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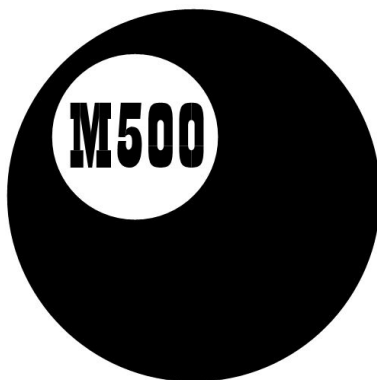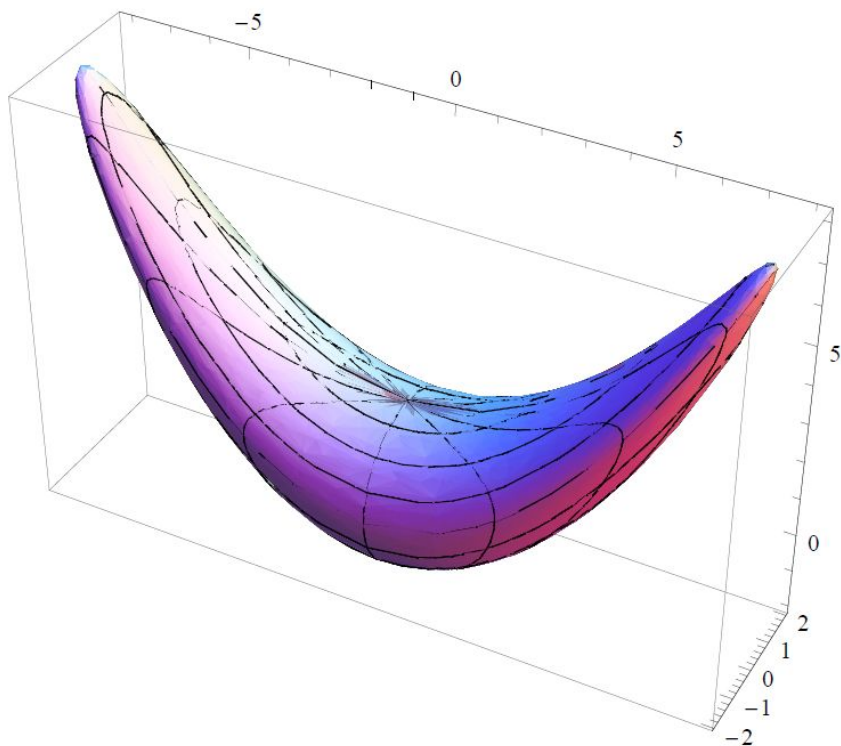
M500 173

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

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Pascal Triangle matrices – I
Sebastian Hayes

```
1
1    1
1    2    1
1    3    3    1
1    4    6    4    1
1    5   10   10    5    1
1    6   15   20   15    6    1
1    7   21   35   35   21    7    1
1    8   28   56   70   56   28    8    1
1    9   36   84  126  126   84   36    9    1
1   10   45  120  210  252  210  120   45   10    1
```

To recapitulate [see 'Zero sum Pascal Triangle', M500 **169** 1]: The entry in row n, column r (n, $r = 0, 1, 2, \ldots$) is given by $^nC_r = n!/(r!(n-r)!)$. The columns are the *figurate numbers*, F_0, F_1, F_2, $\ldots$ (so called because $F_2 = 1$, 3, 6, $\ldots$ give us the triangular numbers, and F_3 the tetrahedral numbers), with general formula:

$$F_0 = 1, \ F_r(n) = \frac{n(n+1)\ldots(n+r-1)}{r!}, \quad r = 1, 2, \ldots, \ n = 0, 1, 2, \ldots.$$

Each column r —I use variable k instead when speaking of the sets of figurate numbers—commences with r blank spaces in the triangular presentation, which explains the difference in the two formulae. Note that the column number r or k, when speaking of the figurate numbers, is one less than the natural number that comes in second place (because we commence with zero); e.g. the fourth column, 1, 4, 10, $\ldots$, is F_3.

Inverses of Pascal matrices

The ^nC_r coefficients are usually presented as a triangle but if you fill in with zeros you get an indefinitely extendible matrix, $(R+1) \times (R+1)$, $R = 1$, 2, 3, $\ldots$.

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & \ldots \\
1 & 2 & 1 & 0 & 0 & \ldots \\
1 & 3 & 3 & 1 & 0 & \ldots \\
1 & 4 & 6 & 4 & 1 & \ldots \\
1 & 5 & 10 & 10 & 5 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}.$$

This matrix will be called Pascal$_0$. It is a so-called lower triangular matrix: all entries above the leading diagonal are zero. The determinant of a lower or upper triangular matrix is just the product of the leading diagonal, in this case $1! = 1$. To see this, think of the definition of a determinant. The determinant of a trivial 1×1 matrix is, by definition, just this entry and that of a 2×2 matrix is, by definition, $ad - bc$. For square matrices of higher degree, you take an entry, then mentally blot out row and column and multiply by the determinant of what's left. You work along a row or column in this way imposing the $\pm$ grid

$$
\begin{array}{ccccc}
+ & - & + & - & \ldots \\
- & + & - & + & \ldots \\
+ & - & + & - & \ldots
\end{array}
$$

It is a remarkable feature of square matrices (*only* square matrices?) that you get the same result whatever row or column you choose. We choose the top row since it is nearly all zeros. So we obtain the corner entry x (the determinant of the inner square matrix) and so on until we reach the bottom right corner.

So Pascal$_0$ has an inverse since for any R it has a non-zero determinant. This inverse is pretty obviously the same as Pascal$_0$ with the $\pm$ grid imposed, i.e.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
-1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
1 & -2 & 1 & 0 & 0 & 0 & \ldots \\
-1 & 3 & -3 & 1 & 0 & 0 & \ldots \\
1 & -4 & 6 & -4 & 1 & 0 & \ldots \\
-1 & 5 & -10 & 10 & -5 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
$$

(There is incidentally no difference between a right and a left inverse for square matrices.)

This matrix, Pascal$_0^{-1}$, though banal at first sight, is not wholly without interest. Note that every row except the first must sum to zero since it multiplies the column $(1, 1, 1, \ldots)$ and must give the result zero.

Is Pascal$_0$ the only indefinitely extendible matrix whose inverse is the same as itself apart from the imposition of the $\pm$ grid? No;

$$\begin{pmatrix} a & \dfrac{a^2 - 1}{c} \\ c & a \end{pmatrix}$$

will do for a 2×2 matrix without looking any further.

If we set $a = 1$, we get a lower triangular or upper triangular matrix (the identity matrix counts as both at once). At first it looks as if Pascal_0^{-1} is the simplest non-trivial example but this turns out not to be the case.

Pascal_1^{-1}

There is perhaps not a lot of use for the inverse of Pascal_0 but if we proceed to construct a whole family of Pascal matrices by slicing off the leading diagonal and then taking the inverse, there are some interesting results.

Clearly, Pascal_1 is just Pascal_0 without the 1s, i.e.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 3 & 0 & 0 & 0 & \dots \\ 1 & 4 & 6 & 4 & 0 & 0 & \dots \\ 1 & 5 & 10 & 10 & 5 & 0 & \dots \\ 1 & 6 & 15 & 20 & 15 & 6 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

This matrix is also lower triangular and has determinant $(R + 1)!$ for any $(R + 1) \times (R + 1)$ square matrix ($R = 0, 1, 2, \dots$).

Some entries of the inverse can be guessed at once. It is fairly obvious that the inverse of a lower triangular matrix must also be lower triangular. Also the leading diagonal of Pascal_1^{-1} is going to be the Harmonic Sequence, $1, 1/2, 1/3, \dots$, since we must get a 1 along the leading diagonal when Pascal_1 and its inverse are multiplied together.

