expressed as follows:

$$\sin 2x \;=\; (2)(\sin x)(\cos x)$$
$$\sin 3x \;=\; (3 - 4\sin^2 x)(\sin x)$$
$$\sin 4x \;=\; (4 - 8\sin^2 x)(\sin x)(\cos x)$$
$$\sin 5x \;=\; (5 - 20\sin^2 x + 16\sin^4 x)(\sin x)$$
$$\sin 6x \;=\; (6 - 32\sin^2 x + 32\sin^4 x)(\sin x)(\cos x)$$
$$\sin 7x \;=\; (7 - 56\sin^2 x + 112\sin^4 x - 64\sin^6 x)(\sin x)$$

etc., etc.

Unfortunately, this is not the way that formulae books usually show multiple angle identities: for example, see the Schaum's Outline Series Mathematical Handbook, by Murray R. Spiegel. However, from formulae in books it is easy to see the origin of the above expressions, with coefficients taking a combinatorial form in the general case. The easiest way to derive these identities is using complex variables, with the coefficients arising from a binomial theorem expansion. However, for those who have never studied complex variables the above expressions at least show how, with some tedious algebra, the general problem could be solved entirely in a real variable.

The case of the pentagon could be solved directly, because

$$5 - 20\sin^2 x + 16\sin^4 x \;=\; 0$$

happens to be a quadratic in $\sin^2 x$. However, there is no need to go this far, and anyway the polynomial equations deriving from the higher-order cases could not be solved directly.

Barry Lewis

After writing out my first attempt [M500 **171** 14] I had a brainwave, which gives this elegant proof of the general case.

The roots of the equation

$$x^{n-1} + x^{n-2} + \cdots + x + 1 \ = \ 0$$

are just

$$\omega, \ \omega^2, \ \ldots, \ \omega^{n-1},$$

where each term is a complex nth root of unity. These, together with 1 are simply the vertices of the regular polygon in question. So the roots of the equation

$$(1-y)^{n-1} + (1-y)^{n-2} + \cdots + (1-y) + 1 \ = \ 0$$

are

$$1 - \omega, \ 1 - \omega^2, \ \ldots, \ 1 - \omega^{n-1},$$

and the product of these is just the constant term in the equation, so that

$$(1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1}) \ = \ (-1)^{n-1}n,$$

and taking absolute values gives the desired result:

$$|1-\omega||1-\omega^2|\ldots|1-\omega^{n-1}| \ = \ n.$$

But this inspires the following related problems . . . .

Problem 172.2 – Chords again

Barry Lewis

A regular polygon of n sides is inscribed in a unit circle. Suppose that A_1, A_2, ..., A_n are the vertices of this polygon.

Prove, first of all, the corresponding result about the sum of the same diagonals, that is, that

$$A_1 A_2 + A_1 A_3 + \cdots + A_1 A_n \ = \ \frac{n(n-1)}{2}$$

and secondly that if P is any other point on the unit circle then

$$(PA_1)^2 + (PA_2)^2 + \cdots + (PA_n)^2 \ = \ 2n.$$

Sebastian Hayes

If we consider a circle of unit radius, the product of the chords for a triangle is $(\sqrt{3})^2 = 3$ and for a square it is $(\sqrt{2})(\sqrt{2})2 = 4$. And if we consider a diameter as a regular 2-gon, the formula even works in this case. It is tempting to conjecture that the rule is general and there should be some way of showing this using *elementary* methods only (an elementary proof is not necessarily easy). But, as far as I know, no such proof exists.

We can derive at once a recursive formula in $\sin x$, for example, for an odd-sided regular $(2n+1)$-gon: $(2\sin^2 180°/(2n+1))(2\sin^2 2 \times 180°/(2n+1))\ldots$, stopping at the nth $\sin^2$ term, and for an even-sided polygon there is a similar formula stopping at $2\sin 90° = 2$. We then apply the formula for double angles and sort out the result. However, to show that this reduces to n, as first proved by Chebyshev, is neither simple nor elementary. Presumably a proof by induction is possible but the algebra would be horrendous.

