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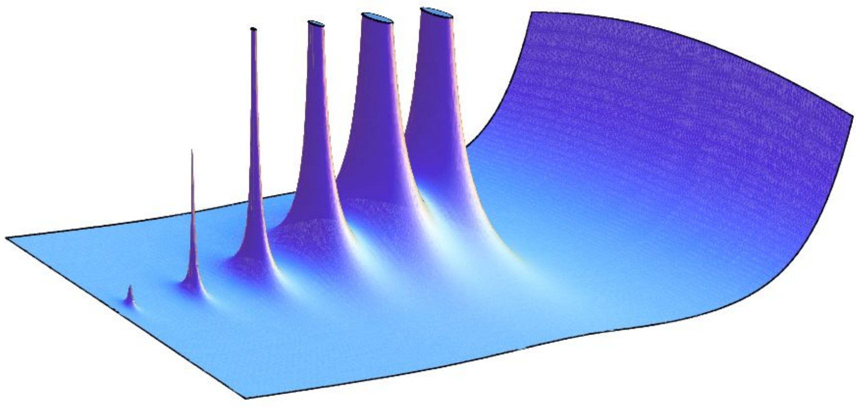

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Pascal Triangle matrices – II

Sebastian Hayes

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

To recapitulate [see Part I, M500 **173** 1]: The entry in row n, column r, is

$$ {}^nC_r \;=\; \frac{n!}{r!(n-r)!}. $$

The columns are the *figurate numbers*, F_0, F_1, F_2, ... (so called because $F_2 = 1, 3, 6, \ldots$ give us the triangular numbers, and F_3 the tetrahedral numbers), with general formula

$$ F_r(n) \;=\; \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}. $$

The coefficients are usually presented as a triangle but if you fill in with zeros, you get an indefinitely extendable matrix,

$$ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 \\ 1 & 5 & 10 & 10 & 5 \end{pmatrix}. $$

This matrix will be called Pascal_0.

Squares of Pascal Matrices

Like any other matrix Pascal_0 can be squared, i.e. we multiply row by column, term by term.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & \dots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & \dots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & \dots \\ 4 & 4 & 1 & 0 & 0 & \dots \\ 8 & 12 & 6 & 1 & 0 & \dots \\ 16 & 32 & 24 & 8 & 1 & \dots \end{pmatrix}.$$

The first column consists of the powers of two, 1, 2, 2^2, ..., as it must do since every row of Pascal's Triangle is being multiplied by unity and then summed, and the sum of any row n of Pascal's Triangle is 2^n. And the diagonals in the above are just the figurate numbers multiplied by the appropriate power of 2. Thus the third diagonal is $2^2(1, 3, 6, \dots)$. Once again we have the familiar pattern of an initial value fixed once and for all and a set of 'conversion factors' which always consist of the figurate numbers F_k, $k = 0, 1, 2, \dots$, in some combination or other.

Transpose squares

When taking matrix squares, the full lines such as 1 3 3 1 do not actually confront each other. I wondered what would happen if they did. The first results are

$$1 \cdot 1 = 1$$

$$1 \cdot 1 + 1 \cdot 1 = 2$$

$$1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 = 6$$

$$1 \cdot 1 + 3 \cdot 3 + 3 \cdot 3 + 1 \cdot 1 = 20$$

This does not, at first sight, exhibit any particular pattern. (The reader might like to pause here and deduce a general formula.)

However, after glancing over Pascal's Triangle, I noted that all these numbers, 1, 2, 6, 20, ..., appear as the middle term of a row of Pascal's Triangle with n even. (There are $n+1$ terms in a row.) The mid-term for n even is ${}^nC_{n/2}$. This led to the conjecture that the square of any row $n = 0$, 1, 2, ... is simply ${}^{2n}C_n$. Thus the sum of the square of 1 4 6 4 1 is ${}^8C_4 = 70$. Why should this be so? We can lay out ${}^nC_r$ as an isosceles triangle instead of a right-angled one. Thus

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \quad \dots \end{array}$$

Also, we can work backwards from the mid-point of an odd numbered row, creating another isosceles triangle. Starting with 252, which is the mid point of row 10,

$$\begin{array}{ccccccccccc}
 & & & & & 252 & & & & & \\
 & & & & 126 & & 126 & & & & \\
 & & & 56 & & 70 & & 56 & & & \\
 & & 21 & & 35 & & 35 & & 21 & & \\
 & 6 & & 15 & & 20 & & 15 & & 6 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Entries in Pascal's Triangle are built up by adding two from the row above, or, in this presentation, from the row below. We have

$$\begin{aligned}
(1)252 &= (1)126 + (1)126 \\
&= (1)56 + (1)70 + (1)70 + (1)56 \\
&= (1)56 + (2)70 + (1)56 \\
&= (1)21 + (3)35 + (3)35 + (1)21.
\end{aligned}$$

The ${}^nC_r$ coefficients 1, 1 1, 1 2 1, 1 3 3 1, ... reappear as they are bound to do because of the way Pascal's Triangle is constructed. We find, if we start with the middle of the tenth row, ${}^{10}C_5 = 252$, that by the time we get to the fifth row we have as coefficients (1 5 10 10 5 1) and so we end up with the matrix square

$$1 \cdot 1 + 5 \cdot 5 + 10 \cdot 10 + 10 \cdot 10 + 5 \cdot 5 + 1 \cdot 1 = 252.$$

In fact, if we build up a diamond shape, the matrix product of any two symmetrically placed rows will always be the same, in this case 252.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & 6 & & 15 & & 20 & & 15 & & 6 & \\
 & & 21 & & 35 & & 35 & & 21 & & \\
 & & & 56 & & 70 & & 56 & & & \\
 & & & & 126 & & 126 & & & & \\
 & & & & & 252 & & & & &
\end{array}$$

More generally, for any even numbered row, there will be $2n+1$ entries—since we start at ${}^nC_0$ —and the central entry will be ${}^{2n}C_n$. Applying the fundamental rule of formation ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$, i.e. every entry in row n is the sum of the two entries above it, we can reduce ${}^{2n}C_n$ by n steps to the full row ${}^nC_n$ with coefficients which turn out to be 1, n, $n(n-1)/2$, $\ldots$, or ${}^nC_r$, $r = 0, 1, 2, \ldots$. Thus every single entry ${}^{2n}C_n$ is the 'square' of row ${}^nC_r$.

These numbers are in fact the leading diagonal of the product of Pascal$_0$ with its transpose Pascal$_0^{\mathrm{T}}$ (which is formed by interchanging rows and columns).

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & \ldots \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \ldots \\ 0 & 1 & 2 & 3 & 4 & \ldots \\ 0 & 0 & 1 & 3 & 6 & \ldots \\ 0 & 0 & 0 & 1 & 4 & \ldots \\ 0 & 0 & 0 & 0 & 1 & \ldots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 2 & 3 & 4 & 5 & \ldots \\ 1 & 3 & 6 & 10 & 15 & \ldots \\ 1 & 4 & 10 & 20 & 35 & \ldots \\ 1 & 5 & 15 & 35 & 70 & \ldots \end{pmatrix}.$$

Another surprising result is that if we take any even numbered row, and take the matrix square of ${}^nC_r$ by ${}^nC_r$ with alternating signs, the result is $\pm$ the middle term. (In the case of an odd numbered row, $n = 3, 5, 7, \ldots$, the result is always zero since entries cancel each other out; e.g. $-1 \cdot 1 + 3 \cdot 3 - 3 \cdot 3 + 1 \cdot 1 = 0$.) For example,

$$\begin{aligned} -1 \cdot 1 + 2 \cdot 2 + -1 \cdot 1 &= 2 \\ -1 \cdot 1 + 4 \cdot 4 - 6 \cdot 6 + 4 \cdot 4 - 1 \cdot 1 &= -6 \\ -1 \cdot 1 + 6 \cdot 6 - 15 \cdot 15 + 20 \cdot 20 - 15 \cdot 15 + 6 \cdot 6 - 1 \cdot 1 &= 20. \end{aligned}$$

(The sign depends on whether the row number n is a multiple of 4 or not.)