The second diagonal starts with $-1/2$ since we must have $1 \cdot 1 + 2x = 0$, $x = -1/2$. We commence

$$\begin{pmatrix} 1 & 0 & 0 \\ -\dfrac{1}{2} & \dfrac{1}{2} & 0 \\ ? & ? & \dfrac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We find that the second diagonal is (surprisingly) constant with value $-1/2$ because in every case we must solve the equation

$$rx + \frac{1}{r+1}\,\frac{r(r+1)}{2} = 0,$$

with solution $x = -1/2$.

We can fill in a row using the zero sum condition—for, once again, every row must yield zero when multiplying the column $(1,\,1,\,1,\,\dots)$.

Pascal_1^{-1} commences

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\[4pt]
-\dfrac{1}{2} & \dfrac{1}{2} & 0 & 0 & 0 \\[4pt]
\dfrac{1}{6} & -\dfrac{1}{2} & \dfrac{1}{3} & 0 & 0 \\[4pt]
0 & \dfrac{1}{4} & -\dfrac{1}{2} & \dfrac{1}{4} & 0 \\[4pt]
-\dfrac{1}{30} & 0 & \dfrac{1}{3} & -\dfrac{1}{2} & \dfrac{1}{5}
\end{pmatrix}.$$

Now any row of the above (continued indefinitely) gives the coefficients of the sum of the powers from 0 to n for power $r = 0,\,1,\,2,\,\dots$ with just this one difference: that the second diagonal must be changed from $-1/2$ to $1/2$. For example, row 2 tells us that

$$\sum_{k=0}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

It is not at all obvious why this is so. Before attempting an explanation, let us examine how this inverse is built up. We can envisage each diagonal of Pascal_1^{-1} as the product of a starting value and a set of conversion factors which in this case take the form ${}^r C_k / r - k + 1$ with $r = 0,\,1,\,2,\,\dots$ and $k = 0,\,1,\,2,\,\dots,\,r$. Each column is stepped back (because we are dealing with a lower triangular matrix) and starts with k zeros followed by unity.

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	...
initial value	1	$-\dfrac{1}{2}$	$\dfrac{1}{6}$	0	$-\dfrac{1}{30}$	...
$r = 0$	1	0	0	0	0	...
$r = 1$	$\dfrac{1}{2}$	1	0	0	0	...
$r = 2$	$\dfrac{1}{3}$	1	1	0	0	...
$r = 3$	$\dfrac{1}{4}$	1	$\dfrac{3}{2}$	1	0	...
$r = 4$	$\dfrac{1}{5}$	1	2	2	1	...
...	...	...	...	...	...	...
r	$\dfrac{1}{r+1}$	1	$\dfrac{r}{2!}$	$\dfrac{r(r-1)}{3!}$	$\dfrac{r(r-1)(r-2)}{4!}$	...

The initial values are determined by the requirement that we start with 1 in the corner and from then onwards every row of Pascal_1^{-1} must sum to zero. Note that if the initial value is zero, as in the case of the fourth column, the column (which becomes a diagonal of Pascal_1^{-1}) will, once multiplied, be zero everywhere notwithstanding the conversion factors.

In fact, if we return to Pascal_0^{-1} we can set out the diagonals in the same way and the initial values turn out to be just $+1$ and -1 and the columns are just the sets of figurate numbers, i.e.

value	$+1$	-1	$+1$	-1	$+1$
$r = 0$	1	0	0	0	0
$r = 1$	1	1	0	0	0
$r = 2$	1	2	1	0	0
$r = 3$	1	3	3	1	0
$r = 4$	1	4	6	4	1

Here, we have a constant first diagonal (presented as a column), namely the leading diagonal with 1, 1, 1, ..., while in the case of Pascal_0^{-1} we had a constant non-zero second diagonal with value $-1/2$.

If we now proceed to Pascal$_2$ and Pascal$_2^{-1}$, i.e. excise the second diagonal of the Pascal Triangle and take the inverse, we obtain

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 \\
1 & 4 & 6 & 0 \\
1 & 5 & 10 & 10 & \cdots
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-\dfrac{1}{3} & \dfrac{1}{3} & 0 & 0 \\
\dfrac{1}{18} & -\dfrac{2}{9} & \dfrac{1}{6} & 0 \\
\dfrac{1}{90} & \dfrac{1}{18} & -\dfrac{1}{6} & \dfrac{1}{10} & \cdots
\end{pmatrix}
$$

This time the leading diagonal of Pascal$_2$ is the set of triangular numbers, F_2, and the determinant is the product $r(r+1)/2!$. The inverse must have leading diagonal $1/F_2$. As always, the sum of any row (except the first) must be zero and we have once more a constant diagonal, the third, with value $1/18$. If we lay out the inverse in the same manner, we obtain

	$k=0$	$k=1$	$k=2$	$k=3$
initial value	1	$-\dfrac{1}{3}$	$\dfrac{1}{18}$	$\dfrac{1}{90}$
$r=0$	1	0	0	0
$r=1$	$\dfrac{1}{3}$	1	0	0
$r=2$	$\dfrac{1}{6}$	$\dfrac{2}{3}$	1	0
$r=3$	$\dfrac{1}{10}$	$\dfrac{1}{2}$	1	1
$r=4$	$\dfrac{1}{15}$	$\dfrac{2}{5}$	1	$\dfrac{4}{3}$
r	$\dfrac{2}{(r+1)(r+2)}$	$\dfrac{2}{r+1}$	1	$\dfrac{r}{3}$

Or, in terms of the sets of figurate numbers we have

$$
\frac{F_0(r+1)}{F_2(r+1)} \quad \frac{F_1(r)}{F_2(r)} \quad \frac{F_2(r-1)}{F_2(r-1)} \quad \frac{F_3(r-2)}{F_2(r-2)}.
$$

One can see an agreeable pattern building up, and I confidently conjecture that Pascal_3^{-1} will have as leading diagonal 1, 1/4, 1/10, ..., or $F_0(r)/F_3(r)$ and that the fourth column will be constant. More generally, for Pascal_j^{-1} the 'conversion factors' will be F_0/F_j, F_1/F_j, F_2/F_j, ... The constant diagonal occurs whenever we get F_j/F_j. The initial values cannot be given by a formula but must be worked out using the zero sum condition.

Indeterminate Coefficients

But why does Pascal_1^{-1}, which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\dfrac{1}{2} & \dfrac{1}{2} & 0 & 0 & 0 \\ \dfrac{1}{6} & -\dfrac{1}{2} & \dfrac{1}{3} & 0 & 0 \\ 0 & \dfrac{1}{4} & -\dfrac{1}{2} & \dfrac{1}{4} & 0 \\ -\dfrac{1}{30} & 0 & \dfrac{1}{3} & -\dfrac{1}{2} & \dfrac{1}{5} \end{pmatrix}$$

give us the coefficients of the sum of the powers?

Assume that for every power r we can express the result of summing the natural numbers from 0 to n with a single set of the form

$$A_0 + A_1 n + A_2 n^2 + \cdots + A_r n^r + A_{r+1} n^{r+1} + \ldots \;=\; f(n, r).$$

However, the 'coefficients' A_k are not constants but functions in r—possibly constant functions.

If $n = 0$, then the grand result is bound to be zero whatever the value of r. So $A_0 = 0$, i.e. there is no coefficient independent of n.

If $n = 1$, then for any power r, the sum of the r-functional coefficients for this r must be unity—a very useful result. (For example, in the case of the sum of the squares we have $1/3 + 1/2 + 1/6 = 1$.)