It is a curious paradox that a simple and concise solution for this problem in basic geometry, as for several others, requires complex numbers: thus the elegant solution given in the *Hungarian Problem Book* [see M500 **171**, page 13] and the equally good one on similar lines by Barry Lewis [this M500, page 10].

Problem 172.3 – Decagon

Sebastian Hayes

Call the vertices of a regular decagon A, B, $\ldots$, J. Show, using geometrical demonstration only, that the difference between AB and AD is the radius of the circle.

————

ADF—This gives a 'pure' geometric solution to the chords problem for the decagon. We know from the pentagon case that $(AC)(AE)(AG)(AI) = 5$.

Also $AF = 2$, $AJ = AB$ and $AH = AD$. Hence the product for the decagon is $10(AB)^2(AD)^2$. But now $AD = AB + 1$ and we have previously seen that $AB = (\sqrt{5} - 1)/2$ [M500 **171** 12]. Hence $(AB)(AD) = 1$.
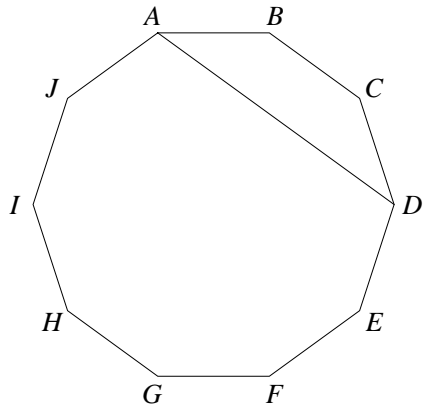
Solution 170.1 – Interesting integral

Show that

$$\int_0^1 \frac{1}{x^x}\,dx \;=\; \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

John Bull

I offer one reasonable solution. It is not fully rigorous, but if Euler can avoid rigour for most of his results, I don't see why others of us should always struggle to provide it!

Using the usual identities for natural logs, found in any formulae book, and the Taylor series expansion for e (which is always convergent) the integral can be expressed as follows:

$$\int_0^1 \frac{1}{x^x}\,dx \;=\; \int_0^1 e^{-x\log x}\,dx$$

$$=\; \int_0^1 \left(1 - \frac{x\log x}{1!} + \frac{x^2\log^2 x}{2!} - \frac{x^3\log^3 x}{3!} + \frac{x^4\log^4 x}{4!} - \ldots\right)dx.$$

Now consider the general integral $\int_0^1 x^m \log^n x\,dx$. This can be integrated by parts as follows:

$$\int_0^1 x^m \log^n x\,dx \;=\; \int_0^1 \log^n x\,d\left(\frac{x^{m+1}}{m+1}\right)$$

$$=\; \left[\frac{x^{m+1}}{m+1}\log^n x\right]_0^1 - \frac{1}{m+1}\int_0^1 x^{m+1}d(\log^n x).$$

At the given limits, the first expression is always zero (or at least at the lower limit the expression tends to zero as x tends to zero). Hence we derive the reduction formula

$$\int_0^1 x^m \log^n x\,dx \;=\; -\frac{n}{m+1}\int_0^1 x^m \log^{n-1} x\,dx.$$

Now consider the special case of $m=n$: $f(n) = \int_0^1 x^n \log^n x\,dx$.

$$f(n) = -\frac{n}{n+1} \int x^n \log^{n-1} x \, dx = \frac{n(n-1)}{(n+1)^2} \int x^n \log^{n-2} x \, dx$$

$$= -\frac{n(n-1)(n-2)}{(n+1)^3} \int x^n \log^{n-3} x \, dx$$

$$\cdots$$

$$= (-1)^n \frac{n(n-1)(n-2)\ldots(1)}{(n+1)^n} \int x^n \log^{n-n} x \, dx$$

$$= (-1)^n \frac{n(n-1)(n-2)\ldots(1)}{(n+1)^n} \left[\frac{x^{n+1}}{n+1}\right]_0^1 = (-1)^n \frac{n!}{(n+1)^{n+1}}$$

