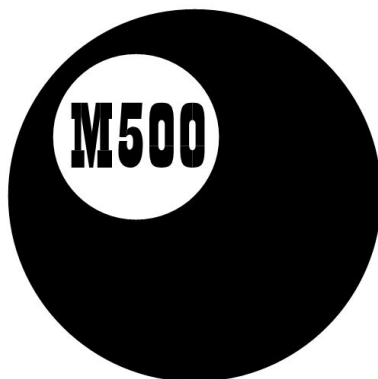


*

ISSN 1350-8539



M500 177



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

Editor – *Tony Forbes (ADF)*

Editorial Board – *Eddie Kent (EK)*

Editorial Board – *Jeremy Humphries (JRH)*

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

Pascal's pyramid, Pascal's hyperpyramid and Pascal's line

Robin Marks

In a previous paper [M500 175 16] we described Pascal's pyramid. We now derive a formula for the number of entries in Pascal's pyramid at a given number of steps from the origin. We extend this to dimensions other than 3 and we shall see that this will enable us to calculate the number of ways to score a total of 6 by throwing up to six dice.

To recall: Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be a set of 3-dimensional position vectors, no two collinear. Note that vector addition is commutative; that is, $\mathbf{e} + \mathbf{f} = \mathbf{f} + \mathbf{e}$. Start at the origin $\mathbf{0} = (0, 0, 0)$. This is step 0. At step 1, add each of the three vectors to $\mathbf{0}$. At step 2, to each of the vector sums obtained at step 1, we add each of the three vectors. At step n , to each of the vector sums obtained at step $n - 1$, we add each of the three vectors. Pascal's pyramid gives the number of ways of choosing a route with p legs in the direction of \mathbf{e} , q legs in the direction of \mathbf{f} , and r legs in direction \mathbf{g} , that is,

$$\frac{(p + q + r)!}{p!q!r!}$$

routes, from the origin to the position given by the vector sum $p\mathbf{e} + q\mathbf{f} + r\mathbf{g}$.

I now wish to calculate the number of different 3-dimensional positions reachable after n steps. This is the same as the number of different vector sums possible containing n vectors, each vector belonging to $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. We start at step 0 at the origin, $\mathbf{0}$. This is located at a single 3-dimensional position. At step 1 there are three positions $\mathbf{0} + \mathbf{e}$, $\mathbf{0} + \mathbf{f}$ and $\mathbf{0} + \mathbf{g}$, that is, the positions \mathbf{e} , \mathbf{f} and \mathbf{g} . At step 2 we arrive at one of the six positions

$$\mathbf{e} + \mathbf{e}, \mathbf{f} + \mathbf{f}, \mathbf{g} + \mathbf{g}, \mathbf{e} + \mathbf{f}, \mathbf{e} + \mathbf{g}, \mathbf{f} + \mathbf{g}.$$

At step 3 we arrive at one of the ten positions

$$\begin{aligned} &\mathbf{e} + \mathbf{e} + \mathbf{e}, \mathbf{f} + \mathbf{f} + \mathbf{f}, \mathbf{g} + \mathbf{g} + \mathbf{g}, \\ &\mathbf{e} + \mathbf{e} + \mathbf{f}, \mathbf{e} + \mathbf{e} + \mathbf{g}, \mathbf{e} + \mathbf{f} + \mathbf{f}, \mathbf{e} + \mathbf{f} + \mathbf{g}, \mathbf{e} + \mathbf{g} + \mathbf{g}, \mathbf{f} + \mathbf{f} + \mathbf{g}, \mathbf{f} + \mathbf{g} + \mathbf{g}. \end{aligned}$$

Let us, for example, add six vectors to $\mathbf{0}$ in the order $\mathbf{e}, \mathbf{g}, \mathbf{e}, \mathbf{e}, \mathbf{f}, \mathbf{f}$. The vector sum is: $\mathbf{e} + \mathbf{g} + \mathbf{e} + \mathbf{e} + \mathbf{f} + \mathbf{f}$. The same vector sum would be arrived at by adding these six vectors in any order, since vector addition is commutative. Let us, then, arrange the vector sum in alphabetical order. In this example this gives $\mathbf{e} + \mathbf{e} + \mathbf{e} + \mathbf{f} + \mathbf{f} + \mathbf{g}$. The question now is, how

many different alphabetically-sorted vector sums are there at step 6? Well, there are three different vectors and six addition steps. Imagine moving from the origin to the positions specified by the vector sum expression, step by step.

We start at step 0 at the position given by the vector $\mathbf{0}$, that is, the origin. We choose to add e s first. This is represented by the \star at the top left in the following table.

Step	Vector		
	e	f	g
0	\star		
1	#		
2	#		
3	#	\star	
4		#	
5		#	\star
6			#

The symbol \star marks the vector chosen to be added next; # marks addition of a vector to the sum.

We add the first e . In the table we step down one row. We add the second and third e s. In the table we step down two more rows. Since there are no more e s, we add f s. In the table we move right one column. We add the two f s. In the table we step down two more rows. There are no more f s. We now only have g s to consider. In the table we move right, reaching the final column. We add the remaining g . In the table we step down one row to the final row.

What have we achieved? Well, in the 7×3 table we started at step 0 at the top left, moved down a total of $7 - 1 = 6$ rows and to the right a total of $3 - 1 = 2$ columns, and ended up at the bottom right. Any combination of six steps down and two moves to the right in the table will represent a vector sum. There are $(6 + 2)! / (6! 2!) = 28$ such combinations. Hence the number of different vector sums is 28. Hence there are 28 numbers in Pascal's pyramid at step 6.

As a matter of interest we can use the same reasoning to calculate the number of ways to throw a total score of 6 with up to three dice: in the table above the three columns now represent the three dice; the number of # signs in the column represents the score on each die (no # signs in a column means the die has not been thrown). Imagine selecting one die from the three at random, throwing it, and if the score is less than 6 throwing

another, and so on until either a total score of 6 is achieved (success) or all the three dice have been thrown (failure). To check this we can calculate the number of ways of throwing a particular combination of numbers as

$$\frac{3!}{a! b! c! d! e! f! g!},$$

where a is the number of dice not thrown (score zero), b is the number of dice showing 1, c is the number of dice showing 2, d is the number of dice showing 3,

$$\text{We can throw a 1, a 2 and a 3 in } \frac{3!}{0! 1! 1! 1! 0! 0! 0!} = 6 \text{ ways.}$$

$$\text{We can throw three 2s in } \frac{3!}{0! 0! 3! 0! 0! 0! 0!} = 1 \text{ way.}$$

$$\text{We can throw a single 6 in } \frac{3!}{2! 0! 0! 0! 0! 0! 1!} = 3 \text{ ways.}$$

$$\text{We can throw a 4 and two 1s in } \frac{3!}{0! 2! 0! 0! 4! 0! 0!} = 3 \text{ ways.}$$

$$\text{We can throw two 3s in } \frac{3!}{1! 0! 0! 2! 0! 0! 0!} = 3 \text{ ways.}$$

$$\text{We can throw a 4 and a 2 in } \frac{3!}{1! 0! 1! 0! 1! 0! 0!} = 6 \text{ ways.}$$

$$\text{We can throw a 5 and a 1 in } \frac{3!}{1! 1! 0! 0! 0! 1! 0!} = 6 \text{ ways.}$$

Total: 28 ways, which agrees with our formula $\frac{(6+2)!}{6! 2!} = 28$.

Generalized Pascal's hyperpyramid

Let d be a positive integer. In general we can construct a Pascal's hyperpyramid using a set of d vectors, non-collinear and d -dimensional, $\{\mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h}, \dots\}$. After n steps this leads to a table sized $(n+1) \times d$, as shown above. Moving from the upper left to the lower right in the table involves n steps down and $d-1$ steps right. The number of different vector sums after n steps is therefore $(n+(d-1))!/(n!(d-1)!)$.

Six-dimensional Pascal's hyperpyramid

When $d=6$ we get the 6-dimensional Pascal's hyperpyramid. The 6-dimensional Pascal's hyperpyramid gives the number of ways of choosing a route with p legs in the direction of \mathbf{e} , q legs in the direction of \mathbf{f} , r legs in direction \mathbf{g} , s legs in direction \mathbf{h} , t legs in direction \mathbf{i} , and u legs in the direction of \mathbf{j} , that is, $(p+q+r+s+t+u)!/(p! q! r! s! t! u!)$ routes, from the origin to the position given by the vector sum $p\mathbf{e} + q\mathbf{f} + r\mathbf{g} + s\mathbf{h} + t\mathbf{i} + u\mathbf{j}$.