We can work out the specific values for a particular r simply by using Pascal's Triangle and the method of indeterminate coefficients. For example:

$$1 + 2^3 + 3^3 + \cdots + n^3 = f(n, 3),$$

$$1 + 2^3 + 3^3 + \cdots + (n+1)^3 = f(n+1, 3).$$

But this difference is just the last term $(n+1)^3$ i.e.

$$n^3 + 3n^2 + 3n + 1 = f(n+1, 3) - f(n, 3)$$

$$= A_1 + A_2(2n+1) + \cdots + A_r\left(rn^{r-1} + \frac{r(r-1)}{2}n^{r-2} + \dots\right)$$

$$+ A_{r+1}\left((r+1)n^r + \dots\right).$$

Equating coefficients, we find that all A_k with $k > 4$ must be zero and that $A_4 = 1/4$. Working backwards we obtain the other coefficients. In matrix form, this is equivalent to solving

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 \\ 4 & 3 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A_4 \\ A_3 \\ A_2 \\ A_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}.$$

This is Pascal$_1$ turned on its head as it were, and the inverse Pascal$_1^{-1}$ also gets skewed round, but the individual entries are not affected.

Continuing in this way we can solve for any particular power r, but what one really wants is a general formula. Very early on, after reading Barry Lewis's original article (M500 **162**), I guessed that the general formula for coefficient A_{r-k+1} was

$$A_{r-k+1} = (1 - \text{ coefficients obtained so far }) \frac{{}^rC_k}{r-k+1}.$$

Here r varies with the power while $k = 0, 1, 2, \ldots, r$. It is to be understood that the coefficents within the brackets are previously obtained functions in r which have been given the current value of k.

For powers higher than the $(r+1)$th the coefficients must be zero, so we commence with $A_{r-0+1}(k=0)$, which gives $(1-0)\dfrac{{}^rC_0}{r-0+1} = \dfrac{1}{r+1}$.

For $k = 1$, $A_{r-1+1} = \left(1 - \dfrac{1}{2}\right)\dfrac{r}{r} = \dfrac{1}{2}$.

For $k = 2$, $A_{r-2+1} = \left(1 - \left(\dfrac{1}{3} + \dfrac{1}{2}\right)\right)\dfrac{^rC_2}{r-2+1} = \dfrac{r}{12}$.

So the formula seems to be working. Further coefficients are

$$A_{r-2} = 0,$$
$$A_{r-3} = -\frac{r(r-1)(r-2)}{6 \cdot 5!},$$
$$A_{r-4} = 0,$$
$$A_{r-5} = -\frac{r(r-1)(r-2)(r-3)(r-4)}{6 \cdot 7!}.$$

This led to the conjecture that all further coefficients A_{r-k} with k odd are of the form $-r(r-1)\ldots(r-k+1)/(6 \cdot k!)$.

Now this procedure of evaluating the coefficients is practically identical with the matrix treatment given earlier, with entries worked out according to 'initial values' and 'conversion factors'. But what about this one anomaly that the second diagonal in Pascal_1^{-1} is $-1/2$ instead of $1/2$? All the trouble comes because we must have unity and not zero in the bottom right (or top left) corner of the matrix. (If we put zero we have a singular matrix and the whole thing collapses.) Instead of the coefficients summing to unity, a row in the inverse matrix sums to zero—*except* the original corner 1×1 'matrix', namely (1), which sums to 1.

Proof? What about proof? In fact, I found the recursive formula for the sum of the powers, so easily obtained, devilish to prove and, although I eventually did obtain an *elementary* proof (i.e. one that does not employ calculus or other advanced techniques), it is rather tiresome and will not be given here. (Barry Lewis has himself communicated an elementary proof which only uses the theory of equations to me privately.)

Note that this trick of summing values of a function from 0 to $n+1$ and subtracting the sum from 0 to n can be applied to any function in n. For someone ambitious enough, it should be possible to work out a recursive formula for summing from 0 to n *any* of the sets of figurate numbers to any power r —not just F_1, the natural numbers, to power r. And then there are plenty of other functions in n and r ....

Part II will appear in M500 **174** (June 2000).

Problem 169.2 – Chords – An interesting observation

John Bull

This note derives some interesting results from the chords problem, although the results do not lead to an analytical solution (at least, no more directly than other published solutions [M500 **171**, **172**]).
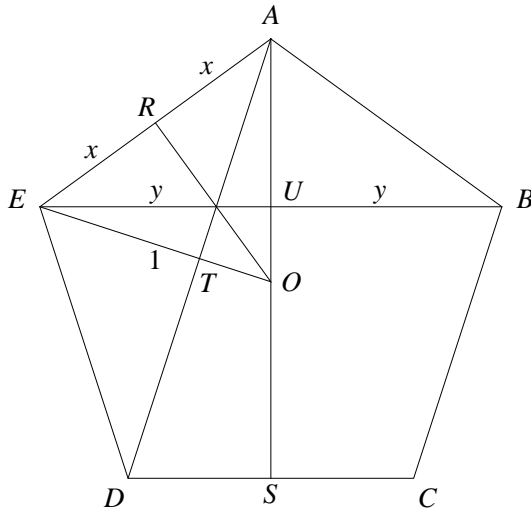

Assuming all the preliminaries, we can arrive at the position shown in the diagram, where triangle ATO is similar to ASD, and triangle ARO is similar to AUE. It follows that

$$\frac{AT}{AO} \;=\; \frac{AS}{AD} \;=\; \frac{y}{1} \;=\; \frac{\sqrt{4y^2 - x^2}}{2y}, \quad \frac{AR}{AO} \;=\; \frac{AU}{AE} \;=\; \frac{x}{1} \;=\; \frac{\sqrt{4x^2 - y^2}}{2x},$$

$$x^2 \;=\; 4y^2 - 4y^4, \qquad\qquad y^2 \;=\; 4x^2 - 4x^4.$$

This is quite a remarkable symmetry. The graphs of these two equations are shown on the next page.

For the original chords problem we would now need to show that $(2^2 xy)^2 = 16x^2y^2 = 5$. A result can be shown numerically, but again one has to resort to trigonometry to make progress with an analytical solution.
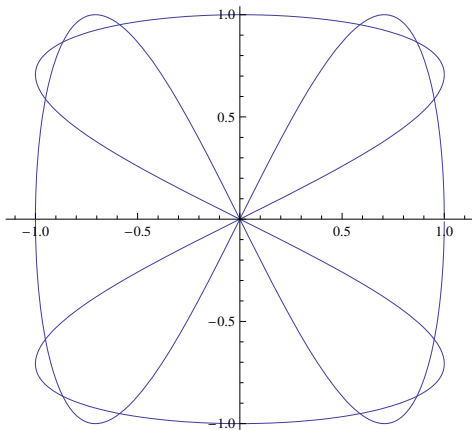
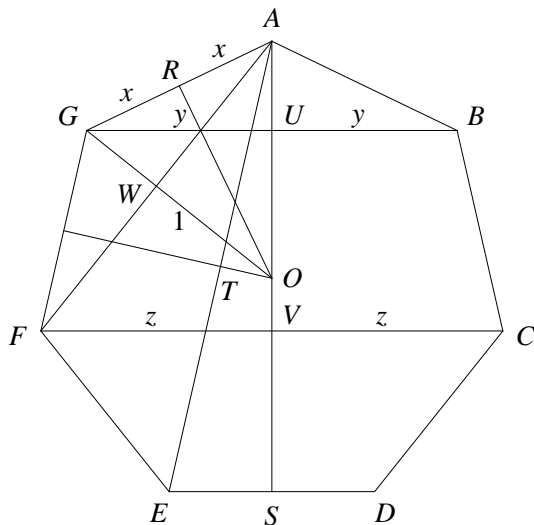Substitutions $x = \sin\alpha$ and $y = \sin\beta$ give the simultaneous equations

$$x = 2y\sqrt{1-y^2} = \sin\alpha = 2\sin\beta\cos\beta = \sin 2\beta,$$
$$\alpha = 2l\pi + 2\beta \quad \text{or} \quad (2k+1)\pi - 2\beta,$$
$$y = 2x\sqrt{1-x^2} = \sin\beta = 2\sin\alpha\cos\alpha = \sin 2\alpha,$$
$$\beta = 2l\pi + 2\alpha \quad \text{or} \quad (2k+1)\pi - 2\alpha.$$

Unfortunately, this does not help very much, although the result is interesting. Now extend the method to a 7-gon.

