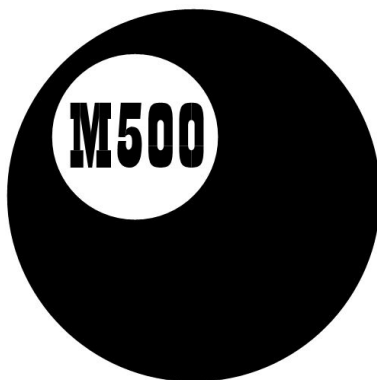


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ISSN 1350-8539



M500 179



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

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Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

Recurrence relations

Robin Marks

This article is about recurrence relations of the form

$$U_n = \frac{1}{k} (U_{n-1} + U_{n-2} + \cdots + U_{n-k}).$$

In particular, I shall look at the interesting question: ‘Is there a limit to U_n as n approaches infinity, and if so, what is it?’ I first tried to answer these questions for the third order recurrence relation

$$U_n = \frac{1}{3}(U_{n-1} + U_{n-2} + U_{n-3}),$$

but the methods in M203 for finding the limits of series did not seem to work. I have persevered and I believe I have shown in what follows that there is a limit to U_n as n approaches infinity, and that the limit depends on the initial values.

It is still not entirely clear to me why the values $1/6$, $2/6$ and $3/6$ appear in the third order inverse matrix, $1/10$, $2/10$, $3/10$, and $4/10$ appear in the fourth order inverse matrix, $1/15$, $2/15$, $3/15$, $4/15$ and $5/15$ appear in the fifth order inverse matrix, \dots , although I can think of a ‘hand-waving’ sort of argument. Perhaps M500 readers will explain.

I have used consecutive initial values of U_n to calculate the values of the coefficients A, B, C, \dots . This was for convenience. Non-consecutive initial values could have been used, although in some cases this would not work; for example in the fifth order case we could not have used the five initial values $U_{-1} = 0$, $U_0 = 0$, $U_1 = 0$, $U_2 = 0$, $U_3 = 0$ to generate the series obtained. I was surprised to see how many times $U_n = 0$ appears in each series; six times in the fourth order series, ten times in the fifth order and fifteen times in the sixth order series.

I did the calculations using Microsoft Excel spreadsheet functions as well as Visual Basic for Excel subroutines. I translated a Fortran subroutine for getting the roots of quartic and quintic equations, obtained from

<http://www.uni-koeln.de/math-nat-fak/phchem/deiters/quartic/quartic.html>, into Visual Basic, and I wrote a Visual Basic routine for multiplying the (complex) roots together in all the combinations necessary to obtain the coefficients of powers of x for the fifth order and sixth order equations. I will be happy to pass these routines on if anyone is interested.

Finally, as a motivation for trying to solve these recurrence relations, a possibly realistic problem could be as follows.

Suppose a laser sends out a light pulse.

The pulse hits a succession of k equally-spaced partially-reflecting mirrors. Each mirror reflects a fraction $1/k$ of the original pulse energy back to the laser, and transmits any remaining energy. (The last mirror reflects all of the light energy hitting it.)

If all the light energy returning to the laser is once more reflected back to the mirrors with 100 per cent efficiency, how much of the pulse energy is present at the laser (and at each mirror) at each successive clock time, and, in particular, at the limit of infinite time?

Order-2 recurrence relation

First consider the order-2 recurrence relation

$$U_n = \frac{1}{2}U_{n-1} + \frac{1}{2}U_{n-2}.$$

Try $U_n = Ax^n$ as a solution. This gives $Ax^n = Ax^{n-1}/2 + Ax^{n-2}/2$. Cancelling Ax^{n-2} gives $x^2 = x/2 + 1/2$. So the quadratic auxiliary equation is $x^2 - x/2 - 1/2 = 0$. Let the roots be r_1, r_2 . The general solution is $U_n = Ar_1^n + Br_2^n$, A, B constants. The equation factorizes as $(x - 1)(x + 1/2)$, giving real roots $1, -1/2$.