I found this result, innocuous though it seems, impossible to prove; so I have set it as a problem for M500 readers! [Problem 173.1, M500 **173** 16.]

${}^nC_r - {}^nC_{r-1}$

Pascal's Triangle is built up by setting the initial entry in the left hand top corner at 1 and assuming everything else in the row is zero. Adding two consecutive entries in a row gives you the entry below the second one, i.e. ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$. Instead of taking the sum I wondered what would happen if I took the difference.

The difference between two successive entries in a row turns out to be the same as the difference between the two in the row above but missing out one, i.e.

$${}^{n}C_r - {}^{n}C_{r-1} \;=\; {}^{n-1}C_r - {}^{n-1}C_{r-2}.$$

For example, 6 − 4 from the row 1 4 6 4 1 = 3 − 1 from the row 1 3 3 1. What happens if we continue in this way? The following curious pattern emerges

	r	$r-1$	$r-2$	$r-3$	$r-4$	$r-5$	$r-6$	$r-7$
n	1	−1	0	0	0	0	0	0
$n-1$	1	0	−1	0	0	0	0	0
$n-2$	1	1	−1	−1	0	0	0	0
$n-3$	1	2	0	−2	−1	0	0	0
$n-4$	1	3	2	−2	−3	−1	0	0
$n-5$	1	4	5	0	−5	−4	−1	0
$n-6$	1	5	9	5	−5	−9	−5	−1
$n-k$	1	$k-1$	...					

The third column is the set of triangular numbers less 1, i.e. 1 − 1, 3 − 1, 6 − 1, ..., with three initial entries *hors série*. And so we go on with the tetrahedral and higher figurate numbers appearing somewhat camouflaged because the third column, $r-2$, is (triangular −1) each time. Note that every row is symmetrical about the middle—apart from sign—and that every row (n odd) has 0 as entry in the middle (why?).

Using this table we can read off the difference between any two successive entries in a row in terms of any earlier row simply by multiplying the coefficients above by the appropriate entries in Pascal's Triangle. For example, suppose we want to have the difference between ${}^{8}C_4$ and ${}^{8}C_3$ (70 and 56 respectively) in terms of row 5. The result is, applying

1	2	0	−2	−1	0	to
5	10	10	5	1		

$$= \; 5 + 20 + 0 - 10 - 1 \;=\; 14 \;=\; 70 - 56.$$

The procedure works even when the difference is negative: if we took ${}^{8}C_5$ and ${}^{8}C_4$ we would end up with −14, which is given by the matrix product

1	2	0	−2	−1	0
1	5	10	10	5	1

$$= \; 1 + 10 + 0 - 20 - 5 + 0 \;=\; -\,14.$$

Pascal's Triangle is in fact just a special case of a matrix obeying the follow-ing rules: 1) the top left corner entry is unity, or ${}^{0}X_0 = 1$; 2) any entry, row n column r, is the sum of the entry above and to the left, or ${}^{n}X_{r-1} + {}^{n}X_{r} = {}^{n+1}X_{r}$.

In the case of Pascal's Triangle, we assume that the second entry in the top line is zero (as are all succeeding entries) but if we start with $1-1$, we get the indefinitely extendable matrix just given, which tells us the coefficients to use for ${}^{n}C_{r} - {}^{n}C_{r-1}$. If we start with $1 - 2$, we get the coefficients that apply to the case ${}^{n}C_{r} - {}^{2n}C_{r-1}$, namely

1	−2	0	0	0	0	0	0	0
1	−1	−2	0	0	0	0	0	0
1	0	−3	−2	0	0	0	0	0
1	1	−3	−5	−2	0	0	0	0
1	2	−2	−8	−7	−2	0	0	0
1	3	0	−10	−15	−9	−2	0	0

The entries per row are no longer symmetrical, and a zero appears amongst them every third row, not every other row. The leading diagonal is -2 and not -1. Like Pascal's Triangle Matrix, this matrix is indefinitely extendible to the right and below, and indeed we could imagine it extending to the left and above also with zero everywhere. It is not immediately evident whether it has an inverse but this seems likely since the first few square inner matrices have non-zero determinants.

${}^{n}C_{r} \times F_{k}$

If we make columns (or rows) alternate in sign, what happens when a column of figurate numbers (discounting the initial zeros) multiplies a row term by term? As stated in my article Zero sum Pascal Triangle (M500 **169**) if k (or r) $< n$, the result is zero. Thus, for example,

$$(1\ -3\ 6\ -10\ 15) \times (1\ 4\ 6\ 4\ 1) \ = \ 1 - 12 + 36 - 40 + 15 \ = \ 0.$$

More specifically, we have the theorem:

If a column k from Pascal's Triangle is multiplied term by term by a row from Pascal's Triangle with alternating signs, $\pm {}^{n}C_{r}$, and then slid across, the result is $k+1$ zero sums and then the coefficients of $\pm {}^{n-(k+1)}C_{r}$ (which sum to zero).

Thus

$$\begin{array}{rrrrrcrl} -1 & 4 & -6 & 4 & -1 & = & 0, & n=4,\ r=0,\ 1,\ \ldots,\ 4 \\ 1 & 2 & 3 & 4 & 5 & = & 0 & \\ & 1 & 2 & 3 & 4 & = & 0 & \\ & & 1 & 2 & 3 & = & -1 & \\ & & & 1 & 2 & = & 2 & \\ & & & & 1 & = & -1 & \end{array}$$

Here (1, 2, 3, ...) is F_1. Proof of the theorem is by repeated induction. In that article I did not discuss what happens if we set $k = n$, or let $k > n$. If $k = n$, we find that the result is always $+1$ or -1, in fact, $(-1)^n$, e.g.

$$\begin{array}{lcl} (1 \quad -2 \quad 1) & 1 & \\ & 3 & = \ (-1)^2 \ = \ 1. \\ & 6 & \end{array}$$

And for $k > n$, we find the same numbers of Pascal's Triangle reappearing like perennials in a flower bed. Thus, trying the next set of figurate numbers, (1 4 10), against (1 -2 1) we find we get result 3, the next set (1 5 15) gives 6, and so on—i.e. we get the triangular numbers 1, 3, 6, 10, ... The full results are set out in matrix form below.

$$\begin{array}{crrrrrcrrrrrr} F_0 & 1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & 1 & 1 \\ F_1 & 1 & 2 & 3 & 4 & 5 & \ldots & 0 & -1 & -2 & -3 & -4 & -5 \\ F_2 & 1 & 3 & 6 & 10 & 15 & \ldots & 0 & 0 & 1 & 3 & 6 & 10 \\ F_3 & 1 & 4 & 10 & 20 & 35 & \ldots & 0 & 0 & 0 & -1 & -4 & -10 \\ F_4 & 1 & 5 & 15 & 35 & 70 & \ldots & 0 & 0 & 0 & 0 & 1 & 5 \\ F_5 & 1 & 6 & 21 & 56 & 126 & \ldots & 0 & 0 & 0 & 0 & 0 & -1 \end{array}$$

$$= \begin{array}{rrrrrrc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -2 & 1 & 0 & 0 & 0 & \ldots \\ 1 & -3 & 3 & -1 & 0 & 0 & \ldots \\ 1 & -4 & 6 & -4 & 1 & 0 & \ldots \\ 1 & -5 & 10 & -10 & 5 & -1 & \ldots \end{array}$$