With reference to the diagram, triangle ATO is similar to ASE, ARO is similar to AUG, and AWO is similar to AVF. This gives

$$\frac{AT}{AO} = \frac{AS}{AE} = \frac{z}{1} = \frac{\sqrt{4z^2 - x^2}}{2z}, \quad x^2 = 4z^2 - 4z^4,$$

$$\frac{AR}{AO} = \frac{AU}{AG} = \frac{x}{1} = \frac{\sqrt{4x^2 - y^2}}{2x}, \quad y^2 = 4x^2 - 4x^4,$$

$$\frac{AW}{AO} = \frac{AV}{AF} = \frac{y}{1} = \frac{\sqrt{4y^2 - z^2}}{2y}, \quad z^2 = 4y^2 - 4y^4.$$

For the original chords problem we would now need to show that $(2^3 xyz)^2$ $= 64x^2y^2z^2 = 7$. Again, a numerical result can be derived. The graph has the form shown on the right.

Following through as before with substitutions
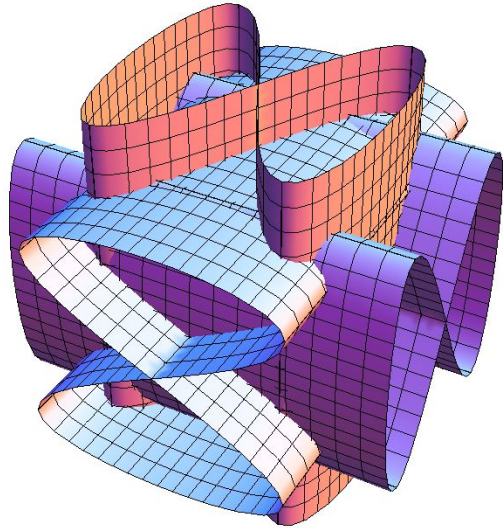
$$x = \sin\alpha,$$

$$y = \sin\beta$$

and

$$z = \sin\gamma,$$

we derive the simultaneous equations

$$\sin\alpha = \sin 2\beta,$$
$$\sin\beta = \sin 2\gamma,$$
$$\sin\gamma = \sin 2\alpha.$$

The general result for an odd n-gon is now obvious, although apparently this does not help very much towards an analytical solution. However, it is quite remarkable that the problem for an odd n-gon ends up with a series of simultaneous equations of the form:

$$\sin\theta_1 = \sin 2\theta_2, \quad \sin\theta_2 = \sin 2\theta_3, \quad \ldots, \quad \sin\theta_{(n-1)/2} = \sin 2\theta_1.$$

For interest, the numerical results (subject to rounding) are:

5-gon:	$x = 0.587785,$	$y = 0.951057,$	
7-gon:	$x = 0.433884,$	$y = 0.781832,$	$z = 0.974928.$

Solution 170.2 – Rational square

R. M. Boardman

> *PQRS* is a square. We require a point *O* such that the distances *OP*, *OQ*, *OR* and *PQ* are integers.

It is easier to solve the problem for rational distances and then multiply by the LCM of the denominators to convert the distances to integers.

The methods used to solve this problem date back to Pierre de Fermat (1601–1665). They find individual solutions rather than the general solution, and no more general methods are known.

First choose *A* on *PQ* such that *OA* is perpendicular to *PQ* and choose *B* on *QR* such that *OB* is perpendicular to *QR*.

It is well known that the sides of a right-angled triangle satisfy $x^2 + y^2 = z^2$ and that integer solutions to this equation may be generated by $x = u^2 - v^2$, $y = 2uv$, $z = u^2 + v^2$. If u and v have no common factor and are not both odd, these numbers have no common factor. Otherwise, they still generate a solution but it may have a common factor. If we want a specific common factor, we multiply by an extra term w.
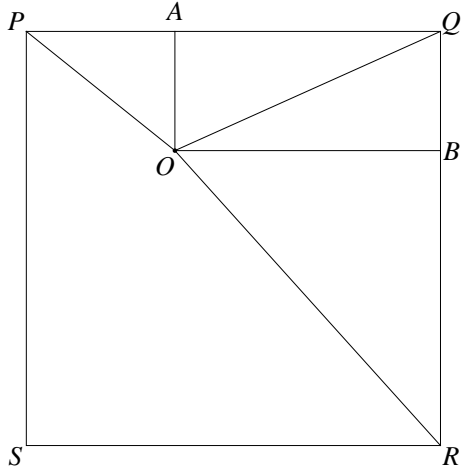
The triangle *OQP* is made up of two right-angled triangles with a common side *OA*. A triangle of this type can be generated by:

$$OA \;=\; 2uvw, \quad AP \;=\; (u^2 - v^2)w, \quad QA \;=\; v(u^2 - w^2),$$

$$OP \;=\; (u^2 + v^2)w, \quad OQ \;=\; v(u^2 + w^2).$$

When combined with the rest of the figure, this ensures that all the lines are integers except *OR*. Again using Pythagoras' theorem we have

$$(OR)^2 \;=\; (QA)^2 + (QA + AP - OA)^2.$$

That is

$$
\begin{aligned}
(OR)^2 \;=\;& v^2(u^2 - w^2)^2 + \left(-2uvw + (u^2 - v^2)w + v(u^2 - w^2)\right)^2 \\
=\;& u^4 w^2 + v^4 w^2 \\
& + v^3 \left(4uw^2 - 2w(u^2 - w^2)\right) \\
& + v \left(-4u^3 w^2 + 2u^2 w(u^2 - w^2)\right) \\
& + v^2 \left(2u^2 w^2 - 4uw(u^2 - w^2) + 2(u^2 - w^2)^2\right).
\end{aligned}
$$

Dividing by w^2 gives

$$
\begin{aligned}
\frac{(OR)^2}{w^2} \;=\;& u^4 + v^4 + v^3 \left(4u - \frac{2u^2}{w} + 2w\right) \\
& + v \left(-4u^3 + \frac{2u^4}{w} - 2u^2 w\right) \\
& + v^2 \left(-2u^2 + \frac{2u^4}{w^2} - \frac{4u^3}{w} + 4uw + 2w^2\right).
\end{aligned}
$$

Fermat found a rational solution to this by replacing the left hand side with $\left(v^2 + mv + n\right)^2$. When expanded, the coefficient of v^4 is 1 in both cases and we can choose m and n such that the coefficients of v^3 and v^2 are also the same. That is,

$$
n \;=\; \frac{1}{2}\left(-4u^2 + \frac{u^4}{w^2} + w^2\right), \qquad m \;=\; 2u - \frac{u^2}{w} + w.
$$

When we insert these values, the terms in v^4, v^3 and v^2 cancel and only the terms in v and the constant term are left:

$$
-\frac{(u^2 - w^2)^2(u^2 - 2uw - w^2)(u^2 + 2uw - w(4v + w))}{4w^4}.
$$

We equate this to zero and solve for v giving

$$
v \;=\; \frac{u^2 + 2uw - w^2}{4w}.
$$

If we insert this value for v back in the equation for $(OR)^2$ it makes it into a perfect square.

$$
(OR)^2 \;=\; \frac{(5u^4 + 4u^3 w - 6u^2 w^2 - 4uw^3 + 5w^4)^2}{256\,w^2}.
$$

Hence, inserting this value for v into the original values for OA, QA and AP we get a 2-parameter solution to the original problem:

$$OA = \frac{1}{2} u(u^2 + 2uw - w^2),$$

$$AP = \frac{(u^2 + 2uw - w^2)(-u^2 + 2uw + w^2)}{16w},$$

$$QA = \frac{(u^2 - w^2)(u^2 + 2uw - w^2)}{4w}.$$

This will in general be a rational solution and can be converted to an integer solution by multiplying by the lowest common multiple of the denominators. For example, putting $u = 2$, $w = 1$ gives $OA = 7$, $AP = 15/16$, $QA = 21/4$. Multiplying throughout by 16 gives $OA = 112$, $AP = 15$, $QA = 84$. The side of the square is $84 + 15 = 99$. In this case O is outside the square.