and illustrating this for a few specific terms,

$$f(1) = \int_0^1 x \log x \, dx = -\frac{1}{2^2}, \quad f(2) = \int_0^1 x^2 \log^2 x \, dx = \frac{2!}{3^3},$$

$$f(3) = \int_0^1 x^3 \log^3 x \, dx = -\frac{3!}{4^4}, \quad \ldots.$$

Substitute these successive results back into the original series; then

$$\int_0^1 \frac{1}{x^x} = \int_0^1 e^{-x \log x} \, dx$$

$$= \int_0^1 \left(1 - \frac{x \log x}{1!} + \frac{x^2 \log^2 x}{2!} - \frac{x^3 \log^3 x}{3!} + \frac{x^4 \log^4 x}{4!} - \ldots\right) dx$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

This is indeed an interesting integral! One feels that there ought to be a neat and simple illustration of the correspondence between the form of the integral and the form of the summation, but this insight eludes me.

———————

Similar solutions were sent by **Colin Davies**, **Barry Lewis** and **Mike Nugent**.

Mike Nugent writes: Having established the given equality, I wondered if there is any other function $f(x)$ with the property that $\int_0^1 f(x)dx = \sum_{n=1}^{\infty} f(n)$. Does anyone know whether such a function exists? I will explore this and see what I discover.

Solution 170.3 – Reciprocals

When is it true that $\dfrac{1}{a+b} = \dfrac{1}{a} + \dfrac{1}{b}$, or $\dfrac{1}{a+b+c} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$, or $\dfrac{1}{a+b+c+d} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} + \dfrac{1}{d}$, or ...?

Brian O'Donnell

In its most general form it can be stated as: solve

$$1 \Big/ \sum_{i=1}^{n} z_i \;=\; \sum_{i=1}^{n} \frac{1}{z_i} \text{ for all } n \in \mathbb{N}^+, \ z_i \in \mathbb{C}, \ z_i \neq 0.$$

Note that the trivial, but not irrelevant, case $n = 1$ is included. For $n = 1$, the solution is $z_1 \in \mathbb{C}$, $z_i \neq 0$. For $n = 2$, we solve

$$\frac{1}{z_1 + z_2} \;=\; \frac{1}{z_1} + \frac{1}{z_2} \;\Rightarrow\; z_1^2 + z_1 z_2 + z_2^2 \;=\; 0 \;\Rightarrow\; z_1 \;=\; z_2\left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right),$$

w.l.o.g. as the expressions are symmetric in the z_is.

Solutions for $n > 2$ can be obtained by reducing them to one or other of the above two cases, as follows:

For $n = 2(m+1)$, $m \in \mathbb{N}^+$,

$$1 \Big/ \sum_{i=1}^{n} z_i \;=\; \sum_{i=1}^{n} \frac{1}{z_i} \;\Rightarrow\; z_1 \;=\; z_2\left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right),$$

$z_3 + z_4 = 0$, $z_5 + z_6 = 0$, ..., $z_{n-1} + z_n = 0$ (w.l.o.g.).

For $n = 2m + 1$, $m \in \mathbb{N}^+$,

$$1 \Big/ \sum_{i=1}^{n} z_i \;=\; \sum_{i=1}^{n} \frac{1}{z_i} \;\Rightarrow\; z_1 \in \mathbb{C}, \ z \neq 0,$$

$z_2 + z_3 = 0$, $z_4 + z_5 = 0$, ..., $z_{n-1} + z_n = 0$ (w.l.o.g.).

Do these cover all possible solutions? No. The hint comes from the solution for $n = 2$. Note that the three values involve the cube roots of unity, which reminds us that the sum of the roots of unity is zero and the sum of the reciprocals of the roots is also zero (being the complex conjugates). So for $k < n$, we can employ the kth roots of unity—they need not be cancellations in pairs as in the expressions above. Here is a solution for $n = 10$—one of an uncountably infinite number of solutions.

$$\left(4i - 4i - 7 + 7 + \alpha(w_0 + w_1 + w_2 + w_3) + 1 - \left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)\right)^{-1}$$
$$= \frac{1}{4i} - \frac{1}{4i} - \frac{1}{7} + \frac{1}{7} + \frac{1}{\alpha w_0} + \frac{1}{\alpha w_1} + \frac{1}{\alpha w_2} + \frac{1}{\alpha w_3} + \frac{1}{1} + \frac{1}{\left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)},$$

where α is any non-zero complex number and the w_j are the four fourth roots of unity.