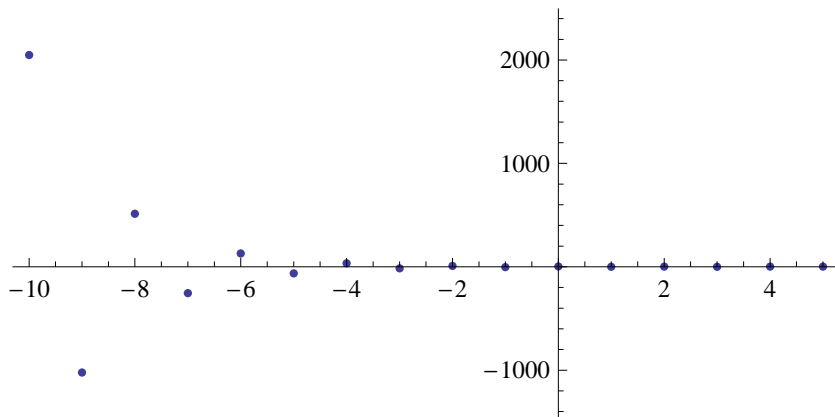
Thus $U_n = Ar_1^n + Br_2^n$. So $U_0 = A + B$, $U_1 = Ar_1 + Br_2 = A - B/2$. Written as a matrix equation, this is

$$\begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} U_0 \\ U_1 \end{bmatrix}.$$

If we start with initial values, for example, $U_0 = 3, U_1 = 0$, then we can use the inverse matrix to determine A and B :

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix};$$

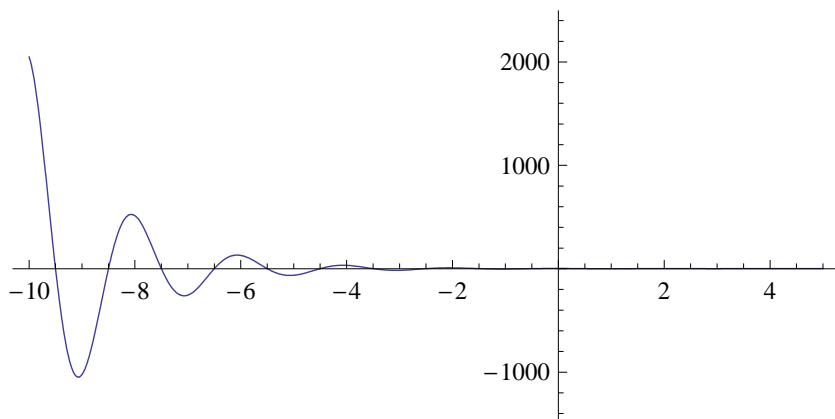
thus $A = 1, B = 2$; $U_n = 1 + 2(-1/2)^n$. This function is plotted at the top of the next page and the actual values are listed in the left-hand column of page 4.



I now want to extend the domain to all real values. The general solution is $U(x) = Ar_1^x + Br_2^x$ but because the root r_2 is negative, real values do not always exist for fractional x . An inspiration came to me one day at 3:30 am! If we convert the negative root to polar coordinates using $-1 = e^{\pi i}$,

$$r_2^x = \left(\frac{-1}{2}\right)^x = \left(\frac{1}{2}\right)^x e^{\pi i x} = \left(\frac{1}{2}\right)^x (\cos \pi x + i \sin \pi x).$$

Then take the real part, $\Re U(x) = A + B/2^x \cos \pi x$, which is graphed below. There is a sample of actual values listed in the right-hand column of the next page.