After step 6 we have

$$\frac{(n + (d - 1))!}{n!(d - 1)!} = \frac{(6 + (6 - 1))!}{6!(6 - 1)!} = \frac{11!}{6!5!} = 462$$

different vector sums, and therefore 462 entries in the Pascal's hyperpyramid. The Pascal's hyperpyramid entry when, for example, $p = 1$, $q = 1$, $r = 1$, $s = 1$, $t = 1$ and $u = 1$, is

$$\frac{(p + q + r + s + t + u)!}{p!q!r!s!t!u!} = \frac{6!}{1!1!1!1!1!1!} = 6! = 720.$$

Another example of a Pascal's hyperpyramid entry is $p = 0$, $q = 0$, $r = 6$, $s = 0$, $t = 0$ and $u = 0$:

$$\frac{(p + q + r + s + t + u)!}{p!q!r!s!t!u!} = \frac{6!}{0!0!6!0!0!0!} = 1.$$

Pascal's triangle

When $d = 2$ we get the familiar 2-dimensional Pascal's triangle. The 2-dimensional Pascal's triangle gives the number of ways of choosing a route with p legs in the direction e and q legs in the direction f , that is, $(p + q)!/(p!q!)$ routes, from the origin to the position given by the vector sum $pe + qf$. The number of entries after step 6 is $(6 + (2 - 1))!/(6!(2 - 1)!) = 7$. The entry for, for example, $p = 2$ and $q = 4$ is $(p + q)!/(p!q!) = 6!/(2!4!) = 15$.

Pascal's line

When $d = 1$ we get what we will call Pascal's line. The 1-dimensional Pascal's line gives the number of ways of choosing a route with p legs in the direction e ; that is, $p!/p!$ routes, from the origin to the position given by the vector pe . The number of entries after step 6 is $(6 + (1 - 1))!/(6!(1 - 1)!) = 1$. The entry for, for example, $p = 6$ is $p!/p! = 6!/6! = 1$. In fact Pascal's line is just a line of equally-spaced 1s.

Ways to throw a total of 6 with up to 6 dice

We are now able to calculate the number of ways to throw a total of 6 with up to six distinguishable dice thrown one at a time, according to the following rules: Choose one of the six dice. (We could use a seventh die to choose.) Throw it. If a 6 is thrown we stop. Otherwise continue by throwing one of the remaining dice. If a total score of 6 is achieved, stop. If not continue until all six dice are used, or the total score equals or exceeds 6. If a total score of 6 is achieved, make a note of the score on each thrown die.

Answer: We can imagine a table similar to the sorted one above containing # signs, but containing 7 rows, including step 0, and 6 columns, one each for the vectors e, f, g, h, i, j . Each column represents a die. The number of # signs in a column represents the number of spots shown on that die. We have $d = 6$ and $n = 6$. So the number of ways to throw a total of 6 with up to 6 dice is

$$\frac{(n + (d - 1))!}{n!(d - 1)!} = \frac{(6 + (6 - 1))!}{6!(6 - 1)!} = \frac{11!}{6!5!} = 462.$$

Check: To check this we can calculate the number of ways of throwing a particular combination of numbers as $6!/(a!b!c!d!e!f!g!)$, where a is the number of dice not thrown (score zero), b is the number of dice showing 1, c is the number of dice showing 2, d is the number of dice showing 3, ...

$$\text{We can throw six 1s in} \quad \frac{6!}{0!6!0!0!0!0!} = 1 \text{ way.}$$

$$\text{We can throw a single 6 in} \quad \frac{6!}{5!0!0!0!0!0!} = 6 \text{ ways}$$

(one way with each die).

$$\text{We can throw two 3s in} \quad \frac{6!}{4!0!0!2!0!0!0!} = 15 \text{ ways.}$$

$$\text{We can throw three 2s in} \quad \frac{6!}{3!0!3!0!0!0!0!} = 20 \text{ ways.}$$

$$\text{We can throw a 4 and a 2 in} \quad \frac{6!}{4!0!1!0!1!0!0!} = 30 \text{ ways.}$$

$$\text{We can throw a 5 and a 1 in} \quad \frac{6!}{4!1!0!0!0!1!0!} = 30 \text{ ways.}$$

$$\text{We can throw four 1s and a 2 in} \quad \frac{6!}{1!4!1!0!0!0!0!} = 30 \text{ ways.}$$

$$\text{We can throw a 4 and two 1s in} \quad \frac{6!}{3!2!0!0!1!0!0!} = 60 \text{ ways.}$$

$$\text{We can throw a three 1s and a 3 in} \quad \frac{6!}{2!3!0!1!0!0!0!} = 60 \text{ ways.}$$

$$\text{We can throw two 1s and two 2s in} \quad \frac{6!}{2!2!2!0!0!0!0!} = 90 \text{ ways.}$$

$$\text{We can throw a 1, a 2 and a 3 in} \quad \frac{6!}{3!1!1!1!0!0!0!} = 120 \text{ ways.}$$

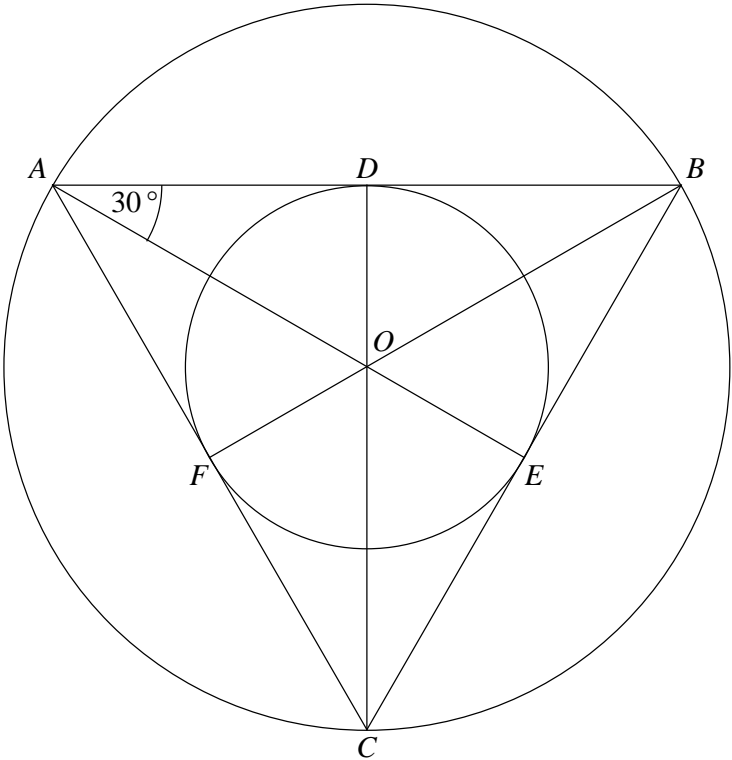
$$\text{Total: } 462 \text{ ways, which agrees with our formula: } \frac{(6 + 5)!}{6!5!} = 462.$$

Two theorems with some applications

David L. Brown

THEOREM 1 *The radius of the circumcircle of an equilateral triangle is twice the radius of its incircle.*

Proof It is clear that the centre of the circumcircle and the centre of the incircle of $\triangle ABC$ are the same point, O , say. Join O to the vertices and let D, E, F be the points at which radii of the circumcircle meet the sides of $\triangle ABC$. By construction, $\angle ODA = \angle ODB = \angle OEB = \angle OEC = \angle OFC = \angle OFA = 90^\circ$ and $AD = DB = BE = EC = CF = FA$ (because O is the centre of the circumcircle of $\triangle ABC$). In the right triangle AOD (for example), angle $OAD = 30^\circ$. Therefore $\sin 30^\circ = DO/AO = 1/2$. Therefore $AO = 2DO$.