I suspect that non-trivial solutions also exist for non-complex quaternions, if only to cancel them in the way that the complex roots of unity disappeared, but on this I'd invite editorial comment.

Problem 172.4 – Grandfather clock

Colin Davies

Further to Problem 168.1 [M500 **168** 20] about relative speed of hands, here is another clock problem.

My long case (grandfather) clock needs winding about every eight days, so we try to wind it every Saturday when the idea comes to mind. The difficulty is that one or both of the two holes into which the winding handle goes are periodically covered up by one or both hands. So I think of winding the clock, but find that the time is 18 minutes past 9, and a hole is covered so I can't wind it. That seems to happen a great deal.

The minute hand covers a hole between 15 and 21 minutes and between 39 and 45 minutes during an hour. The hour hand covers a hole between 7.44 and 9.06, and again between 2.48 and 9.10.

So, if it's Saturday, and I remember that the clock needs winding, what is the probability that I can wind the clock? An approximate answer is easy, but what is the exact answer?

My mother, Southern to the bone, once told me, "All Southern literature can be summed up in these words: 'On the night the hogs ate Willie, Mama died when she heard what Daddy did to sister.'"—Pat Conroy.

All literature can be summed up in these words: 'Darling, I love you. But do be careful, there's an angry-looking mammoth just outside the cave.' —Heard on R5 mathematics programme.

Letters to the Editors

Sixteen matches

Dear Tony,

This problem [M500 **170** 28] first appeared in my *Mathematics for Primary Schools* (Nelson, 1963) and I am wondering whether EK still has a copy.

It's a pity that the reader is *told* that there are sixteen matches, because my solution read something like:

'A mathematician would immediately count the matches. On realising that there are 16 he would see at once the four remaining squares must meet only at the corners. That is to say, they cannot have any sides in common. It then becomes quite easy.'

Yours sincerely,

John Halsall

―――――――――――

Assuming the mathematician is of the type that can count.—**ADF**

―――――――――――
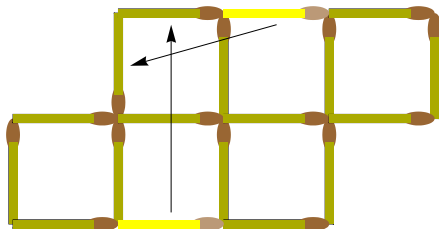
Problem 170.1

Dear Jeremy,

About the interesting integral **170.1**. I could not integrate $1/x^x$, but I have a friend in Minsk who has a friend in Pinsk, etc., who has a friend in the University of Warwick. The solution is to take $\log_x$ of x^{-x}, change the base to e, and get $e^{-x \log x}$. Then ... [see John Bull's article, pages 12–13, for the rest].

Colin Davies

―――――――――――

Fractiles

Dear Sue,

Thought I would make contact with you as I do not know anyone else who might be of assistance.

I have only just joined M500 so I am a little green as to how to benefit from being a member.

I have been doing maths with the OU for the past two years, MU120 and MST121. I cannot say that I am all that good at maths but one thing has caught my interest and I wondered if you knew anyone that could give me any help with creating and designing *fractiles*. What I would really like to do is to put a few equations onto computer (I have MATHCAD), and begin to design simple fractiles. I feel sure that someone in the membership might be able to help.

Please advise if you can. I would be very grateful. Thank you in advance.

Yours sincerely,

Dave Allaire

V.A.T.

Dear Tony,

Here is an anecdote to illustrate the lonely world we mathematicians live in.