While this method is elegant and gives a family of solutions, it does not give all solutions. A 'brute force' computer search finds the following:

OA	QA	Side of square
24	45	52
48	140	195
112	84	99
168	99	148
280	165	228
297	396	700

Problem 173.1 – Binomial coefficients squared
Sebastian Hayes

Show that

$$\sum_{r=0}^{n} (-1)^r \, (^nC_r)^2 = \begin{cases} 0 & \text{for } n \text{ odd} \\ \pm^n C_{n/2} & \text{for } n \text{ even.} \end{cases}$$

Here, ^nC_r is the binomial coefficient $n!/(r!(n-r)!)$. The sign of the even result depends on whether n is divisible by 4 or not. For example,

$$1^2 - 4^2 + 6^2 - 4^2 + 1^2 = 6.$$

An early protractor

Alan Slomson

I was intrigued by Dirk Bouwens's description (M500 **171** 18) of the method he was told by a builder for constructing angles. The trigonometry involved is not too difficult.

    We have a semicircle centred on O, $BC = BA$, and the claim is that if the point P divides AO in the ratio $\lambda : 1-\lambda$ then OD divides the angle AOE in approximately the same ratio.

    Suppose that the length of OA is 1 unit. Then BC has length 2 and so, by Pythagoras, OC has length $\sqrt{3}$. For convenience we put $\mu = 1-\lambda$, and applying Pythagoras to the triangle COP, we have that $CP = \sqrt{\mu^2 + 3}$. In the triangle OPD, we have

$$\sin \beta \;=\; \sin(\pi - \gamma) \;=\; \sin \gamma,$$
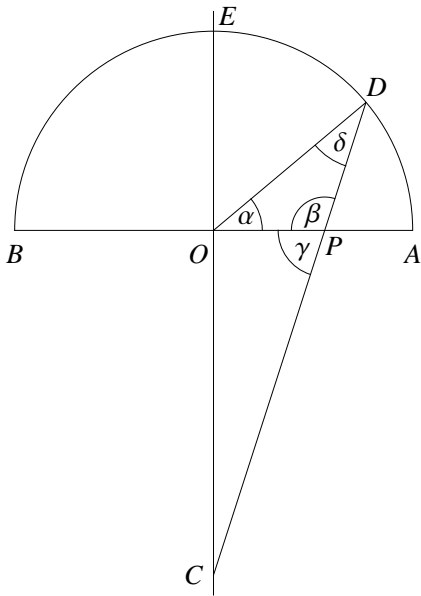
and from the right angled triangle COP we see that

$$\sin \gamma \;=\; \frac{\sqrt{3}}{\sqrt{\mu^2 + 3}}.$$

Now, applying the sine formula to triangle OPD we have $\dfrac{\sin \delta}{\mu} = \dfrac{\sin \beta}{1}$ and so $\sin \delta = \mu \sin \beta = \dfrac{\sqrt{3}\mu}{\sqrt{\mu^2 + 3}}$. Thus we have that

$$\alpha \;=\; \pi - \beta - \delta \;=\; \pi - \sin^{-1}\left(\frac{\sqrt{3}}{\sqrt{\mu^2 + 3}}\right) - \sin^{-1}\left(\frac{\sqrt{3}\mu}{\sqrt{\mu^2 + 3}}\right). \quad (1)$$

Using the fact that $\sin \alpha = \sin(\beta+\delta) = \sin \beta \cos \delta + \cos \beta \sin \delta$, and replacing μ by $1 - \lambda$, we can see that (1) is equivalent to

$$\alpha \;=\; \sin^{-1}\left(\sqrt{3}\,\frac{\sqrt{1 + 4\lambda - 2\lambda^2} - (1-\lambda)^2}{4 - 2\lambda + \lambda^2}\right). \quad (2)$$

It is easy to check that (2) gives $\alpha = 0$ when $\lambda = 0$, and $\alpha = 90$ when $\lambda = 1$. Also when $\lambda = 2/3$, $\alpha = 60$. So in this last case OD also divides the angle AOE in the same ratio as P divides AO. Indeed in this case the triangles POD and PCA are similar, so the angle POD equals the angle PAC, which is 60 degrees.

The surprising thing is that the graph of (2) for $0 \leq \lambda \leq 1$ is very close to a straight line and so, over the whole range of values of λ, OD divides the angle AOE in approximately the same ratio as P divides AO. Thus the comment that the approximation only works when the line OA is divided into 9 or some multiple of 9 is not correct.

Some numerical data is given below. The maximum error in the approximation occurs for $\lambda \approx 0.18$ when the angle AOD is 0.637 degrees greater than the correct angle. In the table, the first column gives the value of 90λ (the angle which divides 90 degrees in the ratio $\lambda : 1 - \lambda$) and the second column gives the value of α calculated from formula (2).

It would be interesting to know of any printed sources for this problem, and also whether anyone can come up with a mathematical analysis of why the graph of (2) is so close to a straight line.

90λ	0	10	20	30	40	50	60	70	80	90
α	0	10.57	20.62	30.47	40.28	50.11	60	69.96	79.96	90

Mick Bromilow

The problem posed by Dirk Bouwens in M500 **171** 18 is a nice example of mathematics in the raw. I used MATHCAD to investigate the relationship. The problem is to compare the angle DOA in degrees with the function $90(1 - x)$ where x is the distance from O that the line CD cuts the x-axis. I have set the radius of the circle to be 1 unit.
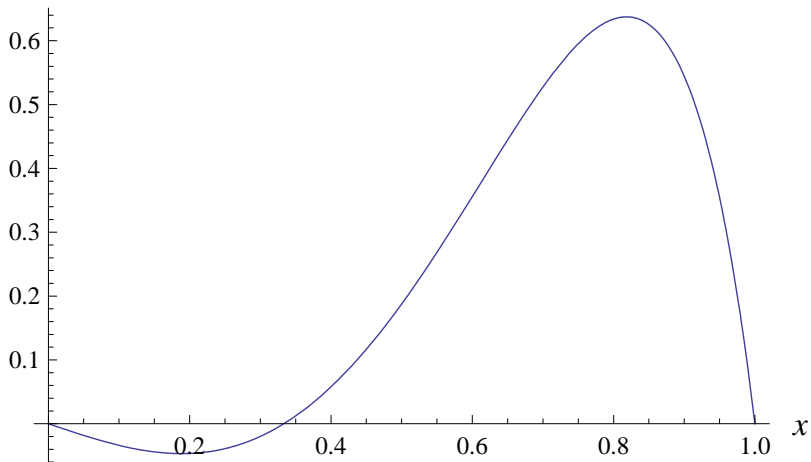
The use of similar triangles (draw a vertical line through D and note that the length of OC is $\sqrt{3}$) shows that

$$\sin \alpha = \frac{-\sqrt{3} \cdot x^2 + \sqrt{9 - 6x^2}}{3 + x^2},$$

where α is the angle DOA, measured in radians. We thus need to consider the function

$$y(x) = \sin^{-1}\left(\frac{\sqrt{9 - 6x^2} - \sqrt{3} \cdot x^2}{3 + x^2}\right) \cdot \frac{180}{\pi},$$

$y(x) - z(x)$

$$\sin^{-1}\left(\frac{\sqrt{9-6x^2}-\sqrt{3}\cdot x^2}{3+x^2}\right)\cdot\frac{180}{\pi}-90(1-x)$$

which gives the angle DOA in degrees. Drawing graphs of this function and of the function

$$z(x) \;=\; 90(1-x)$$

shows that they are remarkably similar over the whole range of values of x between 0 and 1. The plot of $y(x)-z(x)$ shows that the maximum difference is about 0.6 degrees when the angle is approximately 18 degrees.