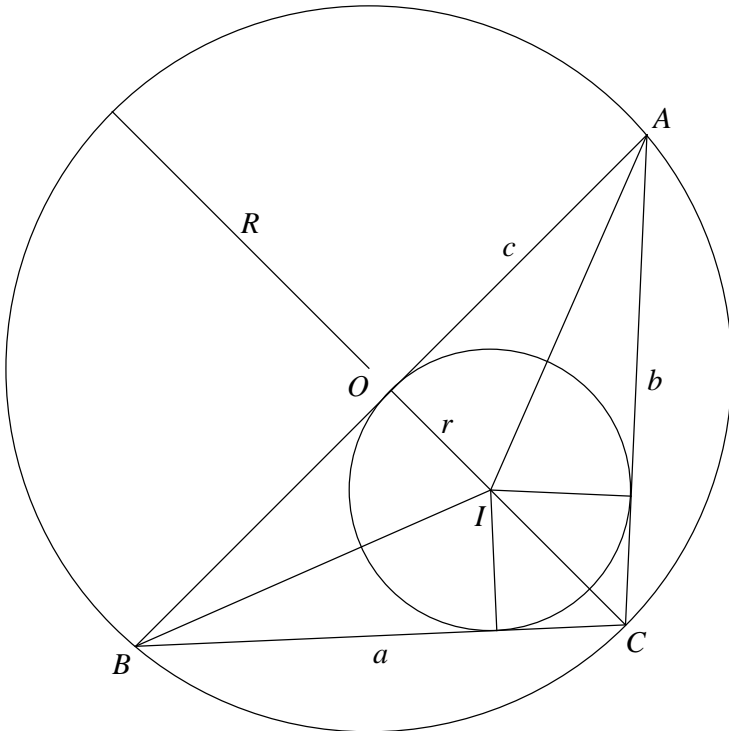
THEOREM 2 *The ratio of the product of the sides of a triangle to the sum of its sides is twice the product of the radii of its incircle and its circumcircle.*

Proof Given triangle ABC with sides of length a , b and c with inscribed circle of radius r (centre I) and circumcircle of radius R (centre O), it is required to prove that $abc/(a + b + c) = 2rR$. If the area of $\triangle ABC$ is Δ , then $\Delta = ab(\sin C)/2$. From the sine rule, $\sin C = c/(2R)$; therefore $\Delta = abc/(4R)$. The sum of the areas of triangles BIC , CIA and AIB is the area of triangle ABC . Hence

$$\Delta = \frac{ra}{2} + \frac{rb}{2} + \frac{rc}{2} = \frac{r}{2}(a + b + c).$$

Therefore

$$\frac{r}{2}(a + b + c) = \frac{abc}{4R} \text{ giving } \frac{abc}{a + b + c} = 2rR.$$



COROLLARY 1 The above result can be expressed as $rR = \frac{abc}{2(a+b+c)}$, or, verbally: *The area of the rectangle contained by the radii of the incircle and circumcircle of a triangle is equal to the ratio of the product of its sides to twice their sum.*

An alternative way of expressing Corollary 1 is: *The area of the rectangle contained by the radii of the incircle and circumcircle of a triangle is equal to the ratio of the volume of the cuboid formed by the sides of the triangle to twice the perimeter of the triangle.*

COROLLARY 2 If ABC is an equilateral triangle, then $a = b = c = p$ (say), $r = R/2$ and the centres of the circumcircle and inscribed circle are at the same point (Theorem 1); hence $p^3/(3p) = R^2$ giving $p^2 = 3R^2$ from which we can say: *The square contained by a side of an equilateral triangle is three times the square contained by the radius of its circumcircle.*

COROLLARY 3 If triangle ABC is right-angled at C then $c = 2R$; therefore

$$\frac{abc}{a+b+c} = 2Rr = cr.$$

Therefore $r = \frac{ab}{a+b+c}$ and hence we can say that: *The radius of the incircle of a right-angled triangle is equal to the ratio of the product of the non-hypotenuse sides to the sum of the three sides.*

An alternative way of stating Corollary 3 is: *The radius of the incircle of a right-angled triangle is equal to the ratio of the rectangle contained by the non-hypotenuse sides to the perimeter.*

COROLLARY 4 From Corollary 3 we know that, for a right-angled triangle with hypotenuse equal to c , $r = \frac{ab}{a+b+c}$. But $c = 2R$; hence for the special case where $a = b = q$ (say), $r = \frac{q^2}{2q+2R}$ giving $q^2 = 2r(q+R)$. Therefore we can say that: *The square of one of the equal sides of a right isosceles triangle is twice the rectangle contained by radius of the incircle and the sum of the remaining equal side with the radius of the circumcircle.*

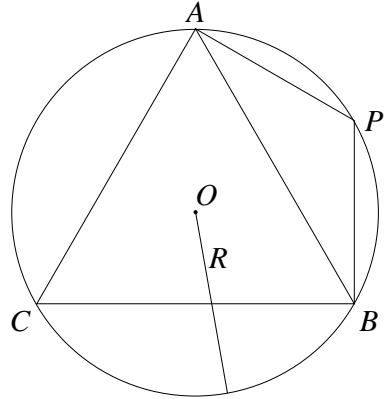
COROLLARY 5 If q is the length of one of the equal sides of a right isosceles triangle and R is the length of the radius of its circumcircle, then $c = 2R$. But $c^2 = 2q^2$. Therefore $q = R\sqrt{2}$. Hence it could be stated that: *The equal sides of a right isosceles triangle are $\sqrt{2}$ times the radius of its circumcircle.*

CONSTRUCTIONS

1. To construct a line of length $n\sqrt{3}$. From Corollary 2, $p^2 = 3R^2$. Therefore $p = R\sqrt{3}$, so construct a circle with radius $R = n$. From any point P on the circumference, find A and B that

$$PA = PB = n.$$

Then $AB = n\sqrt{3}$, because it is a side of the equilateral triangle ABC .



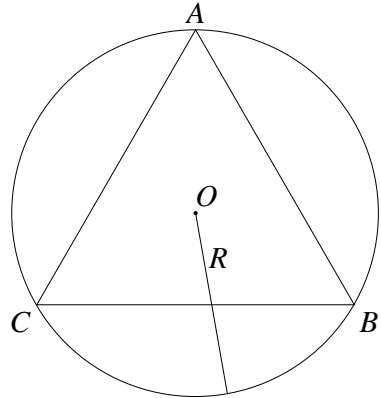
2. To construct a line of length $p/\sqrt{3}$. From Corollary 2 we have that

$$R = \frac{p}{\sqrt{3}}.$$

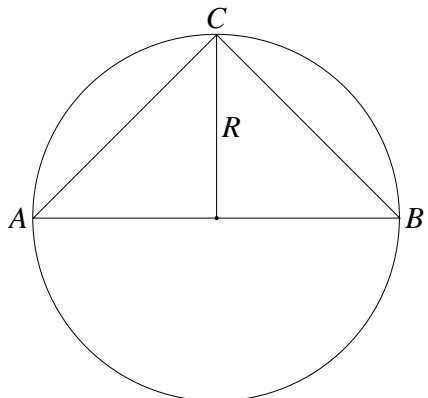
Hence we have the following method. Construct an equilateral triangle with side of length p and its circumcircle, the radius of which will be of length $p/\sqrt{3}$, making

$$AB = BC = CA = p;$$

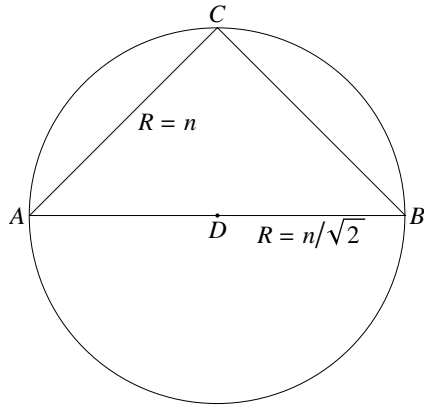
therefore $R = p/\sqrt{3}$.



3. To construct a line with length equal to $n\sqrt{2}$, we apply Corollary 5 as follows. Put $n = R$. Construct a circle with radius R and let the perpendicular bisector of a diameter AB meet the circumference at C . Then CA (or CB) will be of length $R\sqrt{2}$, since $\angle ACB = 90^\circ$.



4. To construct a line equal to $n/\sqrt{2}$. Construct triangle ABC , right-angled at C and with $AC = CB = n$. Find the midpoint, D . Then $AD = DB = n/\sqrt{2}$ because the radius of the circumcircle will be the midpoint of the hypotenuse and, according to Corollary 5, $R = n/\sqrt{2}$, where $n = p$.