The general formula is

$$\sum_{r=0}^{n} F_k(r)(-1)^r \, {}^nC_r \ = \ (-1)^n \, {}^kC_{k-n}.$$

We define ${}^nC_{-r}$ as 0 for all n. Then if $k < n$, the total is 0; if $k = n$, the total is ± 1; if $k > n$, the total is ${}^kC_n$. A curious result is that if we introduce the relevant set of figurate numbers back to front as it were, we get the same sums in reverse order i.e.

$$\begin{array}{rrrrrcr} -1 & 4 & -6 & 4 & -1 & = & 0 \\ 1 & & & & & = & -1 \\ 2 & 1 & & & & = & 2 \\ 3 & 2 & 1 & & & = & -1 \\ 4 & 3 & 2 & 1 & & = & 0 \\ 5 & 4 & 3 & 2 & 1 & = & 0 \end{array}$$

The triangle has simply been turned upside down; $(-1, 4, -6, 4, -1)$ is a symmetric set: in the case of an antisymmetric set such as $(-1, 3, -3, 1)$ we would get the signs of ${}^{n-(k+1)}C_r$ reversed, but that is the only difference.

If we define F_{-k} as indicating this, we can extend the original matrix above the top line and make a sort of mirror image.

$$\begin{array}{rrrrrr} -1 & 5 & -10 & 10 & -5 & 1 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -3 & 3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{rrrrrr} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

According to Eli Maor (e, *The Story of a Number*, ch. 8) it was this trick of extending Pascal's Triangle above the top line and alternating the signs (though not in this way) that suggested to Newton that the Binomial Expansion could work for negative indices. Newton tried

$$\begin{array}{rclrrrrrl} n & = & -3 & 1 & -3 & 6 & -10 & 15 & \ldots \\ n & = & -2 & 1 & -2 & 3 & -4 & 5 & \ldots \\ n & = & -1 & 1 & -1 & 1 & -1 & 1 & \ldots \end{array}$$

Reciprocal Pascal

If we take reciprocals of Pascal's Triangle and alternate the sign of columns, we obtain

1							1
1	−1						0
1	−1/2	1					1 + 1/2
1	−1/3	1/3	−1				0
1	−1/4	1/6	−1/4	1			1 + 2/3
1	−1/5	1/10	−1/10	1/5	−1		0
1	−1/6	1/15	−1/20	1/15	−1/6	1	1 + 3/4

The odd numbered rows (the first is row zero) are symmetric sets and thus, since the signs alternate, sum to zero.

For n even the sum is $1+\dfrac{\frac{n}{2}}{\frac{n}{2}+1}$ or $\dfrac{2(n+1)}{n+2}$ with limit 2 as n increases without bound. Thus

$$\sum_{r=0}^{n}\frac{(-1)^r}{{}^nC_r} = \begin{cases} 0 & n \text{ odd} \\ \dfrac{2(n+1)}{n+2} & n \text{ even.} \end{cases}$$

ln 2 *and its successors*

The second diagonal 1, −1/2, 1/3, −1/4, ... is convergent and is today called ln 2.

Several 17th century mathematicians including Newton discovered this series independently as the sum of the area under the hyperbola $y = 1/(1+x)$ between limits 0 and 1. At this time the exponential and logarithmic functions had not been defined as such (though in the air) and the summation was done by slogging it out from first principles (something nobody ever does these days) on the basis of

$$1 \;=\; (1+x)(x-x^2+x^3+\cdots+(-1)^{n+1}x^{n-1})-(-1)^{n+1}x^n$$

with x taking the values 0, $1/n$, $2/n$, ..., n/n.

I wondered whether the other diagonals, i.e. the reciprocals of figurate numbers with alternating signs such as 1, −1/3, 1/6, −1/10, ..., or 1, −1/4, 1/10, ..., converged. In fact they all do.

The ratio test is inconclusive since for every series it is 1, and series with $t_{n+1}/t_n = 1$ can diverge, since the Harmonic Series does. But the conditions for Leibnitz's Alternating Series test are met, a_n —the sequence disregarding sign—is decreasing and $\lim a_n = 0$. Actually, it might be worth proving convergence from first principles for the benefit of those who, like myself, were unfamiliar with this test.

I choose 1, $-1/2$, $1/3$, ... but the argument applies to all the figurate sets with $\pm$-alternating terms. Chop up the series of partial sums t_1, t_2, t_3, ... into odd and even numbered series. Thus t_{odd} starts 1, $1-1/2+1/3$, $1-1/2+1/3-1/4+1/5$, ... This series is decreasing since the difference between successive terms is always of the form $1/(2n+1) - 1/(2n) < 0$. However, the series is bounded below, by 0, since it can be written as a sum of positive quantities $(1-1/2)+(1/3-1/4)+\ldots\ +1/(2n+1)$. Thus, by the Axiom of Completeness, this series converges to limit L_1 say. In much the same way one can show that t_{even} is increasing but less than 1, so it converges to a limit L_2 say. But t_{even} up to term $2n$ is just t_{odd} up to term $2n-1$ plus $-1/(2n)$. Then limit $t_{\text{even}} = \lim t_{\text{odd}} + \lim -1/(2n) = \lim t_{\text{odd}} + 0$. Thus the two limits are equivalent and the joint series converges.

Whether $(-1)^n + F_k/2$—the sets of figurate numbers with alternating signs—converge to anything mathematically interesting I do not know.

$\ln 2$ *as multiplying set*

If we remove the last term from the Pascal Triangle, i.e. we use

$$\begin{array}{llll} 0 \\ 1 \\ 1 & 2 \\ 1 & 3 & 3 \\ 1 & 4 & 6 & 4 \end{array}$$

and multiply by $\ln 2 : 1,\ -1/2,\ 1/3\ , -1/4,\ \ldots,$ we obtain

$$\sum_{r=0}^{n-1} (-1)^r \frac{{}^nC_r}{r+1} \;=\; \begin{cases} 0 & n \text{ for } n \text{ even} \\ \dfrac{2}{n+1} & n \text{ for } n \text{ odd.} \end{cases}$$

Removing a second term we obtain 1 for n even, $\dfrac{1}{2(n+1)} - 1$ for n odd.