| n | $U_n = \frac{1}{2}U_{n-1} + \frac{1}{2}U_{n-2}$ | x | $U(x)$ |
|-----|---|------|-------------|
| -13 | -16383.00000 | -10 | 2049.00000 |
| -12 | 8193.00000 | -9.9 | 1818.32783 |
| -11 | -4095.00000 | -9.8 | 1443.38633 |
| -10 | 049.00000 | -9.7 | 978.77660 |
| -9 | -1023.00000 | -9.6 | 480.62335 |
| -8 | 513.00000 | -9.5 | 1.00000 |
| -7 | -255.00000 | -9.4 | -416.53638 |
| -6 | 129.00000 | -9.3 | -740.01609 |
| -5 | -63.00000 | -9.2 | -950.62009 |
| -4 | 33.00000 | -9.1 | -1042.78075 |
| -3 | -15.00000 | -9 | -1023.00000 |
| -2 | 9.00000 | -8.9 | -907.66392 |
| -1 | -3.00000 | -8.8 | -720.19316 |
| 0 | 3.00000 | -8.7 | -487.88830 |
| 1 | 0.00000 | -8.6 | -238.81167 |
| 2 | 1.50000 | -8.5 | 1.00000 |
| 3 | 0.75000 | -8.4 | 209.76819 |
| 4 | 1.12500 | -8.3 | 371.50805 |
| 5 | 0.93750 | -8.2 | 476.81004 |
| 6 | 1.03125 | -8.1 | 522.89037 |
| 7 | 0.98438 | -8 | 513.00000 |
| 8 | 1.00781 | -7.9 | 455.33196 |
| 9 | 0.99609 | -7.8 | 361.59658 |
| 10 | 1.00195 | -7.7 | 245.44415 |
| 11 | 0.99902 | -7.6 | 120.90584 |
| 12 | 1.00049 | -7.5 | 1.00000 |
| 13 | 0.99976 | -7.4 | -103.38409 |
| 14 | 1.00012 | -7.3 | -184.25402 |
| 15 | 0.99994 | -7.2 | -236.90502 |
| 16 | 1.00003 | -7.1 | -259.94519 |
| 17 | 0.99998 | -7 | -255.00000 |
| 18 | 1.00001 | -6.9 | -226.16598 |
| 19 | 1.00000 | -6.8 | -179.29529 |
| 20 | 1.00000 | -6.7 | -121.22207 |
| 21 | 1.00000 | -6.6 | -58.95292 |
| 22 | 1.00000 | -6.5 | 1.00000 |

* Initial condition

Order-3 recurrence relation

Consider the order-3 recurrence relation

$$U_n = \frac{1}{3}U_{n-1} + \frac{1}{3}U_{n-2} + \frac{1}{3}U_{n-3}.$$

Try $U_n = Ax^n$ as a solution. This gives

$$Ax^n = \frac{1}{3}Ax^{n-1} + \frac{1}{3}Ax^{n-2} + \frac{1}{3}Ax^{n-3}.$$

The cubic auxiliary equation is $x^3 = x^2/3 + x/3 + 1/3$. Let the roots be r_1, r_2, r_3 . The general solution is $U(n) = Ar_1^n + br_2^n + cr_3^n$, A, b, c constants. As before, $r_1 = 1$, and dividing by $x - 1$ gives a conjugate pair of complex roots $r_2 = -1/3 + \sqrt{2}i/3$ and $r_3 = -1/3 - \sqrt{2}i/3$.

Now express r_2 and r_3 in polar coordinates. Let

$$r = \sqrt{(1/3)^2 + (\sqrt{2}/3)^2} = \frac{\sqrt{3}}{3} \approx 0.57735027$$

and

$$\theta = \arccos -\frac{1}{3r} = \arccos -\frac{\sqrt{3}}{3} \approx 2.18627604.$$