I was recently in the Factory Shop looking for a ceiling light for a children's bedroom. After a considerable amount of parental haggling a new-fangled halogen light was decided on. Nobody knew what halogen had to do with it but then one shouldn't ask questions.

The price was £80, actually £79.95 or something like that. With V.A.T. the bill was £96.45 or thereabouts. But I had done my own mental calculation. Knowing (as a mathematician would) that V.A.T. is 7/40 and that the price was just under £80, then the V.A.T. should be just under £14. On challenging the shop assistant I found that she had just added £17.50 to the price!

Finally, have you noticed that $1/3!+1/5! = 7/40$? Therefore if you want to calculate what a price will become when V.A.T. is added you merely have to multiply by $\sinh 1$.

Yours sincerely,

John Reade

Millennium

Sir,

I've only become a subscriber recently and only have volumes **170** and **171** to hand. However it is easy to understand why there is a shortage of contributors. You appear unwilling to tackle important current issues. Someone ought to tell Messrs Griffiths & Margolis that Fermat died over 300 years ago. Pythagoras, I'm fairly sure, died even earlier.

To take a further example, does Mr Singmaster really think that the last useful thing to say about Time happened in 1963. Time is a serious matter—it won't stand beating that way.

Of course, the issue which should be high on your agenda, the issue which has captured everyone's imagination—everyone it seems except the docile antiquarians of M500—is the Millennium issue.

Y2K compliancy, I mean. How can you neglect it? I trust that I have shamed you. This extensive subject extends well beyond computer hardware, named software packages and operating systems. It goes right down to the smallest domestic issues in our lives.

A friend of mine—a mathematician working in Paris—has looked at it from all sides. Presently he is considering his bottle opener. Is it compliant? Will its operations continue across the boundary?

Complicated and multi-purpose tools they are these days. As well as unbottling, uncanning and uncorking, his little item tells him the time on one face and performs simple calculations on another. I'm told it came from Taiwan.

Should it fail—what then? Even if it only appears likely that it might the evening could be ruined. Here is a tip (it originated in Australia)—it might help. Have all the containers opened in advance, then—and this is the antipodean contribution (the clever bit)—lay a matchstick across the opening. It sets up a flow pattern which causes the liquid to retain its gases and its freshness. Don't open them too early—the effect only lasts a short time.

Some time after midnight has struck, if you are able, retest your tools. Remember you are now on the other side of the time boundary. And don't forget, should they pass the test, to label the tools with a sticker declaring Y2K compliance so that they will be ready for the next millennium and you won't encounter the same problem.

For those interested in how the matchstick trick works—may I suggest you enrol for the M337 course in Complex Analysis. After the groundwork is covered in sections A, B & C, unit D2 applies the Flow Mapping Theorem to obstacles. A small extension of the Joukowski function will be needed to

analyse the matchstick trick, but this is not beyond a dedicated student.

Returning once again to your organ, I see in issue **170** Mr Greatrix gives us devices for calculating the millionth place of π, and in the right place, at the right time, we would be grateful for them. But what consolation will they be to the man who can't open a bottle on the fateful day?

Yours sincerely,

Arthur Quigley

———————————

Tony Forbes writes—Arthur is quite right for taking us to task. As a responsible organization we should have added our contribution to the debate. So here it is. Admittedly a little late, we think we now have a reasonable understanding of what all the fuss was about.

There are actually three different date-related computer problems.

The Y2000 bug. Presumably you spent the change-over period from 1999 to 2000 in the only sensible manner: locked in your home, electricity switched off and with plentiful supplies of coal, alcohol, candles, £50 notes, etc. Hopefully, if you are reading this in February, 2000, things will be back to normal, whatever that may mean, and it should be safe to come out.

The Millennium bug. This happens during the transition from 31st December 2000 to 1st January 2001. Even if they survived the 1999/2000 change, computers and computer-controlled equipment will fail catastrophically in 2001 because of quick fixes to the Y2000 bug by incompetent programmers. Your software may work fine in 2000. But if the programmer, being lazy, just patched in something like `IF YEAR=0 THEN YEAR=2000`, you are in trouble and you won't know about it until next January.