Reciprocal figurate numbers as multiplying sets

We now try out the sets of figurate numbers as multiplying sets, with alternating sign applying them to ${}^nC_r$;

$$F_0 \;=\; 1,\;\; -1,\; 1,\;\; -1,\;\; \ldots$$

and the result when applied to ${}^nC_r$ is 0.

$$F_1 \;=\; 1,\;\; -\frac{1}{2},\; \frac{1}{3},\;\; -\frac{1}{4},\; \frac{1}{5},\;\; \ldots,\;\; (-1)^{n-1}\frac{1}{n},\;\; \ldots,$$

with result when applied

$$\begin{array}{llllllll}
1 & & & & & & & 1\\
1 & 1 & & & & & & 1/2\\
1 & 2 & 1 & & & & & 1/3\\
1 & 3 & 3 & 1 & & & & 1/4\\
{}^nC_0 & {}^nC_1 & {}^nC_2 & {}^nC_3 & \ldots & {}^nC_n & & 1/(n+1);
\end{array}$$

$$F_2 \;=\; 1,\;\; -\frac{1}{3},\; \frac{1}{6},\;\; -\frac{1}{10},\;\; (-1)^{n-1}\frac{2}{n(n+1)},$$

$$\begin{array}{llllllll}
1 & & & & & & & 1\\
1 & 1 & & & & & & 2/3\\
1 & 2 & 1 & & & & & 1/2 = 2/4\\
1 & 3 & 3 & 1 & & & & 2/5\\
{}^nC_0 & {}^nC_1 & {}^nC_2 & {}^nC_3 & \ldots & {}^nC_n & & 2/(n+2);
\end{array}$$

$$F_3 \;=\; 1,\;\; -\frac{1}{4},\; \frac{1}{10},\;\; -\frac{1}{20},\;\; (-1)^{n-1}\frac{3!}{n(n+1)(n+2)}$$

$$\begin{array}{llllllll}
1 & & & & & & & 1\\
1 & 1 & & & & & & 3/4\\
1 & 2 & 1 & & & & & 3/5\\
1 & 3 & 3 & 1 & & & & 3/6\\
{}^nC_0 & {}^nC_1 & {}^nC_2 & {}^nC_3 & \ldots & {}^nC_n & & 3/(n+3).
\end{array}$$

The sum is, for figurate numbers $F_k, k = 0,\ 1,\ 2,\ 3,\ \ldots,$

$$\sum_{r=0}^{n} {}^nC_r \frac{1}{F_k(r+1)} \;=\; \frac{k}{n+k},$$

$r = 0, 1, 2, \ldots, k = 0, 1, 2, \ldots, n = 1, 2, 3, \ldots.$

Note that the formula works even for $k = 0$ on the understanding that $F_0 = 1,\ 1,\ 1,\ \ldots$, with alternating signs, for the result is then $0/(n+0) = 0$.

Non-regular dice

David Singmaster

> [**ADF**—When I set 'Problem 171.1 – Cylinder' (Throw a short, fat cylinder, such as a coin, up in the air and let it fall on to a flat surface. It will almost certainly land on one of its faces, 'heads' or 'tails'. Do the same with a long, thin cylinder and it is far more likely to land on its curved surface. At what radius-to-height ratio will the probabilities be equal?) in M500 **171**, I was completely unaware that there exists a considerable body of work that deals with this very difficult question. In fact there is no satisfactory answer. The same applies to all other solids which have an element of asymmetry. If we don't count limiting cases such as spheres and discs, the five regular polyhedra seem to be the only solid objects where there is an adequate theoretical model for allocating probabilities. The problem is unsolved even for a simple $2 \times 1 \times 1$ brick.]

This deals with determining the probability of the various faces of a die which is not a regular polyhedron. The immediate approach is a simple geometric model—the probability of a face should be proportional to the solid angle subtended by that face viewed from the centroid. However, this fails to agree with reality and a number of authors have attempted to explain the real situation by more complex modellings of the physical situation.

Scott Beach *Musicdotes.* Ten Speed Press, Berkeley, California, 1997, p. 77. Says Jeremiah Clarke (*c.*1674–1707), the organist of St. Paul's Cathedral and a composer best known for the Trumpet Voluntary (properly the Prince of Denmark's March, long credited to Purcell) became enamoured of a lady above his station and was so despondent that he decided to commit suicide. Being somewhat indecisive, he threw a coin to determine whether to hang himself or drown himself. It landed on the ground and stuck on edge! Failing to recognise this clear sign, he went home and shot himself! (The text is given as a Gleaning: A loss of certainty, submitted by me, in *Math. Gaz.* **66** (No. 436) (1982) 154.)

J. D. Roberts 'A theory of biased dice.' *Eureka* **18** (1955) 8–11. Deals with slightly non-cubical or slightly weighted dice. He changes the lengths by s and ignores terms of order higher than first order. He uses the simple geometric theory.

L. E. Maistrov *Probability Theory. A Historical Sketch.* Academic Press, 1974. Heilbronner says he measured ancient dice at Moscow and

Leningrad, finding them quite irregular—the worst cases having ratios of edge lengths as great as 1.2 and 1.3.

Scot Morris *The Book of Strange Facts and Useless Information.* Doubleday, 1979, p. 105. Says 6 is the most common face to appear on an ordinary die because the markings are indentations in the material, making the six side the lightest and hence most likely to come up. He says that this was first noticed by ESP researchers who initially thought it was an ESP effect. The effect is quite small and requires a large number of trials to be observable. (I asked Scot Morris for the source of this information—he couldn't recall but suspected it came from Martin Gardner. Can anyone provide the source?)

Frank Budden Note 64.17: 'Throwing non-cubical dice.' *Math. Gaz.* **64** (No. 429) (October 1980) 196–198. He had a stock of 15 mm square rod and cut it to varying lengths. His student then threw these many times to obtain experimental values for the probability of side versus end.

David Singmaster 'Theoretical probabilities for a cuboidal die.' *Math. Gaz.* **65** (No. 433) (October 1981) 208–210. Gives the simple geometric approach and compares the predictions with the experimental values obtained by Budden's students and finds they differ widely.

Correspondence with Frank Budden led to his applying the theory to a coin and this gives probabilities of landing on edge of 8.1% for a UK 10p coin and 7.4% for a US quarter.

Trevor Truran. 'Playroom: The problem of the five-sided die.' *The Gamer* **2** (September/October 1981) 16 & **4** (January/February 1982) 32. Presents Pete Fayers' question about a fair five-sided die and responses, including mine. This considered a square pyramid and wanted to determine the shape which would be fair.

Eugene M. Levin. 'Experiments with loaded dice.' *Am. J. Physics* **51:2** (1983) 149–152. Studies loaded cubes. Seeks formulae using the activation energies, i.e. the energies required to roll from a face to an adjacent face, and inserts them into an exponential. One of his formulae shows fair agreement with experiment.

E. Heilbronner. 'Crooked dice.' *J. Recreational Math.* **17:3** (1984–5) 177–183. He considers cuboidal dice. He says he could find no earlier material on the problem in the literature. He did extensive experiments, *à la* Budden. He gives two formulae for the probabilities using somewhat physical concepts. Taking r as the ratio of the variable length to the length of the other two edges, he thinks the experimental data looks like a bit of the normal distribution and tries a formulae of the form $\exp(-ar^2)$. He then tries other formulae, based on the heights of the centres of gravity, finding that if R is the ratio of the energies required to tilt from one side to another, then $\exp(-aR)$ gives a good fit.

Frank H. Berkshire. *The 'stochastic' dynamics of coins and irregular dice.* Typescript of his presentation to BAAS meeting at Strathclyde,1985. Notes that a small change in r near the cubical case, i.e. $r = l$, gives a change about 3.4 times as great in the probabilities. Observes that the probability of a coin landing on edge depends greatly on how one starts it—e.g. standing it on edge and spinning it makes it much more likely that it will end up on edge. Says professional dice have edge $3/4''$ with tolerance of $1/5000''$ and that the pips are filled flush to the surface with paint of the same density as the cube. Further, the edges are true, rather than rounded as for ordinary dice. These carry a serial number and a casino monogram and are regularly changed. Describes various methods of making crooked dice, citing Frank Garcia, *Marked Cards and Loaded Dice*, Prentice Hall, 1962, and John Scarne, *Scarne on Dice*, Stackpole Books, 1974. Studies cuboidal dice, citing Budden and Singmaster. Develops a dynamical model based on the potential wells about each face. This fits Budden's data reasonably well, especially for small values of r. But for a cylinder, it essentially reduces to the simple geometric model. He then develops a more complicated dynamical model, which gives the probability of a 10p coin landing on edge as about 10^{-8}. He has presented this material in a number of recent talks.