The accuracy of the method does not depend on the number of points taken.

Problem 173.2 – Nine darts
JRH

In standard darts play, the aim is to score 501 cumulatively, ending on a double, or ending on a bullseye, which scores 50. One dart can score from 0 to 20, or double those numbers, or treble those numbers, or 25 or 50. How many 9-dart finishes are there, ignoring the order of scoring for the first eight darts?

Re: Problem 171.1 – Cylinder

> A coin is a short, fat cylinder. Throw it up in the air and let it fall on to a flat surface. It will almost certainly land on one of its faces, 'heads' or 'tails'. Do the same with a long, thin cylinder and it is far more likely to land on its curved surface rather than on one of its flat ends. At what radius-to-height ratio will the probabilities be equal.

David Singmaster

I think this problem arose from a paper by Frank Budden [Frank Budden, 'Note 64.17: Throwing non-cubical dice', *Mathematical Gazette* **64** (No. 429) (October 1980) 196–198] in which he reported that he had some square rod which he cut into various lengths of cuboids and had his students throw them and record the frequency of landing on a square end. He had no theory for this. I wrote a response [David Singmaster, 'Theoretical probabilities for a cuboidal die', *Math. Gaz.* **65** (No. 433) (October 1981) 208–210] giving a simple geometric probability—the probability of a face is proportional to the solid angle it subtends, viewed from the centroid. However, the theory did not agree with the observed results! I corresponded with Budden about this and he suggested that the cylinder had easier geometry. For a cylinder of height h and radius r, let $s = 2r/h$ be the 'shape' parameter. Then Budden applied my geometric theory to get that the probability of landing on the side (i.e. not one of the circular faces) is $P(s) = (1 + s^2)^{1/2}$.

For the present problem, setting this equal to 1/2 gives $s = \sqrt{3}$, or the diameter should be $\sqrt{3}$ times the height. However, this formula is highly suspect as applying it to a 10p coin, which has $s = 12.3$, we get that the probability of landing on edge is $P = 8.1\%$, but in experience it is essentially zero! I wrote to Victor Bryant giving two anecdotes about coins landing on edge. The first appeared in *Math. Gaz.* **66** (No. 436) (1982). Since then I have seen a number of other comments, for example, J. M. Sharpey-Schafer, a student on an OU maths course in 1971, tossed a coin 1000 times and on the 653rd throw, it landed on edge [*The Guardian* (20 July 1989) 31].

I have discovered earlier and later work, mostly on dice, but some work has considered cylinders. Correspondence with Joe Keller reveals that Fred Mosteller, at Harvard, tried the same type of experiments as Budden, but with pieces cut from a broom handle, i.e. with cylinders, and proposed the same theory that I and Budden did, though it's not clear if he ever published anything. I mentioned the problem to Sir Hermann Bondi, who has published an article on the cylindrical problem, which considers three different modes of bouncing. Other shapes have also arisen.

Problem 173.3 – Primality testing

Tony Forbes

Let A be an algorithm for deciding whether numbers are prime or composite. There are two inputs to A: n and P; n is the number to be tested and P is any finite set of primes. The output of A is given by

$$A(n, P) = \begin{cases} 0 & n \text{ if } n \text{ is composite} \\ 1 & n \text{ if } n \text{ is prime} \\ 2 & n \text{ if } A \text{ cannot tell whether } n \text{ is prime or composite} \end{cases}$$

Furthermore, A is valid if and only if all the numbers in P are primes.

Is it the case that if $A(p, P) = 1$ for all numbers p in P, then every number in P is prime?

————————————

The sort of thing I have in mind is this: Suppose you want to prove that n is prime. One way of doing this is to factorize $n-1$. Suppose $n-1 = FR$, where $F = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ and $p_1, p_2, \ldots, p_r$ are prime. Suppose also that for each prime factor p of F, there is an integer b co-prime to n such that $\gcd(b^{(n-1)/p} - 1, n) = 1$ and $b^{n-1} \equiv 1 \pmod{n}$. Suppose furthermore that $F > \sqrt{n}$. Then n is prime.

Let A represent this procedure and let P contain $p_1, p_2, \ldots, p_r$. For A to be valid we need to prove that $p_1, p_2, \ldots, p_r$ are prime. We use A. But then we need prime factors $q_1, q_2, \ldots q_s$, say, of $(p_1-1)(p_2-1)\ldots(p_r-1)$; so let us assume P contains these as well. Similarly, we use A to prove that $q_1, q_2, \ldots, q_s$ are prime, and therefore we add prime factors of $(q_1-1)(q_2-1)\ldots(q_s-1)$ to P. And so on. Eventually we will have enough factors in P to prove that n is prime. However, for the algorithm to work we need to have proved that all the numbers in P really are prime.

I maintain that we can logically and consistently verify all the primes $p \in P$ by applying the algorithm to them, one by one. If $A(x, P) \neq 1$ for just one number $x \in P$ then possibly P contains composite numbers and in any case the algorithm is not valid for any number m that makes use of x as a factor of $m - 1$.

On the other hand, if $A(p, P) = 1$ for all $p \in P$, it is clear that P must contain only primes. I believe that similar reasoning applies to *any* primality-proving algorithm. Am I right?

Recurrence systems using the TI80 calculator
Patrick Meehan

Students who are considering taking MST121 will come across, in the course materials, a method of calculation known as *recurrence systems*, also called *sequences*. These sequences are based on the idea of using an answer of a previous calculation as part of the formula for the present calculation. In effect, a feedback loop is set up with each calculation relying on previous calculations. In MST121 there are three main types of sequences; arithmetic, geometric, and linear. Their formulas are shown below:-

$$U_0 = a, \quad U_n = U_{n-1} + c, \quad n = 1, 2, 3, \ldots$$
$$U_0 = a, \quad U_n = kU_{n-1}, \quad n = 1, 2, 3, \ldots$$
$$U_0 = a \quad U_n = kU_{n-1} + c, \quad n = 1, 2, 3, \ldots$$

Each of these formulas is of the open form variety. There is also a closed form formula for each type; only the open form formulas will be under discussion here. The value a is the starting value of the sequence. For the arithmetic and linear sequences, c is the value added each time to the previous answer (the step size). For the geometric and linear sequences, k is a predefined constant which is used to multiply the answer of the previous calculation. The variable n holds the value of how many calculations are to be carried out and is also used as an index to obtain the answer to the previous calculation which will be held in a table. Hence, U_n represents a table of values and by varying the value of n any particular answer can be retrieved. When using MathCAD 5.0 in MST121 to calculate sequences the results of each calculation will be stored in a table which you can scroll through. However, it is not always convenient to power up a PC in order to use MathCAD; a calculator is often close at hand. Unfortunately, for students who have taken MU120, having gained considerable ability in using a TI80 calculator you will find that you are unable to directly calculate sequences. Only more advanced models such as the TI83 have this capability. The program listed below, called ITERATE, will enable calculations of the basic open form sequences to be carried out on a TI80.

The program has four parts to it. The first part is just a simple menu front end which enables you to choose between an arithmetic, geometric, and linear sequence calculation. Each type of sequence calculation is coded as a stand alone program which can then be called, as a subroutine, from the front end menu program.