COROLLARY 6 From Corollary 4 we know that $r = \frac{ab}{a + b + c}$. If $a = b = q$ (say), then $c^2 = 2q^2$, so

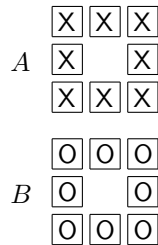
$$r = \frac{q^2}{2q + c} = \frac{q^2}{2q + q\sqrt{2}} = \frac{q}{2 + \sqrt{2}}.$$

From this relationship we are able to construct any multiple of the reciprocal of $2 + \sqrt{2}$. This relationship can be expressed as $q = (2 + \sqrt{2})r$; so: *The equal sides of a right isosceles triangle are $2 + \sqrt{2}$ times the radius of its incircle.*

Problem 177.1 – Eight cubes

Tony Forbes

Eight cubes each have a pair of opposite faces marked with ‘X’ and ‘O’. They are placed in a box in configuration A. Get them into configuration B. The only legal move is to roll or slide a cube into the vacant space (leaving a hole behind it for the next move). No lifting out and putting back.



Twenty-two moves are sufficient; for example,

LURR DLDL UuRD IRdR IRuu DL,

where upper case letters L, R, U, D denote roll a cube to the left, to the right, up, down respectively and the corresponding lower case letters l, r, u, d denote the ‘slide’ moves.

Although one can use ‘brute force’ methods to verify that it cannot be done in less than 22 moves, we are after something more enlightening. Can you devise a short, human-readable proof that 22 is best possible?

Solution 175.2 – Reciprocal inequalities

Determine functions $f(n)$ and $g(n, s)$ such that

$$\left(a_1 + \frac{1}{a_1}\right) + \left(a_2 + \frac{1}{a_2}\right) + \cdots + \left(a_n + \frac{1}{a_n}\right) \geq f(n)$$

and

$$\left(a_1 + \frac{1}{a_1}\right)^s + \left(a_2 + \frac{1}{a_2}\right)^s + \cdots + \left(a_n + \frac{1}{a_n}\right)^s \geq g(n, s),$$

where n and s are positive integers and a_1, a_2, \dots, a_n are any positive numbers such that $a_1 + a_2 + \cdots + a_n = 1$.

John Bull

In each case the left hand side is a minimum when $a_1 = a_2 = \cdots = a_n$. As there are n variables, and all the variables sum to 1, the expression must be minimum when $a_1 = a_2 = \cdots = a_n = 1/n$. There are n components in the expression, so the functions must be

$$f(n) = n \left(\frac{1}{n} + n\right) = 1 + n^2, \quad g(n, s) = n \left(\frac{1}{n} + n\right)^s.$$

Problem 177.2 – Four spheres

Colin Davies

Here is an interesting problem in solid geometry. It is alleged to be ‘An Aenigma set by women for women’, and is quoted in a book I just read, *Dr Johnson’s London* by Liza Picard (Weidenfeld & Nicolson, 2000) as coming from *The Ladies’ Diary or Woman’s Almanac* for 1760.

If three spheres of brass are in contact, and their diameters are 8, 9 and 10 inches, and they support a fourth sphere weighing 12 lb, what quantity of weight does each supporting sphere sustain?

I think there is a lack of data here. Presumably the contacting spheres are on a flat plane, but I think we need to know either the diameter of the 12 lb sphere, or, assuming the spheres are solid, the density of brass.

‘The Electronic Frontier Foundation is offering a \$100,000 award to the first person to discover a ten million-digit prime number. Entropia are trying to find the largest known prime number...’—*The Times*. [Spotted by JRH.]

Solution 175.5 – abc

Suppose $a + b + c = ab + bc + ca = 0$. Prove that

$$a^n + b^n + c^n = \begin{cases} 3(abc)^{n/3} & \text{if } n \text{ is a multiple of } 3 \\ 0 & \text{otherwise.} \end{cases}$$

Peter Fletcher

The way to do this problem is to somehow get an equation involving $a^n + b^n + c^n$ on one side and terms involving abc on the other; and hopefully simplify the terms in abc to either $3(abc)^{n/3}$ or 0.

We have

$$\begin{aligned} a^{n-1}(a + b + c) &= a^n + a^{n-1}b + a^{n-1}c = 0, \\ b^{n-1}(a + b + c) &= ab^{n-1} + b^n + b^{n-1}c = 0, \\ c^{n-1}(a + b + c) &= ac^{n-1} + bc^{n-1} + c^n = 0. \end{aligned}$$

From these three equations,

$$a^n + b^n + c^n = -(a^{n-1}b + a^{n-1}c + b^{n-1}a + b^{n-1}c + c^{n-1}a + c^{n-1}b).$$

Next,

$$\begin{aligned} a^{n-2}(ab + bc + ca) &= a^{n-1}b + a^{n-2}bc + a^{n-1}c = 0, \\ b^{n-2}(ab + bc + ca) &= ab^{n-1} + b^{n-1}c + ab^{n-2}c = 0, \\ c^{n-2}(ab + bc + ca) &= abc^{n-2} + bc^{n-1} + ac^{n-1} = 0. \end{aligned}$$

From these three equations,

$$\begin{aligned} -(a^{n-1}b + a^{n-1}c + b^{n-1}a + b^{n-1}c + c^{n-1}a + c^{n-1}b) \\ = a^{n-2}bc + ab^{n-2}c + abc^{n-2}. \end{aligned}$$

Therefore,

$$a^n + b^n + c^n = a^{n-2}bc + ab^{n-2}c + abc^{n-2}.$$

This can be factorized to give

$$a^n + b^n + c^n = abc(a^{n-3} + b^{n-3} + c^{n-3}).$$

Replacing n by $n - 3$ and multiplying by abc gives

$$abc(a^{n-3} + b^{n-3} + c^{n-3}) = (abc)^2(a^{n-6} + b^{n-6} + c^{n-6}).$$

Repeating this k times and adding all these equations so that terms which appear on both sides cancel out gives

$$a^n + b^n + c^n = (abc)^k(a^{n-3k} + b^{n-3k} + c^{n-3k}). \quad (1)$$

If $n = 3k$, i.e. a multiple of three, so that $k = n/3$, then

$$a^n + b^n + c^n = (abc)^{n/3}(a^0 + b^0 + c^0) = 3(abc)^{n/3}.$$

If n is not a multiple of 3, two cases arise. If $n = 3k + 1$, the second factor on the right of (1) becomes $a + b + c = 0$. If $n = 3k + 2$, the second factor on the right of (1) is $a^2 + b^2 + c^2$. But $a + b + c = 0$, so that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$$

and, since $ab + bc + ca = 0$, $a^2 + b^2 + c^2 = 0$.

Solution 175.1 – Nested roots

Prove that

$$\sqrt{1 + x\sqrt{1 + (x+1)\sqrt{1 + (x+2)\sqrt{\dots}}}} = \frac{x^2 - 1}{\frac{x^2 - 1}{\frac{x^2 - 1}{\dots} + 2} + 2} + 2.$$

John Bull

We have

$$\begin{aligned} f(x) &= x + 1 = \sqrt{1 + 2x + x^2} = \sqrt{1 + xf(x+1)} \\ &= \sqrt{1 + x\sqrt{1 + (x+1)f(x+2)}} \\ &= \sqrt{1 + x\sqrt{1 + (x+1)\sqrt{1 + (x+2)f(x+3)}}} \end{aligned}$$

etc., etc. Also

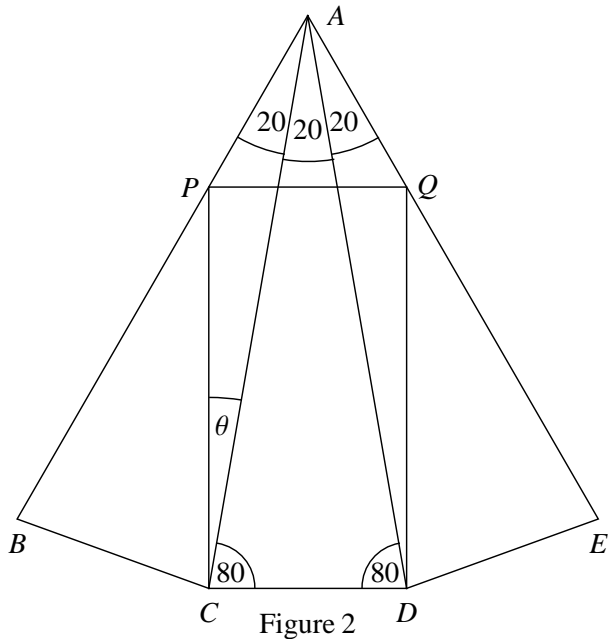
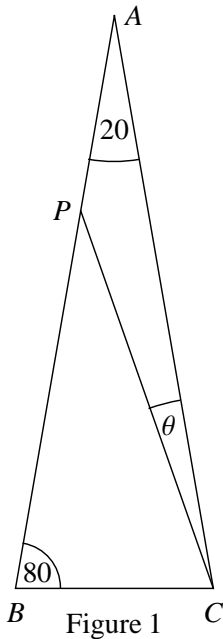
$$\begin{aligned} f(x) &= x + 1 = \frac{(x+1)(x-1)}{x+1} + 2 = \frac{x^2 - 1}{f(x)} + 2 \\ &= \frac{x^2 - 1}{\frac{x^2 - 1}{f(x)} + 2} + 2 = \frac{x^2 - 1}{\frac{x^2 - 1}{\frac{x^2 - 1}{f(x)} + 2} + 2} + 2 \end{aligned}$$

etc., etc.