David Singmaster. 'On cuboidal dice.' Written in response to the cited article by Heilbronner and submitted to *JRM* in 1986 but never used. The experimental data of Budden and Heilbronner are compared and found to agree. The geometric formula and Heilbronner's empirical formulae are compared and it is found that Heilbronner's second formulae gives the best fit so far.

I had a letter in response from Heilbronner at some point, but it is buried in my office.

Joseph B. Keller. 'The probability of heads.' *American Mathematical Monthly* **93:3** (March 1986) 191–197. Considers the dynamics of a thin coin and shows that if the initial values of velocity and angular velocity are large, then the probability of one side approaches 1/2. One can estimate the initial velocity from the amount of bounce—he finds about 8 ft/sec. Persi Diaconis examined coins with a stroboscope to determine values of the angular velocity, getting an average of 76π rad/sec. He considers other devices, e.g. roulette wheels, and cites earlier work on these lines.

Frank H. Berkshire. *The die is cast. Chaotic dynamics for gamblers.* Copy of his OHP's for a talk, June 1987. Similar to his 1985 talk.

J. M. Sharpey-Schafer. Letter: 'On edge.' *The Guardian* (20 July 1989). An OU course asks students to toss a coin 100 times and verify that the distribution is about 50:50. He tried it a 1000 times and the coin once landed on edge.

D. Kershaw. Letter: 'Spin probables.' *The Guardian* (10 August 1989. Responding to the previous letter, he says the probability that a spun coin will land on edge is zero, but this does not mean it is impossible.

A. W. Rowe. Letter. *The Guardian* (17 August 1989). Asserts that saying the probability of landing on edge in zero admits 'to using an over-simplified mathematics model'.

K. Robin McLean. 'Dungeons, dragons and dice.' *Math. Gaz.* **74** (No. 469) (October 1990) 243–256. Considers isohedral polyhedra and shows that there are 18 basic types and two infinite sets, namely the duals of the 5 regular and 13 Archimedean solids and the sets of prisms and antiprisms. Then notes that unbiased dice can be made in other shapes, e.g. triangular prisms, but that the probabilities are not obvious, citing Budden and Singmaster, and describing how the probabilities can change with differing throwing processes.

Joe Keller, in an e-mail of 24 February 1992, says Frederick Mosteller experimented with cylinders landing on edge 'some time ago', probably in the early 1970s. He cut up an old broom handle and had students throw them. He proposed the basic geometric theory. Keller says Persi Diaconis proposed the cuboidal problem to him *c.*1976. Keller developed a theory based on energy loses in rolling about edges. Diaconis made some cuboidal dice and students threw them each 1000 times. The experimental results differed both from Diaconis' theory (presumably the geometric theory) and Keller's theory.

Hermann Bondi. The drop of a cylinder. *European J. Physics.* 1994. Considers a cylindrical die, e.g. a coin. Considers the process in three cases: inelastic, perfectly rough planes; smooth plane, for which an intermediate case gives the geometric probabilities; imperfectly elastic impacts.

In late 1996 through early 1997, there was considerable interest in this topic on NOBNET due to James Dalgety and Dick Hess describing the problem for a cuboctahedron. I gave some of the above information in reply.

Re: Problem 171.1 – Cylinder

Gordon Alabaster

A coin is a short fat cylinder. With what radius-to-height ratio will it have an equal probability of falling on its curved edge or its flat faces?

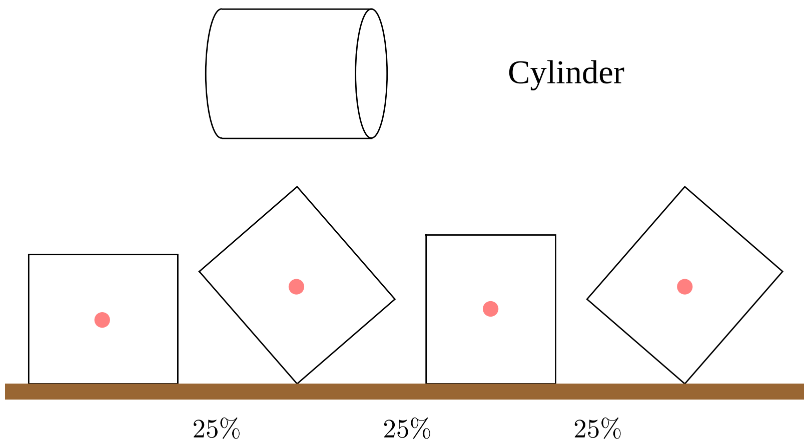
I found myself thinking about this while trying to get to sleep, as such things work marvels for me.

Consider a cylinder with a length about equal to its diameter. (I did not remember the question to the letter.) With the cylinder standing on its curved surface, consider a view perpendicular to the axis of circular symmetry. This will present a rectangular outline with the diameter as height and length as width. (See top of diagram.)

Since the cylinder has circular symmetry when viewed from an end, the above side view is the only one we need consider.

The cylinder is standing on its curved surface. Tip it over slowly to the left and as long as the centre of gravity (CoG) is still vertically over the rectangle base, if released it will return to equilibrium standing on its curved surface. Once the CoG is vertically over the left side, if released it will move to the equilibrium position lying on its face. The same happens if tipped to the right.

We are looking for the probability of the cylinder landing on the bottom or top of this rectangle against landing on either of the two sides. This view has rectangular symmetry, so let's just consider 180 degrees of starting positions, from lying on the left face, through lying on the bottom, to lying on the right face. (See bottom of diagram.) The other 180 degrees would be from lying on the right face, through lying on the top, to lying on the left face.



As indicated in the diagram (bottom row of images), the solution will have equal probability of landing on curved surface or face if it has equal probability for each indicated range over the 180 degrees being considered. This occurs when tipping the rectangle to the unstable equilibrium points is a 45 degree tip, which makes the rectangle a square. That is when the cylinder length is equal to the diameter.

In terms of the question, 'A cylinder with a radius-to-height ratio of 1 to 2 will have an equal probability of falling on its curved edge or its flat faces.'

Solution 172.1 – 345 triangle

> An equilateral triangle encloses a point. The point is 30 metres from one corner, 40 metres from another corner, and 50 metres from the remaining corner. What is the length of the triangle's side?

Dave Ellis

Thanks for publishing my letter in M500 **172**. I'm expecting some of your readers to come up with neat analytical solutions, but, if it's of interest, here's how I did it.

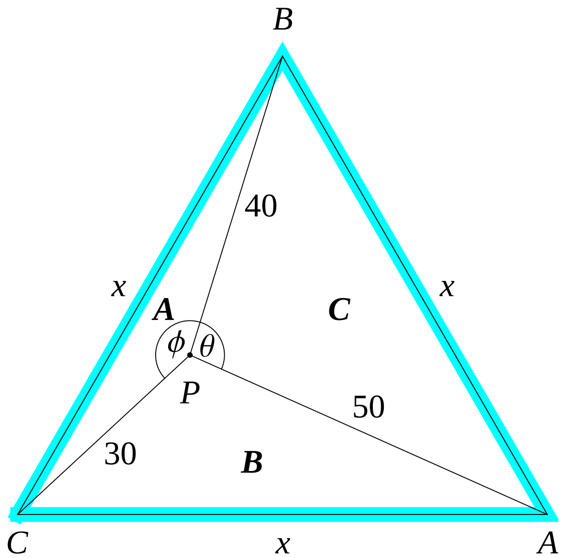
If the triangle's side is x, the total area is $0.25x^2\sqrt{3}$. Then [see diagram on next page]

$$\begin{aligned}\text{Area } \boldsymbol{A} &= \sqrt{(1225 - 0.25x^2)(0.25x^2 - 25)}\\ \text{Area } \boldsymbol{B} &= \sqrt{(1600 - 0.25x^2)(0.25x^2 - 100)}\\ \text{Area } \boldsymbol{C} &= \sqrt{(2025 - 0.25x^2)(0.25x^2 - 25)}\end{aligned}$$

When a value for x is found that sets the total area equal to the sum of areas $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, we have a solution.