Even fully compliant software is not immune. The problem is that computers have to communicate with humans, and humans cannot possibly cope with four-digit dates. Fortunately, there is a solution. We choose a suitable integer, 48, say, and adopt the convention that all numbers less than 48 represent 21st century dates. Conversely, years from 48 to 99 are assumed to belong to the 20th century. Put simply, we define a function $Y(x)$:

$$Y(x) \;=\; \left\{ \begin{array}{ll} 2000 + x & \text{if } 0 \le x < 48 \\ 1900 + x & \text{if } 48 \le x \le 99. \end{array} \right.$$

This device works brilliantly and the idea has been adopted by a number of software producers. You can continue to work with two-digit years and your computer will always slot them into the correct century. Until the year 2048, when things will once again go horribly wrong; this time because the current year, 48, will become 1948 instead of 2048. That's the ***Y2K bug***.

The Trachtenberg system

Ken Greatrix

Introduced into Britain in the 1960s, this system of multiplication was destined to revolutionize the teaching of maths (it seems to have done so in post-war Switzerland—but ...). It was designed by Professor Jakow Trachtenberg, a Russian-born pacifist who suffered at the hands of Hitler's Nazi Germany. It was during his time in the concentration camps that he kept his brilliant mind alert by working on mathematical problems. Thus was born his speed system of multiplication which does not require any knowledge of tables.

The system requires that you learn the following techniques:

(a) Decide if a number is odd or even.

(b) Divide a number by two and ignore fractions. From now on we will use the word *half* to denote this operation.

(c) Double a number.

(d) Add 5 to a number.

(e) Subtract a number from 9 or 10.

(f) For each number of the multiplicand, recognise its neighbour; i.e. the number immediately to the right.

(g) Subtract 1 or 2 from a number.

Here is a summary of the techniques for multiplying by the numbers up to 12. Obviously you add any carry-over digits. There are also hints that the right-hand digit has an imaginary neighbour of 0, and there are as many imaginary 0s to the left as needed to complete the multiplication. In the table below, the middle steps also refer to the left-hand number; the last step occurs under the imaginary 0. As in standard arithmetic, work from right to left.

Multiplier	Procedure
0	Any number multiplied by 0 is 0.
1	Write down the number unchanged.
2	Double the number.
3	*First step*: subtract from 10 and double, add 5 if odd. *Middle steps*: subtract from 9 and double, add 5 if odd, add half the neighbour. *Last step*: Take half of the left-hand number and subtract 2.

4	*First step*: subtract from 9, add 5 if the number is odd. *Middle steps*: subtract from 9, add 5 if the number is odd and add half the neighbour. *Last step:* take half the left-hand number and subtract 1.
5	Use half the neighbour, add 5 if the number is odd.
6	To the number add 5 if odd, add half the neighbour.
7	Double the number, add 5 if it is odd, add half the neighbour.
8	*First step*: subtract from 10 and double. *Middle steps*: subtract from 9, double, and add half the neighbour. *Last step*: subtract from 10.
9	*First step*: subtract from 10. *Middle steps*: subtract from 9 and add the neighbour. *Last step*: subtract 1 from the left-hand number.
10	For each number write down the neighbour (hence the imaginary neighbour of 0 at the right-hand side).
11	To each number, add the neighbour.
12	Double the number, add the neighbour.

From *The Trachtenberg Speed System of Basic Mathematics*, translated by Ann Cutler and Rudolph McShane (Pan books 1973; reprinted by Guernsey Press 1999).

Complex complex complex

> We asked for valid English sentences containing a word that is repeated often. See also M500 **171**, page 19.

Barbara Lee

Eric's job is to patrol the perimeter of the cylindrical round-tower. Each evening he does his round round round the round round-tower.

Plump Paul and Fat Fergus accompany him on Tuesdays, the three of them singing in turn as they go. The strains of their round round round round round the round round-tower can be heard after dusk.

Note. A cylindrical round-tower is an ancient Irish building.