A new solution to a classic problem

John Bull

The following problem is a classic, having been published in many mathematical puzzle books. An early reference was [1], but an example currently in print can be found in [2].



With reference to Figure 1, suppose $\triangle ABC$ is isosceles with $AB = AC$ and $\angle BAC = 20^\circ$. And suppose P is on side AB such that $AP = BC$. Determine $\angle ACP = \theta$.

The classic solution, as published in [2], is as follows:

Construct Figure 2, using three $80^\circ : 80^\circ : 20^\circ$ triangles, and let Q be on AE so that $\angle APQ = 60^\circ$. Then $\triangle APQ$ is equilateral and $PQ = AP = CD$. Congruent triangles give $CP = DQ$. The quadrilateral $PQDC$ therefore has opposite sides equal and is a parallelogram. But by the figure's symmetry, each of the four angles on the quadrilateral is a right-angle (as can easily be proved by congruent triangles). Therefore $\angle ACP = 90^\circ - \angle ACD = 90^\circ - 80^\circ = 10^\circ$.

Here is another solution that, as far as I am aware, has never been published before.

Construct Figure 3, so that $\triangle SAP$ is the same shape as $\triangle ABC$ and $AS = PS = BA = CA$. Angle $PAS = 80^\circ$ and $\angle BAC = 20^\circ$, so $\angle CAS = 60^\circ$. Therefore $\triangle ACS$ is equilateral. Angle $ASC = 60^\circ$ and $\angle ASP = 20^\circ$, so $\angle PSC = 40^\circ$. Also $AS = CA = SC = PS$, so $\triangle SPC$ is isosceles. Hence, $\angle CPS = 70^\circ$. From $\triangle CPA$ it is now easy to calculate that $\angle ACP = 10^\circ$.

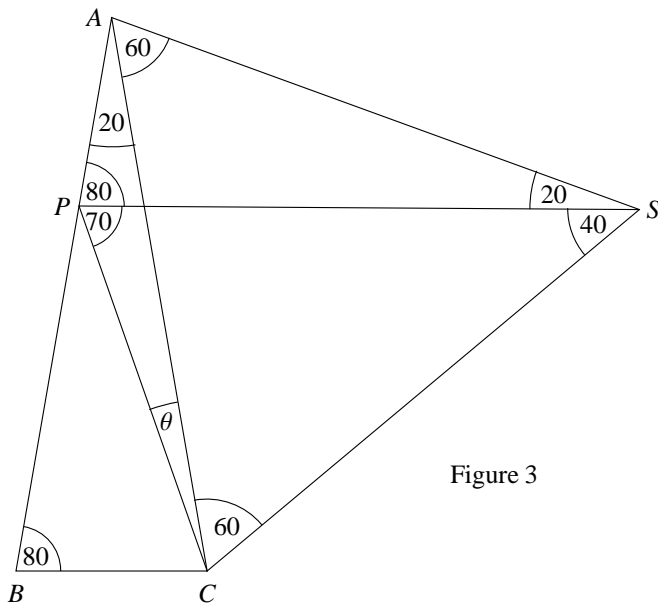


Figure 3

Which solution is 'best' is a matter of personal preference. However, the moral of the story is that just because one particular solution is published in many books, many times, it doesn't mean that there isn't some other equally neat and appealing solution out there somewhere.

References

- [1] Karl-Georg Jacobson and Andrejs Dunkels, Problem 1071, *Journal of Recreational Mathematics*, **14** (1982) 140; **15** (1983) 149.
- [2] Joseph D. E. Konhauser, Dan Velleman and Stan Wagon, *Which Way did the Bicycle Go ... and Other Intriguing Mathematical Mysteries*, Mathematical Association of America, Dolcian Mathematical Exposition **18**.
- [3] Tim Sole, *The Ticket to Heaven, and other Superior Puzzles*, Penguin.

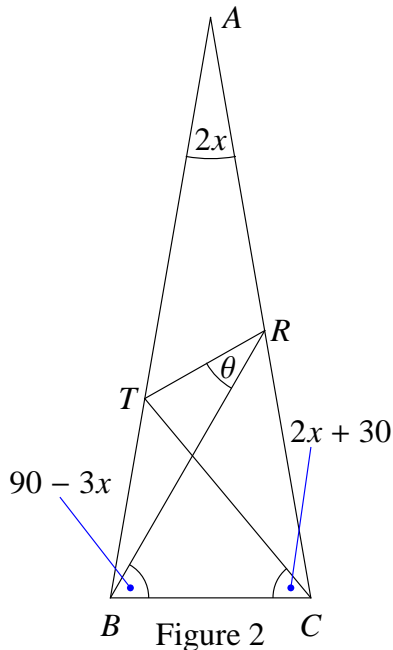
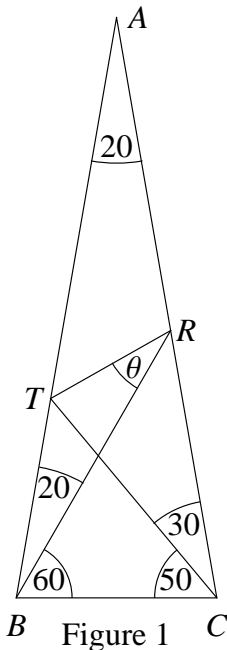
Problem 177.3 – Mahatma’s triangle

John Bull

There is another [see p. 14] related classic problem known as Mahatma’s triangle. In this problem, shown in Figure 1, below, there is an isosceles $80^\circ : 80^\circ : 20^\circ$ triangle as before, but this time with T on AB , and R on AC , such that $\angle BCT = 50^\circ$ and $\angle CBR = 60^\circ$. The problem is to find $\angle BRT = \theta$.

The problem can be generalized to that shown in Figure 2, where $\angle BAC = 2x$, $\angle BCT = 2x + 30^\circ$ and $\angle CBR = 90^\circ - 3x$. Again, the problem is to find $\angle BRT$.

Full solutions to Mahatma’s triangle can be found in Tim Sole’s *The Ticket to Heaven, and other Superior Puzzles*, but there may be ‘better’ solutions than the ones in the book! Maybe M500 readers would like to offer some.



‘Nottingham City Transport – Driver Recruitment. Excellent re-numeration package, ...’—Advertisement on a Nottingham bus. [Spotted by Martin Luck.]

Solution 175.3 – Series squared

Prove that
$$\left(\sum_{k=2}^{\infty} \frac{k^2 - 2}{k!} \right)^2 = \sum_{k=2}^{\infty} \frac{k^3 - 5}{k!}.$$

Peter Fletcher

The easiest way to do this problem is to evaluate both sides separately and show that they equal the same number.

Denote by Θ the differential operator $x \frac{d}{dx}$. For any non-negative integer n , we have

$$\sum_{k=0}^{\infty} \frac{k^n x^k}{k!} = \Theta^n e^x. \quad (1)$$

This is proved by induction. First,

$$\sum_{n=0}^{\infty} \frac{k^0 x^k}{k!} = e^x = \Theta^0 e^x,$$

so (1) is true if $n = 0$. Suppose (1) holds for n and consider $n + 1$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{k^{n+1} x^k}{k!} &= \sum_{n=0}^{\infty} x \frac{d}{dx} \left(\frac{k^n x^k}{k!} \right) = x \frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{k^n x^k}{k!} \right) \\ &= x \frac{d}{dx} (\Theta^n e^x) = \Theta^{n+1} e^x, \end{aligned}$$

as required.