It remains only to define a search range for x. By inspection it must be more than 50, so this will be the minimum value in the search range. The equation for Area $\boldsymbol{A}$ imposes the most severe constraint on the maximum value for x, because $1225 - 0.25x^2$ could leave us trying to find the square root of a negative number, and we're not looking at complex solutions here. This indicates x must not be greater than 70, giving a search range of 50 to 70.

I chose to conduct the search using the bisection method to zero in on the correct answer. I elected to work to three decimal places. Here's the code:




```
DECLARE FUNCTION func!  (x!)
min = 50:  max = 70:  '  search range
CLS
DO
    mean = (max + min) / 2
    f = func(mean)
    IF f > 0 THEN
    max = mean
    ELSE
    min = mean
    END IF
LOOP UNTIL ABS(f) < .001
PRINT USING "##.###"; mean
FUNCTION func!  (x)
    y = .25 * x * x
    TotalArea = y * SQR(3)
    AreaA = SQR((1225 - y) * (y - 25))
    AreaB = SQR((1600 - y) * (y - 100))
    AreaC = SQR((2025 - y) * (y - 25))
    func = TotalArea - AreaA - AreaB - AreaC
END FUNCTION
```

This appears to run instantaneously, the result popping onto the screen before you blink! And the answer is: 67.664 metres.

An alternative approach is to use the cosine rule to calculate each of the three angles at the centre of the triangle in terms of x. The sum of these angles is 2π radians, so we have another equation which is easily solved by iterative methods:

$$2\pi - \cos^{-1}\left(\frac{4100-x^2}{4000}\right) - \cos^{-1}\left(\frac{3400-x^2}{3000}\right) - \cos^{-1}\left(\frac{2500-x^2}{2400}\right) = 0$$

I coded and ran this, and got the same results as before equally quickly.

Martyn Hennessy

Firstly, the triangle is scaled by a factor of 1/10 to make the algebra easier to handle. So let $y = x/10$ in the diagram on page 18. Using the cosine formula:

$$\begin{aligned} y^2 &= 3^2 + 4^2 - 2\cdot 3\cdot 4\cos\phi, \\ y^2 &= 4^2 + 5^2 - 2\cdot 4\cdot 5\cos\theta, \\ y^2 &= 3^3 + 5^2 - 2\cdot 3\cdot 5\cos(\phi+\theta). \end{aligned}$$

This gives

$$\cos\phi = \frac{25-y^2}{24}, \qquad (1)$$

$$\cos\theta = \frac{41-y^2}{40}. \qquad (2)$$

Also $\cos(\phi+\theta) = \dfrac{34-y^2}{30} = \cos\phi\cos\theta - \sin\phi\sin\theta$; hence

$$(\sin\phi\sin\theta)^2 = \left(\cos\phi\cos\theta - \frac{34-y^2}{30}\right)^2 = (1-\cos^2\phi)(1-\cos^2\theta). \quad (3)$$

Substituting $\cos\phi$ and $\cos\theta$ from (1) and (2) into (3) gives an expression in y only. After much simplification this reduces to a quadratic in y^2:

$$y^4 - 50y^2 + 193 = 0.$$

So

$$x = 10y = 10\sqrt{25+12\sqrt{3}}, \quad 10\sqrt{25-12\sqrt{3}} \quad (67.66,\ 20.53).$$

The first solution gives the point P inside the triangle while the second gives P outside the triangle.

John Bull

The problem is, as they say, an old chestnut. The most elegant solution I know of can be found in *Mathematical Quickies* by Charles W. Trigg (Dover, 1967, ISBN 0-486-24949-2), Problem 201. It was originally discussed in the *School Science and Maths Magazine*, No. 33 (April 1933), page 450.

ADF—I have it here in front of me. On the diagram (page 18) draw an equilateral triangle PCF so that F is to the north-west of BC. (I have to keep my wits about me because Trigg's triangle is labelled differently.) Draw a triangle CEB with P on CE and angle $CEB = 90°$. Angles PCA and FCB are equal because they are both 60° minus angle PCB. Also $PC = FC = 30$ and $CA = CB$. Therefore triangles PCA and FCB are congruent. Hence $FB = 50$, FPB is a 3 : 4 : 5 triangle, angle FPB is 90°, angle BPE is 30°, BEP is a $1 : \sqrt{3} : 2$ triangle, $BE = 20$, $EP = 20\sqrt{3}$. Thus

$$BC \;=\; \sqrt{20^2 + (30 + 20\sqrt{3})^2} \;=\; 10\sqrt{25 + 12\sqrt{3}}.$$

However, I prefer Martyn's solution because the diagram does not need to be modified. Then along came this next contribution ....

Chris Pile

Whenever M500 drops through my letter-box everything else is neglected until I have, at least, perused the entertainment within. I particularly enjoy the problems because there is often more than one method of solution—as in the case of 170.1.

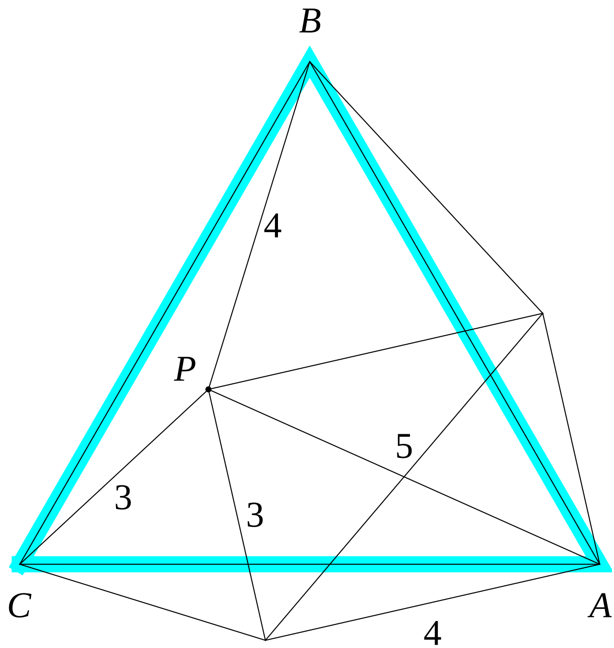
The hard way to do this problem is to use the 'semi-perimeter' formula for the area of a triangle, $\sqrt{s(s-a)(s-b)(s-c)}$, for each of the three small triangles and equate the sum to the area of the equilateral triangle, $x^2\sqrt{3}/4$. The resulting equation can be solved—using a 'simple computer program' to perform the iterations to the desired accuracy. Anyone who does this may wonder why the area of the smallest triangle is exactly equal to 3, and begin to detect the aroma of a scarlet kipper.

The easy way is to use insight, intuition and a bit of Pythagoras.

Start with a 3 : 4 : 5 triangle [we are working in tens of meters] and complete the rectangle. Erect an equilateral triangle on each of the two adjacent sides. Joining the vertices of these triangles and the starting point produces the problem figure. The side length of the large equilateral triangle can then be found:

$$x^2 \;=\; (BQ)^2 + (AQ)^2 \;=\; (3 + \sqrt{12})^2 + 2^2.$$

Hence $x = 6.76643$ (or for the problem as posed, 67.6643 meters).