ITERATE uses the *Lists* of the TI80 to store the results of a sequence calculation. *List* L6 has been chosen as this should rarely be used during your course work on MST121 and therefore should not interfere with your studies. Also letters of the alphabet, such as K, are used so that they do

not clash with the letters used by programs like QUADRAT, which are programmed in MU120. The program uses variable K to index into L6, e.g. if K = 3 then L6(K) means L6(3), the third element in L6. (Note: variable K has no relationship to k as used in the basic formulas above.)

```
ITERATE
   :ClrHome :Disp "1.  ARITH"
   :Disp "2.  GEOM"
   :Disp "3.  LINEAR"
   :Input W
   :If W=1
   :prgmARITH
   :If W=2
   :prgmGEOM
   :If W=3
   :prgmLinear
ARITH
   :ClrHome
   :Input "STEP SIZE ", L
   :For (K, 2,10,1)
   :L6(K-1)+L - L6(K)
   :End
GEOM
   :ClrHome
   :Input "MULTIPLIER ",L
   :For (K, 2,10,1)
   :L*L6(K-1) - L6(K)
   :End
LINEAR
   :ClrHome
   :Input "MULTIPLIER ",L
   :Input "STEP SIZE ",M
   :For (K, 2,10,1)
   :L*L6(K-1)+M - L6(K)
   :End
```

Before any of the programs are run *List* L6 needs to be initialised with the first value which equates to the constant a in the above basic formulas. As can be seen, nine calculations are carried out (K-1=10-1) and the ten results are stored in *List* L6. These simple programs can be modified to carry out more complex calculations. All you have to do to obtain a greater number of calculations is to increase the value in the *For* loop where the number 10 resides. But a word of warning: the TI80 can hold a maximum of 99 elements in each *List* so when editing the above programs make sure

this limit is not exceeded. The ability to store elements in each *List* will also be constrained by the amount of memory (RAM) that is free in your calculator. The TI80 has only 8KB of memory, of which approximately 6KB is actually free for your programs. The more programs you have stored the less memory there will be for temporary storage etc., although it has to be said that the TI80's BASIC seems very efficient at tokenising the BASIC commands and hence is not wasteful of memory. Do not forget to clear your *List* before calculating a new sequence. This is particularly important where you are altering the length of the sequence calculations in the *For* loop. If you are already in the *List* editor then you can scroll to the top of L6 until the cursor is on the heading, i.e. the title L6, then press the Clear button and scroll back down into the *List.* This will have the effect of clearing the *List*.

As an example, if you enter as the first value in L6, 0.4, and then run the LINEAR program, with the value 4 for the multiplier and 0.3 as the constant (step size), L6 will contain the following 10 values: 0.4, 1.9, 7.9, 31.9, 127.9, 511.9, 2047.9, 8191.9, 32767.9, 131071.9. As can be seen when using only small values in a recurrence system the resulting answers can grow to large values very rapidly. This should be taken into consideration when using a calculator as its maximum range can soon be exceeded when calculating long sequences. In this case a computer and a software package like MathCAD would be more appropriate. For MST121 you will find that many sequences can be handled by a TI80 calculator. Frequently you will be looking for some form of relationship or trend, like convergence towards some value, so only the first few values need be calculated, because either convergence occurs rapidly or an implied general trend can be detected which can then be verified, if need be, on a computer.

A major benefit of the ITERATE program, even on calculators like the TI83 (which ITERATE was developed on) is that values are stored as a table of values. The sequence function on calculators like the TI83 only calculate the first seven values of any particular sequence, which is the number that can be seen when entering table view mode. As soon as you scroll down (or up) to see more values the calculator has to start a new calculation, and this occurs for every new value that you scroll on to. For more complex sequences this become very laborious and time consuming, even for a TI83 which has a 6 MHz Z80B processor under its bonnet. ITERATE will first calculate all the required number of sequence calculations which you can then easily scroll through quickly in *List* L6. So, even with the relatively slow processor as used in the TI80, once the calculations have been done there is no processing overhead in scrolling through the tabulated values. Also as the values are stored in L6 it is a trivial matter to plot them. This would be useful in order to verify any conjectures about convergence etc.

Observations on Problem 171.2 − $7n + 1$

Barbara Lee

You could write a book on this problem but I only intend to skim the surface. For each positive integer we have the choice:-

<div align="center">

divide by 2
divide by 3
divide by 6
multiply by 7 and add 1.

</div>

1	$\times 7 + 1$	11	$\times 7 + 1$	21	$\div 3$
2	$\div 2$	12	$\div 6$	22	$\div 2$
3	$\div 3$	13	$\times 7 + 1$	23	$\times 7 + 1$
4	$\div 2$	14	$\div 2$	24	$\div 6$
5	$\times 7 + 1$	15	$\div 3$	25	$\times 7 + 1$
6	$\div 6$	16	$\div 2$	26	$\div 2$
7	$\times 7 + 1$	17	$\times 7 + 1$	27	$\div 3$
8	$\div 2$	18	$\div 6$	28	$\div 2$
9	$\div 3$	19	$\times 7 + 1$	29	$\times 7 + 1$
10	$\div 2$	20	$\div 2$	30	$\div 6$

This gives a cycle of six operations that continues for all positive integers. Note that $7n + 1$ applies to all primes and to all composite odd numbers that are not divisible by 3. Side-stepping for a moment, this pattern will give you a formula for detecting where primes and twin primes can possibly be found, but don't get too excited because this is not a method for finding the exact positions of all the primes without testing individual integers.

The sequence starting at 19 is a loop of 30 integers, odd and even, ranging from 19 to 20168. Other sequences can enter this loop at any of the 30 on offer. The highest entry point is $121008 \div 6 = 20168$. Between 1 and 100 there are about fifteen integers whose sequences enter the 19 loop. They are not evenly distributed but most start as a prime and they enter the loop at 31, 37 or 65.

Some integers are 'bunched'. For example, the sequence from 71 takes 27 iterations to reach 1 and includes 83, 85 and 97.

Numbers 1 and 73 both take one iteration to reach a power of 2. Also 4681, 299593, 19173961 and 1227133513 behave in the same way. Similarly, 5, 185, 6665, 239945 and 8638025 all reach a power of 6 on the first iteration. These two patterns can be continued.

It is only worth testing primes and odd composites not divisible by 3. All other integers will result in a sequence that reduces to 1 or to a previously tested integer.

Numbers 75, 79 and 91 generate sequences that increase much faster than 31, rapidly rising to 11 digits and remaining there for at least 100 iterations. These need further investigation.

Finding a solution to this problem will not be easy.

Lies, damned lies & statistics

Andrew Pettit

Three statisticians and three epidemiologists are travelling by train to a conference. The statisticians ask the epidemiologists whether they have bought tickets. They have.

"Fools!" say the statisticians. "We've only bought one between us!"

When the ticket inspector appears, the statisticians hide together in the toilet. The inspector knocks and they pass the ticket under the door. He clips the ticket and slides it back under the door to the statisticians.

The epidemiologists are very impressed, and resolve to adopt this technique themselves. On the return they purchase one ticket between them, and share the journey with the statisticians, who again ask whether they've all bought tickets.

"No," they reply, "we've bought one to share."

"Fools!" say the statisticians; "we've not bought any."

"But what will you do when the inspector comes?"

"You'll see."

This time when the inspector appears, the epidemiologists hide together in the toilet. The statisticians walk up to the door and knock on it. The epidemiologists slide their ticket under the door, and the statisticians take it and use it as before—leaving the epidemiologists to be caught by the inspector.

The moral of this story is that you should never use a statistical technique unless you are completely familiar with it.

We've all heard that a million monkeys banging on a million typewriters will eventually reproduce the works of Shakespeare. Now, thanks to the Internet, we know this is not true. —Robert Wilensky of the University of California.

[Sent by John Bull.]

Letters to the Editors

Points

Point 1. 1666, the year of the fire in Pudding Lane, when written in Roman numerals is MDCLXVI, each letter once in descending numerical order.

Point 2. Heard on 'Big Break': '400 degrees F is the same as 200 degrees C, true or false?' 'False,' replied the contestant. 'Wrong, it's true, just look on your cooker.' Serves me right for watching, I suppose.