We use (1) to solve the problem. The left-hand side before squaring is

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{k^2 - 2}{k!} &= 3 + \sum_{n=0}^{\infty} \frac{k^2 - 2}{k!} = 3 + [\Theta^2(e^x) - 2e^x]_{x=1} \\ &= 3 + [x^2 e^x + x e^x - 2e^x]_{x=1} = 3 \end{aligned}$$

and the right-hand side is

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{k^3 - 5}{k!} &= 9 + \sum_{n=0}^{\infty} \frac{k^3 - 5}{k!} = 9 + [\Theta^3(e^x) - 5e^x]_{x=1} \\ &= 9 + [x^3 e^x + 3x^2 e^x + x e^x - 5e^x]_{x=1} = 9. \end{aligned}$$

Also solved by **John Bull**.

TeX

Bob Margolis

John Hudson's bit on TeX [M500 175 30] was nice to see, but I'd like to point out that there is hope for people who do not want to install Linux, though that is, itself, a worthy ambition.

There is a CD-ROM, 'TeXLive!' which will install a complete TeX system under Windows 9x or NT, as well as under Unix-alikes. There are other TeX systems available, but that one is probably the best for someone starting out now.

Perhaps it is worth explaining what TeX is, and is not. TeX itself is a document compiler, developed by D. E. Knuth. It accepts a source file which it converts into a generic 'machine code' for display devices. (Such devices include screens, printers and so on.) This generic output file has to be processed before its contents can be realized on any actual device. The programmers amongst you will recognize the TeX processing as the 'compile' phase of program development, whilst the generic output file corresponds roughly to an object file. Producing actual visible output is roughly equivalent to the linker-loader phase of programming.

TeX is not a word processor. Indeed, TeX has no real built in editing facility, it is a command-line program with file input and output, though it can communicate with the user so that limited interactive source error correction can be done, though the corrections are not saved to the source file. It is important to note that there is only one TeX program—the document compiler. LaTeX is not another program, see later. Before an implementation can be called TeX, it must correctly process a test input file called `trip.tex`. This torture device examines a number of dark corners of the code so that there is a reasonable chance that the implementation will typeset documents correctly.

Over the years since a fairly stable release of TeX appeared (in 1983), a number of development environments have appeared to make working with TeX easier for those who hate command-line utilities. (Even those who can only cope with a single mouse button can have TeX.) Lyx and KLyx (the name depends on which Linux desktop environment you use) are two excellent examples, but they also exist for Win 9x and NT. This is being written using one such under NT. The current version of TeX dates from 1989 when, amongst other things, support for languages with large character sets was added. It is heavily used in the academic community and has a considerable following in some commercial publishing houses, apparently more so in continental Europe than in the UK.

Various pundits have been predicting, with great confidence, the imminent death of TeX for some 10 years. It has proved remarkably resilient, possibly because the death of the printed word is not as close as some would wish. For high-class typesetting, particularly of technical material, TeX takes a lot of beating. Moreover, it is free, though copyrighted by

Knuth.

TeX has a small number of built-in commands; things like `\sf \par`, which inserts a paragraph break, and `\sf \def`, which enables new commands to be invented. Most commands, or macros, are ‘secondary’ ones, defined using `\sf \def`. There are a number of macro packages, but the most widely used one is LaTeX, originally produced by Leslie Lamport. The current version is still in development, though perfectly usable, by a team which includes the OU’s Chris Rowley. Many TeX users have only ever prepared documents using the LaTeX macros. They do make life a lot easier. The main advantage is that it is possible to completely separate the design features of a document from the logical structure. The author is free to concentrate on content and its organization into sections, subsections and so on, whilst TeXie designers can concentrate on layout, page size, line lengths and so on.

Not only is TeX a good typesetting/document compiler system, it seems to be the only one which can handle the typesetting of mathematics really well. Most of the ‘wysiwyg’ equation editors are totally hopeless in comparison.

The entire source of the TeX program is published in *TeX – the Program*, by Knuth. It is not object-oriented, though someone is currently completing a Java implementation which is. The language is, more or less, Pascal, but anyone who has met at least one procedural language should be able to read it. It is a large program; the annotated source runs to about 500 pages. All procedural programming is there: linked lists, lexical analysis, mutually recursive procedures, searching, sorting and all the rest of the traditional stuff. There is also a lot about the subtleties of fixed point arithmetic (for accuracy) and of typesetting itself.

In order to make it possible to port TeX to various machines and operating systems, the WEB programming system was developed. This provides a rather more readable method of adapting code than the mess of nested `#if ... #else ... #endif` conditionals so beloved of the C programming community. It also encourages the proper description of how the code works, as you develop it. If curious, *Weaving a Program* by Sewell (Van Nostrand Reinhold) is as good an introduction to the system as any, I think.

To get all these goodies, try looking at the TeX archive network site: www.tex.ac.uk.

‘The obvious mathematical breakthrough would be development of an easy way to factor large prime numbers.’—Bill Gates, *The Road Ahead*. [Spotted by Norma Rosier.]

ADF—We’re divided on this. Factoring primes? Seems trivial. However, surely an integer N is not factorized until (i) all the factors of N are found, and (ii) all the factors are proved prime. If N is prime, (ii) can be difficult. Try ‘factorizing’ $2^{14114} - 3$, for example.

Multiplication

Colin Davies

An article in a recent issue of *IEE News* describes a fascinating method of doing multiplication.

A mathematically challenged tribe cannot multiply and divide except to double and halve, and since they only deal with items like goats and pigs they ignore all fractions. They proceed with their calculations as shown, but, being superstitious, they consider even numbers on the left to be evil and so eliminate them, together with their counterparts in the other column.

A tribesman wanting to purchase 13 goats at £7 each calculates as follows:

$$\begin{array}{r}
 13 \qquad 7 \\
 \hline
 6 \qquad 3 \text{ (eliminate)} \\
 3 \qquad 28 \\
 1 \qquad 56 \\
 \hline
 91 = 7 + 28 + 56
 \end{array}$$

Amazingly they always get it right.

Tony Forbes writes—This technique, or, rather, the same thing with doubling replaced by squaring and addition replaced by multiplication, is well-known to that tribe consisting of people who write computer programs for discovering large primes. The simplest test for probable-primality is to compute $2^N \pmod{N}$ and see if the answer is 2. If N is big, say $N > 10^{10000}$, the task looks impossible but with the square-and-multiply method it is actually quite fast.

Getting back to the original problem, here is what they are really doing: Start with $Y = 0$ and $A = 7$. Scan the binary digits of 13 right-to-left. If 0, double A ; if 1, add A to Y and then double A . Read off the answer at Y when all the binary digits have been processed.

The corresponding procedure for $2^N \pmod{N}$ is: Start with $Y = 1$ and $A = 2$. Scan the binary digits of N right-to-left. If 0, square $A \pmod{N}$; if 1, multiply Y by $A \pmod{N}$ and then square $A \pmod{N}$.

I mention all this because there is an interesting variation: Start with $Y = 0$. Scan the binary digits of 13 *left-to-right*. If 0, double Y ; if 1, double Y and add 7.

Letters to the Editors

Big River

Dear Tony,

The 'solution' to Jeremy Humphries's 'Big River' problem (M500 175) might be hidden in the way he poses the question.

Gravity varies between the equator and the poles and therefore, I assume, the pressure head varies.

When hydrologists talk about 'uphill' they really mean 'from a low to a high total hydraulic head' which, because the elevation head is usually a significant factor, fits our conventional view of what we mean by 'uphill'.

But, if the difference between the elevation heads at St Paul and the Gulf of Mexico is small relative to the difference between the pressure heads, as would be the case of a river meandering over long distances in a north-south direction, 'downhill' in terms of the total hydraulic head might involve 'rising' to an elevation where the total hydraulic head is lower as a result of the lower pressure head (and presumably in the case of the Mississippi the lower velocity head).

Talking about 'flowing against gravity' confuses the issue because gravity is not constant and therefore its contribution to the total hydraulic head varies with latitude.

This may not be the answer to the question and I don't have the figures to test this hypothesis but it would be worth exploring before adding another variable into the equation.

John Hudson

Goldbach again

Goldbach's conjecture [that every large even integer can be expressed as the sum of two primes] can be rephrased, but let's take a look at primes in general first. The primes do follow a pattern, unique to themselves. There is no simple mathematical formula that can generate them all; each new prime has to be calculated individually.