Book review

Alan Turing: The Enigma by Andrew Hodges

Vintage paperback, 1982, 592 pp., ISBN 0-09-911641-3

John Lee

This book is a biography of the mathematician Alan Turing. It was written by Andrew Hodges, also a mathematician, in 1983 after two years research, and reissued in 1992 as above.

The biography takes us through the life of Alan Turing, from his childhood within a middle-class family (which was spent at public school because his parents were in India—his father was a civil servant in Madras) to his final years in Manchester as head of the new computing department until his death in 1954, officially a verdict of suicide, but thought by his mother to be an accident.

His school days give us a good understanding of life in the 1920s – 30s public school system, where he completed his early education before going to Kings College, Cambridge to study for a degree in mathematics.

After graduating, he stayed on at Kings as a lecturer and to undertake further research in mathematics. His paper 'On computable numbers, with an application to the *Entscheidungsproblem*' (*Proc. London Math. Soc.*, Series 2, Volume 42, 230–265) was a significant contribution in 1936. It defeated one of the Hilbert problems set in 1900, which asked 'was mathematics decidable?', meaning 'did there exist a definite method which could, in principle, be applied to any assertion, and which was guaranteed to produce a correct decision as to whether that assertion was true?'

The answer to this question was 'No', there could exist no 'definite method' for solving all mathematical questions.

Alan Turing was 24 years old when he solved this problem in *Computable Numbers*.

The book contains much of what was at the forefront of mathematics in the 1930s – 1940s, but in a non-technical format for the general reader.

It looked as though Turing was to have a life in teaching in Cambridge, but then the war intervened, and he was to find himself at Bletchley Park, as a member of the code breakers, ultimately to head a team in breaking the German 'Enigma' cipher.

The systems used in breaking the codes ranged from educated guesses, index cards, overlays and the *Bombes*, so called because of the ticking noise they made as they worked through the millions of possible combinations to decode the messages.

Much of this information was suppressed after the war and has only recently been removed from the official secrets, permitting the publication of such information and the names of those involved in this work.

After the war ended, he took up a position at Manchester University. He was involved in the development of the first computers there until his tragic death at the age of 41.

I would recommend this book to anyone interested in the history of mathe- matics and the life of a brilliant mathematician.

Problem 174.1 – Four people

JRH

A group of four people, A, B, C and D, have to get across a bridge at night. The bridge cannot take more than two people at a time. They have one torch, and no crossing can be made without the torch. The torch must always be carried. It cannot be thrown, etc. A can cross the bridge in one minute, B in two minutes, C in five minutes and D in ten minutes. When two people cross, they travel at the speed of the slower person.

What is the shortest time for the whole group to get across?

Nine matches

In M500 **172** we asked: Move one match to correct this sum: .

JRH—So far we [JR & Rose] have three, or four if you are a pedant:

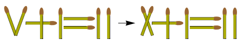
1. Take the vertical one off the + and angle it against the rightmost one, so 5 − 1 = 4.

2. Take the horizontal one off the + and cross it over the leftmost one, so 11 divided by 1 = 11.

3. Take one of the V matches and put it in your pocket, so that 1 + 1 = 2.

4. Ditto with the other V match.

EK—Re. 1: ? Re. 2: Sorry, I can't see it. If anything it becomes 10 / 11 = 11, surely. Re. 3 and 4: Move does not mean remove, nohow. Anyhow 1 does not slope.

The canonical answer is: .

Not only I did not spot that instantly, but also I had seen it before and forgotten. That Al Z gets about quite a bit for an old 'un.

ADF—X + I = II? You have 'X', which is the Roman abbreviation for 1111111111, ten in base 1, and 'II' is eleven in base 10. I'm sorry, you just cannot change the number base whenever you feel like it. Goodness knows what might be if you could. Indeed, the original sum is perfectly valid if you interpret 'II' as six in base 5.

Martin Cooke—  and  both seem correct.

Problem 174.2 – Incredible identity

Show that

$$\sqrt{11+2\sqrt{29}}+\sqrt{16-2\sqrt{29}+2\sqrt{55-10\sqrt{29}}} \;=\; \sqrt{5}+\sqrt{22+2\sqrt{5}}.$$

ADF—A nice little formula to keep you busy for an evening or two! I found it in Henri Cohen's book, *A Course in Computational Algebraic Number Theory* (Springer-Verlag, 1995). He attributes it to Daniel Shanks, who published a number of similar expressions under the title 'Incredible identities' in *Fibonacci Quarterly* (1974).

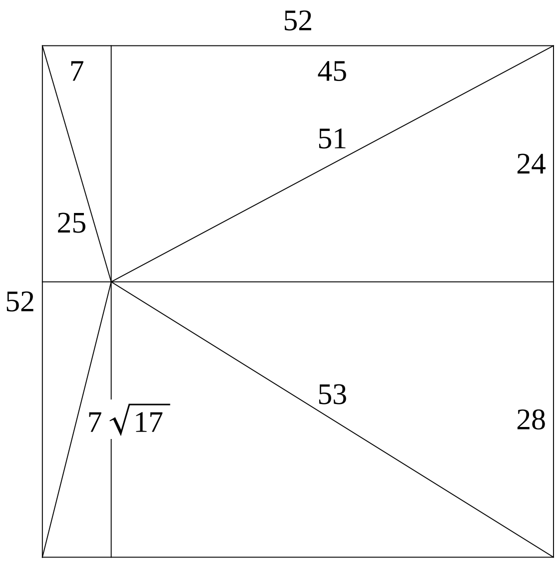
Solution 170.2 – Rational square

> Find a point which is a rational distance from three corners of a unit square.

Chris Pile

I spent some considerable time (more than 10 minutes) on this problem, using trial and error, to no avail. I consulted my list of Pythagorean triples—again with no success. So I gave up and wrote a simple computer program, sat back and waited.

Very soon (less than 10 minutes) the computer produced an answer.



After some considerable time there were no more solutions (apart from reflections and multiples). So I switched off the computer.

Problem 174.3 – Eight wires

Malcolm Maclenan

I am on the third floor with one end of an 8-way cable (all eight wires are identical) and I know the other end is in the basement. Given a continuity meter, what is the least number of trips to the basement I must make to identify each wire?

Solution 172.4 – Grandfather clock

> My grandfather clock needs winding every so often. The difficulty is that one or both of the two holes into which the winding handle goes are periodically covered up by one or both hands. The minute hand covers a winding hole between 15 and 21 minutes and between 39 and 45 minutes during an hour. The hour hand covers a hole between 7:44 and 9:06, and again between 2:48 and 9:10. What is the probability that I can wind the clock?

Ken Greatrix

The minute hand covers a hole for 12 minutes every hour. Let $P(M) = 12/60 = 1/5$. The hour hand covers a hole for 164 minutes in 12 hours. Let $P(H) = 164/720 = 41/180$. During each 82-minute period both hands cover a hole for 13 and 12 minutes, respectively. Let

$$P(B) \;=\; P(H)\left(\frac{13}{82}+\frac{12}{82}\right) \;=\; \frac{5}{72}.$$

Then

$$\begin{aligned} P(W) \;&=\; 1-(P(M)+P(H)-P(B)) \\ &=\; 1-\left(\frac{1}{5}+\frac{41}{80}-\frac{7}{12}\right) \;=\; \frac{77}{120}. \end{aligned}$$

There is almost 2/3 probability that you can wind the clock.