Problem 172.5 – Series
Colin Davies

Here is the answer to a question; the challenge is to determine what the question was:

$$n\text{th term} \ = \ a\left(r^n \pm \frac{1}{r^{n-2}}\right),$$

where $r = (1 + *)/2$ and $a = (5 + *)/10$. The $+$ sign is used for n even and the $-$ sign for n odd. The asterisk denotes a square root, the sight of which would be a significant clue. Does the expression, if the root is inserted, give an exact value for the nth term, and of what series?
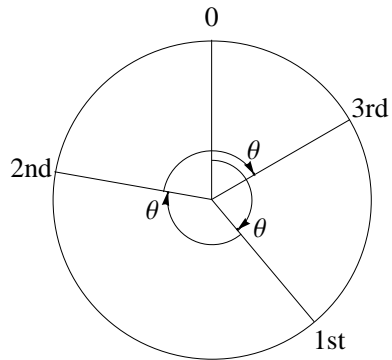
[Adapted from the 'Dipole'/'Micromatters' column of *IEE News*.]

Problem 172.6 – Angle

Sebastian Hayes

We have a circle and we are going to lay off a radius using a *single* angle repeatedly. Each radius must fall into the middle third of the gap it falls into and must not cover a previous radius. This must carry on indefinitely.

What is this angle? Is the solution unique?

A new word
Eddie Kent

Do you know how long the word *sledging* has been around? My American dictionary has no mention of it but it seems to be fairly well established with the *COD*. There it is defined as: Cricket slang—the heaping of insults on an opposing player in order to break his or her concentration [from 'sledge' 'use a sledgehammer on']. But the practice has been around for a while.

For instance take the Revd Lord Frederick Beauclerk, DD, Vicar of St Albans (1773–1850) and great grandson of Charles II and Nell Gwynne. He was known in Soho as Fred Diamond-Eye. As a keen cricketer he admitted to making £600 a year out of the game. Work that out in today's money—I

would say more than half a million. It is rumoured that at least some of this came from backing his opponents, but nothing was suspected at the time.

He would appear at the wicket wearing a scarlet sash and a white beaver hat. His gold pocket watch would be hung on the middle stump as a goad (or sledge) to the bowler. Although Boycott would have little trouble protecting it I would not like to trust anyone less boring. Come to that, the reverend's sermons were said to be excrutiatingly dull, but perhaps he needed the sleep.

However, the cricketing oddity who most excites my imagination is Hesketh K. Naylor, that rarest of birds, an American cricket fan. In the middle of the nineteenth century he was a well-known New York millionaire. He was also impotent. He got his sexual gratification from having teams of women play cricket before him with balloons and without clothes. I am not suggesting this as a way to bring in the crowds but, boy, would it beat sledging!

Anyone not clear what cricket is can take consolation from George Bernard Shaw. When told England were doing well in the Australian Tests he asked "What are they testing?" But he was just an Irish troublemaker. At least Abdul Aziz, the Turkish potentate known as Abdul the Damned, could appreciate the finer points of the game. After watching a match he exclaimed "Remarkable. But what needless exertion! Why do you not compel your slaves and concubines to perform it for you?"

Ah! Why not indeed.

Nine matches

EK

I didn't notice that $4 \times 4 = 16$ until after I had solved it. [The 16-match problem; see page 16.] Here is another. All straight lines are matches; get that? They are not numbers or letters or abstract symbols or dogs; just matches. Move one match to correct this sum. (For interest, I first saw this one at school. Probably on a cigarette card.)
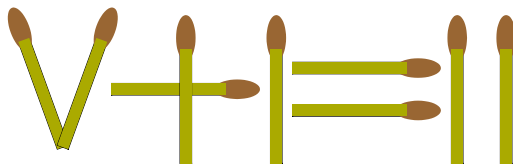
Shell found on beach
EK

Busy editors never have time to read the papers, and there's no need to. If all you want is a concise, accurate summary of the day's news, a quick glance at the headlines is quite sufficient. Here's a selection.