As we run through the natural numbers we sometimes encounter an odd number k that cannot be factorized using any combination of the primes up to $k/2$. For example, after reaching 40 we find that $k = 41$ is not divisible by 2, 3, 5, 7, 11, 13, 17, 19 and so 41 becomes the next prime.

It has already been proved that the number of primes is infinite. They become less frequent as the natural numbers increase with apparently the occasional occurrence of twin primes, one large pair being 1,000,000,000,061 and 1,000,000,000,063.

Back to Goldbach.

If we call each even integer $2n$ then n will include all the natural numbers, even, prime and odd composite. For Goldbach's conjecture to hold, each composite $n \geq 2$ must lie midway between at least one pair of primes. When $n = 2p$, p prime, there appear to be a smaller number of pairs of primes equidistant from n than when n is any other type of composite number.

As Goldbach's conjecture depends upon the distribution of primes it might be easier to aim for a counterexample than a proof. A counterexample would probably be an enormous number, $n = 2p$, that did not lie midway between two primes, and $2n$ would be an even number that could not be expressed as the sum of two primes.

I look forward to hearing that the M500 Editorial Board have received one million dollars from Faber. In the meantime I'll stick to organic chemistry with the Science Faculty.

Barbara Lee

PS. Pascal's pyramid [M500 175 16] is Pascal's tetrahedron from TM361, unit 4, which is useful for calculating trinomial coefficients and leads to the multinomial theorem discovered by Leibniz.—**BL**

M500 *Special Issue*

I was saddened to read that this year's *Special Issue* might be the last.

I completed my degree three years ago but did write a number of articles for the *Special Issue* over the years. Despite my promise to myself never to study again, circumstances have engineered themselves so that I am just coming to the end of the first year of my second degree. This time it's humanities rather than maths and science, but I suspect that somewhere along the line, these topics will reappear in my profile.

This morning when the *Special Issue* arrived, I was literally starting to write down my thoughts on A103 for submission for next year's issue.

I always found these publications of great interest and use—the only way to assess a course in advance is to read the views of those who have done it. Whilst not always 'heeding a warning', generally I never made a course choice before considering the views expressed in your publication.

If this were to be lost, then I think it would be a great shame. Whilst I am an avid user of the Internet, I fear that transferring this to a Web site will only serve to render its demise complete. There is no substitute for the written booklet itself falling through the letter box. People tend to forget to check up-dates, etc., on Web sites.

I appreciate that there seems to be a general lack of interest in providing

articles and this is a shame. I believe that those of us who get something out of the OU should in some small way put something back and your publication enables one to do that.

Two suggestions :

(1) Perhaps contribution of an article of any length should be made a condition of membership? Probably ‘unenforceable’.

(2) An increase in the membership fee of say £5 per annum refundable against the following year’s subscription upon submission of an article? Self regulating—M500 Society does not have to return cash paid (and spent) but could verify each year upon receipt of subscriptions.

At the very least : Please, in considering this matter, try one more time. Let’s have *Special Issue 2001* and prompt people about needing submissions in each issue of M500. Then if things are as ‘bleak’ twelve months hence, it will no doubt be ‘to the Web’ or worse!

Kind regards.

Richard Woolf

OK. It just happens that (apparently—I haven’t seen them) another ten *Special Issue* items have mysteriously turned up. Also the un-editing part of the M500 Committee were pretty chagrined at the unilateral decision the Editorial Board had sneaked past them, so it looks like 2001 is safe.—**Eddie Kent**

The Parrot’s Theorem

I have just read a novel called *The Parrot’s Theorem* by Denis Guedj (Weidenfeld & Nicolson, 2000). Evidently this was a best seller when originally published in France. I suspect it was a best seller in France for much the same reason as Hawking’s *A Brief History of Time* was a best seller here, though it requires far less thought to understand it.

To my mind it does not have a deep or gripping plot, though there is a bit of skullduggery with kidnapping and arson involved. In fact the plot is very weak to my mind, and I had guessed the outcome long before finishing the book. However, the cast of characters listed in the back covers six sides of text, and includes well over 150 mathematicians from Thales in 620 BC to Andrew Wiles now.

If anyone wants an easy read on the history of mathematics, *The Parrot’s Theorem* is quite amusing. And I did learn a few things I did not know before.

Colin Davies

Who wants to be another millionaire?

Eddie Kent

Landon T. Clay is a 74-year-old Boston financier. From 1971 to 1997 he was chairman of the Eaton Vance Corporation, an investment company. Then in 1999 he set up the Clay Mathematics Institute in Boston. Here mathematicians are paid to work on unusual problems during their holidays, or to teach gifted students at summer school. Now, following the lead of Faber & Faber, who have offered a prize of a million dollars for a proof of Goldbach's conjecture (M500 175 33), he has founded the Clay Millennium Prizes.

Clay went up to Harvard to read English and American Literature, but dropped out of the introductory mathematics course. 'I didn't get very far before realizing that academic life was not for me. Before coming to Harvard I was in the Second World War, and when I arrived at college I saw people a lot younger and smarter than me. I felt very ill prepared.

'I didn't make the most of what Harvard had to offer. To me, math is just a spectator sport. Still, even though mathematics was like Mount Everest to me, I'm happy to watch other people do the climbing.'

Clay's prizes are a million dollars each for seven problems that have so far eluded mathematicians. There is the Riemann hypothesis, which states that the zeta function has no non-trivial zeros except on the line $\Re z = 1/2$. Of course like many such conjectures this has been checked for millions of cases, and you might remember that G. H. Hardy once claimed to have proved it, just before setting off on a sea journey, arguing that God would not let him die with such a story uncorrected. He had clearly forgotten what happened to Fermat.

Other problems include P/NP, the computational problem of deciding if certain types of algorithm can run in polynomial time; the Navier–Stokes equations in continuum mechanics; and the Yang–Mills equations in particle physics. Look at www.claymath.org for full details.

The panel choosing the problems includes Sir Andrew Wiles. The chairman is Arthur Jaffe, Landon T. Clay Professor of Mathematics and Mathematical Sciences at Harvard. He explained 'These are all classic math problems and are regarded as the ones that, if solved, would have the biggest influence on mathematics.'

In contrast to the Faber offer on Goldbach, there is no time limit for these prizes. And it is also written into the conditions that if more than one person can be shown to have produced significant work towards a result, the prize will be divided. This is to ensure that results continue to be published as they are found.

Timothy Gowers, Rouse Ball Professor of Mathematics at Cambridge,

in an interview with *The Times*, said that ‘these problems are so enormously important that people are motivated to solve them with or without money. If someone did succeed they would shoot to worldwide fame anyway. There’s an undeniable element of ego, of wanting to impress other mathematicians.’

The Fields Medal of the International Mathematical Union is the main honeypot for mathematicians, since they cannot get the Nobel Prize (for reasons of a somewhat scandalous nature). It was funded in 1936 by the will of the Canadian analyst John Charles Fields, and is given every four years. But there are no millions involved—it is worth just \$15000.

Let us give Clay the last word. ‘I feel that math is not recognized, not appreciated. It’s the queen of the sciences, the key to a fundamental understanding of the world. It’s pretty clear in America that the government is reducing its support for pure science. I think that’s a great mistake. I remember reading about Andrew Wiles having proved Fermat’s last theorem on the front page of the New York Times and feeling very excited. Giving this money is a testimonial that people should be rewarded and excited to labour in this field.’

Solution 175.3 — The first prime

The prime numbers are arranged alphabetically. Which is the first?

Tony Forbes

By ‘alphabetically’ we intended to mean that the numbers are arranged in the usual lexicographical order when translated into their standard English expressions.

One response was 83. Someone else offered both 83 and 811; the same person also showed great initiative by changing the language to German, and came up with the simple answer 3 (drei). I was most impressed until I remembered the German for 8 (acht). Perhaps another language might be more obliging.

Ralph Hancock suggests **8,000,000,081**, eight billion and eighty one.

We are leaving the problem open because there are arguable points. Does one count the ‘and’? A simple computer sort places ‘eight billion and eighty one’ before ‘eight billion eight hundred . . .’ but a human dictionary-compiler would reverse the order. Punctuation may also have an effect. And Eddie reminds me that ‘billion’ was once used (in Britain but not in the USA) to represent 10^{12} rather than 10^9 .