Problem 174.4 – 32 pounds

JRH

You start with £32 and bet against your opponent on the toss of a coin. On each turn you stake half your capital (£32 plus or minus any wins or losses), and your opponent matches your stake. You play six times, and you win half of the plays. What is your capital now?

I notice that my local supermarket is selling kiwi fruit at 15p each, ten for a pound. What's a fair price for nine? Would they insist on £1.35 (because $9 < 10$), or would they let you purchase an imaginary kiwi fruit to make the number up to ten?—**ADF**

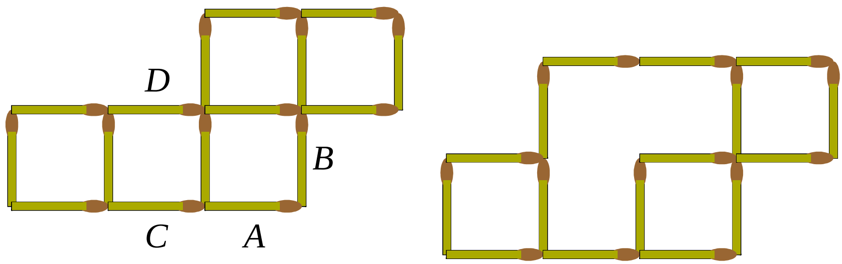
Sixteen matches

Ken Greatrix

In my solution to the problem [M500 **170** 28: move two matches in the left-hand diagram, below, to make four squares], I appealed to the semantics of the situation and thus I offer two variations:

i. 'Move' means 'remove'. Remove A and B, or C and D.

ii. 'Move' means 'move to another position'. The problem says nothing about the size of the squares.



Problem 174.5 – Root 11

Barry Lewis

This problem is concerned with the numbers of the form $(\sqrt{11}+3)^{2n+1}$, where n is a positive integer. For each value of n, we can split such a number into its fractional part f and its integer part I. Prove that these parts have the following properties:

i the fractional part is given by $f = (\sqrt{11}-3)^{2n-1}$;

ii the integer part of $(\sqrt{11}+3)^{2n+1}$, I, is divisible by 2^{n+1};

iii and finally, that $f = \dfrac{1}{2}\left(\sqrt{I^2+2^{2n+3}} - I\right)$.

You must be medically able to wear breathing apparatus and a full, valid driving licence, ... [Advert for Sewage Treatment Process Operators, *Bromsgrove Advertiser*. Spotted by **JRH**.]

Letters to the Editors

Millennium

Dear Tony,

Re Millennium—Issue **172**.

Here we are in February 2000 and we are still worrying about the change-over from 1999 to 2000! Isn't it time to look ahead and make plans for the next major change-over?

The main problem with the 1900s was that many programmers settled for abbreviated two digit years. All computers have now been trained to understand four digit years and so we are safe for the next 8000 years. But what happens in the year 9999 when dates flip back to 0000?

I predict absolute chaos unless we take action soon.

Give thought to our children's children's children's ... children's children.

Ron Potkin

Three people

Dear Jeremy,

I was thinking about shorter questions for the Three People problem [identify as cheaply as possible, at a cost of 1 Mz per word, a liar, a truth-teller and one who alternates between lying and truth-telling, by asking them questions and listening to their answers].

Here is a quite elegant way which costs only 16 Mz.

Ask two of them 'Is Mussolini dead?' *twice*. Two identical questions positively identify any of the three, so there is no need to question the third.

If you are paying for answers only, this costs 4 Mz, as opposed to my earlier solution [M500 **171** 17] which cost 3 Mz. That, too, could be got down to 16 Mz for all words:

As before, one of the three is asked: Is Mussolini dead? Is Mussolini dead? Is your more truthful colleague the taller? But I wouldn't rate these questions as elegant. They still expect that the person asked has knowledge of the other two, though I think that this is assumed in problems of this kind unless stated otherwise. And you might have to measure two people if they were nearly the same height, and so might the person asked—and how many mega-zlotys would he charge for the job?

Ralph Hancock

A curvature anomaly

A cylinder has constant curvature along its length, a cone does not. An ellipse can be formed from the sections of a cylinder or a cone. How can this be?

Ken Greatrix

Decimal currency

Barbara Lee

As long ago as 1853 a House of Commons Committee on a Decimal System of Coinage recommended the adoption of a decimal currency system.

Four different schemes were proposed, all having their relative advantages and disadvantages. The most popular one was based on the sovereign (one pound) which in those days was divided into 240 pennies and each penny into 4 farthings.

The sovereign was to be divided into 10 florins, each florin into 10 cents and each cent into 10 mils. The mil would have been necessary in those days because copper coins included the farthing and halfpenny, which were widely used by poor people. Intermediate coins such as the half-sovereign and a double florin were to be included.

This scheme would still have left us with the pound of the same value as today, and the mil probably abandoned about halfway through the 20th century. The other three schemes were based on the half-sovereign, penny and farthing respectively but all gave rise to many disadvantages. Due to lack of public interest this proposed introduction of a decimal coinage system was never made the subject of an Act of Parliament.

Surprisingly, the 1864 Metric Act of Parliament was passed to permit the use of metric weights and measures, and the teaching of the principles of the metric system in schools in anticipation of this system being adopted in the near future once the decimal coinage was established!

Unfortunately, the use of metric weights for buying and selling resulted in conviction, and we are left with a lot of bad feeling about the loss of the imperial system even though the metric system has been with us in education and science for so many years.

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[*Internet Magazine* article on getting your site to take credit cards. Spotted by **JRH**.]

Twenty-five years ago

From M500 23 & 24

Richard Ahrens—Three men, A, B and C, decide to fight a pistol duel along the following lines. They will first draw lots to determine who fires first, second and third. After positioning themselves at the vertices of an equilateral triangle, they will fire single shots in turn and continue in the same cyclic order until two of them have been hit. The man whose turn it is to fire may aim wherever he pleases. Once a man has been hit, whether killed or not, he takes no further part in the duel. All three men know that A always hits whatever he aims at, B is 80% accurate, and C is 50% accurate.

Assuming that all three adopt the best strategy, and that no-one is hit by a shot not aimed at him, who has the best chance to escape unscathed? What is the exact probability of escape of each of the three men?

Peter Weir—'Stop Press 3' for MST282 gives details of changes to the format of the exam paper. I was shocked to see that no less than 15% of the marks will be given to an essay question. I thought immediately of two reasons for this—firstly that this was an attempt to reduce the high marks that embarrass the Maths Faculty, and secondly that essays are a sop to that infamous *Kettle plan*.

Essays are not new in Maths courses. In M100 1973 a TMA question involved an essay on mode *vs.* arithmetic mean—to get full marks merely involved disguising a simple list of pros and cons as an essay. But 15% is beyond a joke.

Let mathematicians be literate as well as numerate, by all means. But do not erect barriers in the way of specialists. Have mathematical courses, and courses to improve communication and literacy, but not all combined.

Is this evidence of a trend in the OU? Will we be asked to write on such topics as 'Social aspects of the Lebesgue Integral'?

M500 Mathematics Revision Week-end 2000

The **26th M500 Society Mathematics Revision Week-end** will be held at **Aston University**, **Birmingham** over **15 – 17 September 2000**.

Tutorial sessions start at 19.30 on the Friday and finish at 17.00 on the Sunday. On the Saturday night there is a mathematical guest lecture, a disco, and folk singing. The Week-end is designed to help with revision and exam preparation, and is open to all OU students. We plan to present most OU maths courses.

For full details and an application form, send an SAE to **Jeremy Humphries**.

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