M500 **158**, Numbers in swimming pools. My local pool, Liskeard, Cornwall, has the following numbers for depths: 0.850, 1.350 and 1.800. The engineer in me tells me that this means that the depth is controlled to within 0.5 mm. This is obviously wrong because it varies by about ± 2 ins. The figures come from the conversion from imperial to metric. Thus 0.850 is approx. 33.46 ins, a bit over 2 ft 9 ins, and 2 ft 9 ins is 0.8382 m. Writing this as a round number gives 0.850. Similarly $1.350 \approx 53.15$ ins or 4 ft 6 ins is 1.372 m and $1.800 \approx 70.87$ ins or 6 ft is 1.829 m. One decimal place would give more sensible figures. Many years ago, the chain on my motor bike had to have play of approximately 0.5 ins at the point between the two cogs. This meant: try it, if movement is about 0.5 ins, it's OK, otherwise adjust the spacing between the cogs. When the handbook was metricated the measurement became approximately 12.7 mm, giving the impression that the tolerance was now $\pm$ 0.05 mm. What was wrong with 'between 10 and 15 mm'? Lack of common sense is the answer.

M500 **163**, Four-legged table joke. In my local they have four-legged stools and a composite floor around the bar. I have checked the stools, and the legs are the same length. But the floor is not flat, it has small imperfections that make the stools wobble, about an inch at seat height. 'You'll not stop the wobble,' say the locals. 'The mean value theorem says I will,' I reply, remembering the old M203 TV programme of that name and rotating the stool for zero wobble. Now nobody comments on my rotations.

Can any readers supply answers to the following questions?

Euclid's definitions mean that it is impossible to draw a point, a line, a circle or anything using them, e.g., a square. This is akin to Plato's Forms. When we think of a triangle, the thought is perfect but reality is not. The lines, or edges if it is an object, will not be perfectly straight, and the angles will not sum to 180 degrees. When we draw a line, it has thickness, and if viewed through a magnifying glass will show imperfections.

Parallel lines, railway lines for example, are possible in our minds but impossible in practice. If you look along railway lines they are not perfect, bends can be seen in one line that are not in the other line. This is not important if the bends are small, say within 0.5 inches (1.5 cm), as the width of the wheel, with the internal flanges, will keep the wheels on the track. (High speed track may have stricter requirements, but slow speed

lines do show bends.)

Railway lines have a spacing of 4 ft 8.5 ins or 1.435 metres. Now try to imagine that there is a 10 km straight (Plato) railway track. True (Plato/Euclid) parallel railway lines can have a line crossing them (e.g., a sleeper) at 90 degrees to each line. Now suppose there are two more true lines running alongside this track, but the lines are not parallel, the crossing line making an angle of 90 degrees with one line and 89.99 degrees with the other. These lines will cross at a point 8.22 km away.

Question 1. If we stood at the point of correct spacing and looked along the pair of tracks, would we see any difference? The parallel lines would appear to meet, and the other pair would meet (five miles away), but could we tell the difference? Will they appear to meet at the same place or closer than five miles?

Question 2. If we stood at the other end, where they cross, and looked back along the pair of tracks, would we see any difference? The parallel lines would appear to meet, and the other pair would diverge but would we notice it? Would they appear to converge?

Question 3. At what angle of divergence would the lines appear to be parallel?

Pete Charlton

Galileo

Dear Jeremy,

I was interested in David Singmaster's entry at 1581 ['A history of time', David Singmaster, M500 **171**, 1-9]. 'Galileo notes constancy of the pendulum, but he does not use it for timing rolling bodies.' This does not explain why Galileo (or anyone) should have wanted to time rolling bodies, but, having done S100 around 1972, I immediately thought of Galileo's circle-chord theorem. This was the subject of the first home experiment of the course, and involved rolling a ball down a sloping surface and timing it.

Galileo studied pendulums, and noted that the period of a conventional pendulum, in which a bob is constrained to fall down the arc of a circle, is somewhat dependent on amplitude. He worked out by geometry that if the bob could be constrained to fall down the chord of a circle, then provided (and I think he was guessing at Newton's later results) that the acceleration of a falling object was proportional to the force applied to it by gravity (which he also assumed to be constant), the period should not depend on the amplitude. It should be constant for any chord of the circle, including free fall down the diameter. He tried to test this by rolling a ball down a channel; timing it with a water clock. The sloping channel was modelled as a chord of a circle. Experiments using this method do support

the theory, but because the ball picks up rotational kinetic energy as well as translational KE, its acceleration is reduced, and the value obtained for g is lower than it should be.

Galileo may not have realised this problem of rotational KE, but is this the reason for the remark about not timing rolling bodies?

Yours sincerely,

Colin Davies

Lottery odds

Dear Jeremy,

Thank you for sending me M500 **143** along with details of the M500 Society. The reason I wanted that particular issue was because I was told that on page four Andrew Pettit dealt with a probability problem which I have been studying. This was not the case however but the article did help to point me in what seems to be the right direction.

Here is a small taste of some of what I am working on. If you pick n numbers instead of six ($n > 6$) then, choosing from the standard 49 numbers $^{49}C_n$ combinations, what then are the probabilities of six correct, five and the bonus correct, five correct, four correct, three correct, etc. for the general case?

Can anyone help? Many thanks,

Richard Hill

Complex complex complex

Dear Tony,

Two entries for your 'repeated word' sentences. The first was said by the keeper of a cockney pub, the Hand and Glove to a signwriter about a new pub sign: "You've left too much space between 'AND and AND, and AND and GLOVE."

The second concerns an English Grammar Exam taken by two school-boys: "Jim, where John had had 'had', had had 'had had'; 'had had' had had the approval of the examiner."

Dick Boardman

A topical entry for the 'Complex complex complex' series?

'To predict the winner of an American election, study the first important results; that is, the outcome of the primary primary Primary.'

Jean Robertson

Twenty-five years ago

From M500 21/22

Cover—'The Horn of a (Mathematical) Dilemma.' [**ADF**—That was the title attached to a hand-drawn representation of the surface

$$64(z^2 + y^2) + x^4 - 16zx^2 + 4x^2 \;=\; 256,$$

which appeared on the front of M500 **21**. After a certain amount of experimenting I managed to persuade MATHEMATICA to display it for the cover of this issue. It works best in the parametric form

$$\big(8(\cos u)(\cos t), \;\; 2(\cos u)(\sin t), \;\; 8(\cos u)^2(\cos t)^2 + 2\sin u\big),$$

$$0 \le t \le 2\pi, \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}.]$$

Sue White—It's good to get back to the comparative sanity of the Maths Faculty after spending last year studying T100. This was my second foundation course, chosen by a process of elimination—didn't fancy any others, so I thought I'd give it a try. Since I'm no technologist (my husband won't even trust me with a screwdriver) and managed to come out with a distinction (nobody more surprised than me!) based, I suspect, more on knowledge gleaned from M100 than T100, I recommend the course to anyone stumped for a choice. Summer School especially was fun—not much academically, but 12 women being pampered by 200 men is an experience not to be missed.

Harold Moulson—A quadrilateral $ABCD$ has $AB = BC$; angle $ABC = \angle BCD = 90°$; $AD = 2$ units and the area = 4 square units. Determine the length of AB. An obvious answer is 2, i.e. the figure is a square. Are there any other solutions?

M500 Mathematics Revision Week-end 2000

The **26th M500 Society Mathematics Revision Week-end** will be held at **Aston University**, **Birmingham** over **15–17 September 2000**.

We plan to present most OU maths courses. Tutorial sessions start at 19.30 on the Friday and finish at 17.00 on the Sunday. On the Saturday night there is a mathematical guest lecture, a disco, and folk singing. The Week-end is designed to help with revision and exam preparation, and is open to all OU students. For full details and an application form, send an SAE to **Jeremy Humphries**.

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