If it helps, I have a PC program that converts numbers to English. I wrote it years ago, in response to a request in M500 136 (January 1994) by Eddie Kent, who asked for an algorithm to ‘convert a numeric value to a

text string that could be written on a cheque.' If you are interested, I shall try and remember to place a copy at www.ltkz.demon.co.uk/adfchq.zip. It works best on an original IBM 8086 PC, for then it runs so slowly that you can have a lot of fun watching the substitutions take place.

The main procedure of the program is listed below. It calls various subroutines, which perform much as you would expect. The only one that needs any explanation is `AddAfter(n, X)`. This adds the character-string `X` just after the `n`th '#' sign, counting from the right.

Finally, a related question. I would like extend the range of my program. Is it possible to express arbitrarily large numbers in words?

Does the sequence { 'one', 'two', 'three', ... } extend indefinitely?

```

Replace "0", " ZERO#"
Replace "1", " ONE#"
Replace "2", " TWO#"
Replace "3", " THREE#"
Replace "4", " FOUR#"
Replace "5", " FIVE#"
Replace "6", " SIX#"
Replace "7", " SEVEN#"
Replace "8", " EIGHT#"
Replace "9", " NINE#"
AddAfter 2, "TY"
AddAfter 3, " HUNDRED AND AND"
AddAfter 4, " THOUSAND"
AddAfter 5, "TY"
AddAfter 6, " HUNDRED AND"
AddAfter 7, " MILLION"
AddAfter 8, "TY"
AddAfter 9, " HUNDRED AND"
AddAfter 10, " BILLION"
AddAfter 11, "TY"
AddAfter 12, " HUNDRED AND"
AddAfter 13, " TRILLION"
...
AddAfter 29, "TY"
AddAfter 30, " HUNDRED AND"
AddAfter 31, " NONILLION"
AddAfter 32, "TY"
AddAfter 33, " HUNDRED AND"
AddAfter 34, " DECILLION"
AddAfter 35, "TY"
AddAfter 36, " HUNDRED AND"
AddAfter 37, " GARBAGE"
Remove "#"
Remove " ZERO HUNDRED AND"
Remove " AND AND ZEROTY ZERO"
Remove " AND ZEROTY ZERO"
Remove " ZEROTY"
Remove " ZERO"
Replace " AND AND", " AND"
Replace "ON NONILLION", "ON"

Replace "ON OCTILLION", "ON"
Replace "ON SEPTILLION", "ON"
Replace "ON SEXTILLION", "ON"
Replace "ON QUINTILLION", "ON"
Replace "ON QUADRILLION", "ON"
Replace "ON TRILLION", "ON"
Replace "ON BILLION", "ON"
Replace "ON MILLION", "ON"
Replace "ON THOUSAND", "ON"
Replace "ONETY", "TEN"
Replace "TWOY", "TWENTY"
Replace "THREETY", "THIRTY"
Replace "FOURTY", "FORTY"
Replace "FIVETY", "FIFTY"
Replace "EIGHTTY", "EIGHTY"
Replace "TEN ONE", "ELEVEN"
Replace "TEN TWO", "TWELVE"
Replace "TEN THREE", "THIRTEEN"
Replace "TEN FOUR", "FOURTEEN"
Replace "TEN FIVE", "FIFTEEN"
Replace "TEN SIX", "SIXTEEN"
Replace "TEN SEVEN", "SEVENTEEN"
Replace "TEN EIGHT", "EIGHTEEN"
Replace "TEN NINE", "NINETEEN"
AddAtBeginning "#"
Replace "# DECILLION", "#"
Replace "# NONILLION", "#"
Replace "# OCTILLION", "#"
Replace "# SEPTILLION", "#"
Replace "# SEXTILLION", "#"
Replace "# QUINTILLION", "#"
Replace "# QUADRILLION", "#"
Replace "# TRILLION", "#"
Replace "# BILLION", "#"
Replace "# MILLION", "#"
Replace "# THOUSAND", "#"
Replace "# AND", "#"
AddAtEnd "#"
Replace "##", "# ZERO"
Remove "#"

```

Problem 177.4 – e in nine digits

Jeremy Humphries

Find the best approximation of the mathematical constant $e = 2.7182818\dots$ using only the digits 1 through 9 inclusive—once and only once each. Addition, division, multiplication, subtraction, exponentiation, parentheses and decimal points—and nothing else—are allowed.

To give you some idea of what to aim at, we know that there is a solution which is accurate to better than 10^{39} (*sic*) decimal places.

What about π ?

Problem 177.5 – 3 theta

David L. Brown

Show that

$$\frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} = 2.$$

Are there any others? That is, interesting equalities of the form
trigonometric expression involving $\theta =$ positive integer
(apart from Pythagoras's theorem).—**ADF**

Find the missing terms

Colin Davies

Find the missing terms and give the rule of formation.

3, 127, 379, 499, 6079, ?, ?

3, 29, 53, 61, 251, ?, ?

ADF—Here's another.

Milly, ?, ?, ?, Dotty

Some time ago, Jeremy heard a news item on R4 that the five piglets cloned at the PPL Therapeutic Plant, Virginia, have names 'ranging from Milly to Dotty'. We are particularly interested in the rule of formation.

Problem 177.6 – Factorial derivative

Tony Forbes

Prove that if n is a positive integer,

$$\left[\frac{d^n}{dx^n} (1 - x^n)^{1/n} \right]_{x=0} = -(n-1)!.$$

The Science of Secrecy

Once again we welcomed author **Simon Singh** to our September Weekend at Aston University, this time to give a talk based on another of his passionate interests—codes and ciphers. After the lecture, there was a great demand for his books, and supplies soon ran out.

If you would like to buy a signed copy of *Fermat's Last Theorem*, *The Code Book* or *The Science of Secrecy* (which accompanied his recent Channel 4 TV series), you can do so by visiting www.simonsingh.com.

The website also offers translations of *Fermat's Last Theorem* and *The Code Book* in various languages. Furthermore, 30 per cent of your payment will go to the charity Sightsavers.

Twenty-five years ago

From M500 28

David Wells—How to see a million things all at once: make up a metre square of millimetre graph paper.

Bill Shannon—Ship *A* with a maximum speed of 30 knots is chasing ship *B* whose maximum speed is 15 knots. Neither ship has radar. Ship *B* enters an extensive fog bank and from then on is invisible to ship *A*. On the assumption that ship *B* changes course and maintains its new direction unchanged at maximum speed, what plan should ship *A* follow to guarantee that ship *B* will be intercepted.

Thank you all for your contributions to the magazine. We still have a quite a lot in hand, enough for a good start to 2001, so watch out for a bigger-than-average M500 in February. The *Special Issue*, too, is alive and well; why not use the break to write about your OU courses? Send *Special Issue* items to Eddie Kent and articles for the regular M500 to me. Meanwhile, have a Merry Christmas, and best wishes for the New Year, the true dawn of the new millennium.—**ADF**

Pascal's pyramid, Pascal's hyperpyramid and Pascal's line	
Robin Marks	1
Two theorems with some applications	
David L. Brown	6
Problem 177.1 – Eight cubes	
Tony Forbes	10
Solution 175.2 – Reciprocal inequalities	
John Bull	11
Problem 177.2 – Four spheres	
Colin Davies	11
Solution 175.5 – abc	
Peter Fletcher	12
Solution 175.1 – Nested roots	
John Bull	13
A new solution to a classic problem	
John Bull	14
Problem 177.3 – Mahatma's triangle	
John Bull	16
Solution 175.3 – Series squared	
Peter Fletcher	17
TeX	
Bob Margolis	18
Multiplication	
Colin Davies	20
Units	
Eddie Kent	21
Letters to the Editors	
Big River	22
Goldbach again	22
M500 <i>Special Issue</i>	23
<i>The Parrot's Theorem</i>	24
John Hudson	22
Barbara Lee	22
Richard Woolf	23
Colin Davies	24
Who wants to be another millionaire?	
Eddie Kent	25
Solution 175.4 – The first prime	
Tony Forbes	26
Problem 177.4 – e in nine digits	
Jeremy Humphries	28
Problem 177.5 – 3 theta	
David L. Brown	28
Find the missing terms	
Colin Davies	28
Problem 177.6 – Factorial derivative	
Tony Forbes	29
<i>The Science of Secrecy</i>	29
Twenty-five years ago	29