SHELL FOUND ON BEACH —*Evening Argus*

NO WATER – SO FIREMEN IMPROVISE —*Liverpool Daily Post*

WOMAN IN BED WITH CASE OF SALMON —*Liverpool Echo*

GAS RIG MEN GRILLED BY VILLAGERS —*Oxford Times*

CRASH COURSES FOR PRIVATE PILOTS —*Daily Telegraph*

INTERESTING TALL BOYS AT JAMES ADAM'S SALE —*Irish Times*

LORD UPHAM FANCIED by Desmond Hill —*Daily Telegraph*

NEWLY-WEDS, AGED 82, HAVE PROBLEM —*Streatham News*

ALBERT HALL FEELS PINCH —*Guardian*

Q.E.2 THEFT: THREE FINED —*Daily Express*

LESOTHO WOMEN MAKE BEAUTIFUL CARPETS —*Bangkok World*

SEWAGE PLANT ON SEA FRONT MIGHT SMELL, EXPERT SAYS
          —*The Times*

FARMER'S EIGHT HOUR VIGIL IN BOG —*Evening Mail*

SLIM CHICKS LAY BETTER —*Guardian*

SPOTTED MAN WANTED FOR QUESTIONING —*Hackney Gazette*

THREE BATTERED IN FISH SHOP —*Evening News*

BODIES IN THE GARDEN ARE A PLANT SAYS WIFE
          —*Hong Kong Standard*

SPARE OUR TREES – THEY BREAK WIND —*Evening Argus*

GOLDFISH IS SAVED FROM DROWNING —*The Times*

MOUNTING PROBLEMS FOR YOUNG COUPLES —*Western Gazette*

THREATENING LETTERS – MAN ASKS FOR LONG SENTENCE
          —*Scotsman*

MAN FOUND DEAD IN GRAVEYARD —*Evening Standard*

MAGISTRATES ACT TO KEEP THEATRE OPEN —*Evening Citizen*

Twenty-five years ago

From M500 19/20

Helen Gevers—Having completed my last M100 TMA on the day before cut-off date I decided that posting it would be too risky, so I asked a non-OU friend who was going near my tutor's house to deliver it for me. This friend later remarked that he was disappointed to find that the OU was based in an ordinary house, and not in a vast complex of shiny buildings as he had imagined!

Lewis Bradley—Posing as a GPO engineer, i.e. I use a pencil, rubber and slide rule, I tend to use maths on a day to day basis and have even been known to use a computer terminal in moments of stress, but I can say that most of the articles and problems appearing in M500 completely lose me. I know it's silly but what really frightens me is that some of these articles are written by size 14 middle-aged housewives of semi-det. suburbia who would be better employed washing, scrubbing and changing nappies, and I don't know whether to brush up on my maths or my washing, scrubbing, etc.

Jill MacKean—I have heard so many mathematics students complaining about the lack of Honours level courses, but I am extremely grateful to the OU for this very thing.

If there had been a choice I should never, never have tackled *Partial Differential Equations* or *Quantum Mechanics*, and should have missed the two courses that I have enjoyed more than any that I have ever done.

How on earth can you know which courses you would enjoy unless you know what the subject is about? The small amount of knowledge, gained from previous courses, about other subjects is not a very good indication, I find.

I do not quite know why I enjoyed *PDE*. Perhaps I am a masochist.

Bill Shannon—The Polish Embassy were having a party on their National Day of Rejoicing and hurriedly had to hire a pianist to play some suitable music—stuff by Chopin, with maybe a nostalgic piece by Paderewski just for the old folk in memory of days gone by. When the pianist, in tails and white tie sat down at the piano he gave them everything from *St Louis Blues* to *The Young Ones*. But no *Warsaw Concerto*. When they asked him why he said: "That's the way my musical cookies crumble." In deep anger the Poles threw him out onto the street. He hit the pavement screaming and cursing. A passer-by picked him up and asked why he was so incensed and so abusive at leaving the Embassy on his backside. He replied: "Forgive me—but forty square Poles do make one rude."

Contents