Thus

$$\begin{aligned} U(n) &= A + b(r \cos \theta + i \sin \theta)^n + c(r \cos \theta - i \sin \theta)^n \\ &= A + br^n(\cos n\theta + i \sin n\theta) + cr^n(\cos n\theta - i \sin n\theta) \end{aligned}$$

by De Moivre's theorem. Now define $B = b + c$, $C = ib - ic$. Then

$$U(n) = A + Br^n \cos n\theta + Cr^n \sin n\theta$$

and in particular

$$U(-1) = A + Br^{-1} \cos(-\theta) + Cr^{-1} \sin(-\theta) = A - B - \sqrt{2}C,$$

$$U(0) = A + B,$$

$$U(1) = A + Br \cos \theta + Cr \sin \theta = A - \frac{1}{3}B + \frac{\sqrt{2}}{3}C.$$

Written as a matrix equation, this is

$$\begin{bmatrix} 1 & -1 & \sqrt{2} \\ 1 & 1 & 0 \\ 1 & -\frac{1}{3} & \frac{\sqrt{2}}{3} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} U(-1) \\ U(0) \\ U(1) \end{bmatrix}.$$

For initial values, let $U_{-1} = 6$, $U_0 = 0$, $U_1 = 0$. Then we can use the inverse matrix to determine A , B and C :

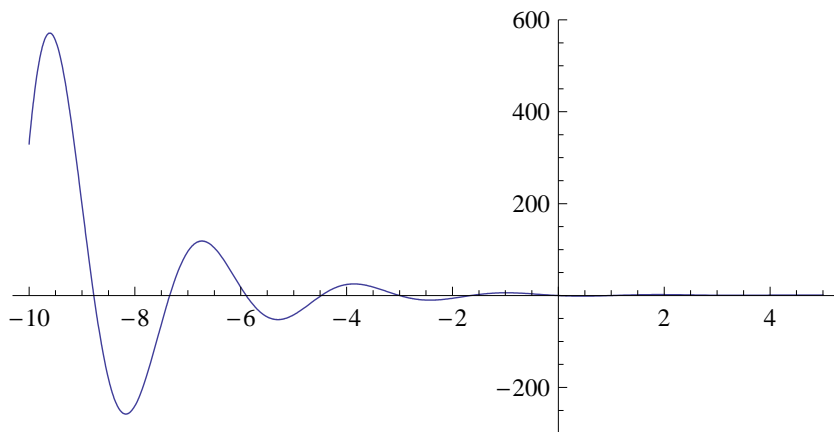
$$\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{2} \\ -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2\sqrt{2} \end{bmatrix}.$$

Thus $A = 1$, $B = -1$ and $C = 2\sqrt{2}$.

The function

$$\begin{aligned} U(x) &= 1 - r^x \cos \theta x - 2\sqrt{2}r^x \sin \theta x \\ &= 1 - \left(\frac{\sqrt{3}}{3}\right)^x (\cos \theta x + 2\sqrt{2} \sin \theta x) \end{aligned}$$

is real for all real values of x and it looks like this.



Order-4 recurrence relation

Consider the order-4 recurrence relation

$$U_n = \frac{1}{4}U_{n-1} + \frac{1}{4}U_{n-2} + \frac{1}{4}U_{n-3} + \frac{1}{4}U_{n-4}.$$

The quartic auxiliary equation is

$$x^4 = \frac{1}{4}(x^3 + x^2 + x + 1).$$

Let the roots be r_1, r_2, r_3 and r_4 . The general solution is

$$U(n) = Ar_1^n + Br_2^n + cr_3^n + dr_4^n,$$

A, B, c, d constants. Solving the quartic (J. E. Hacke, *Amer. Math. Monthly*, Vol. **48** (1941), 327–328, or see <http://www.uni-koeln.de/math-nat-fak/phchem/deiters/quartic/quartic.html>) gives real roots $r_1 = 1$ and $r_2 = -0.6058$ and a pair of complex roots: $r_3 = -0.0721 + 0.6383i$ and $r_4 = -0.0721 - 0.6383i$. The polar coordinates for r_3 and r_4 are given by

$$r = 1\sqrt{(-0.0721)^2 + (0.6383)^2} = 0.6424$$

and

$$\theta = \arccos -\frac{0.0721}{0.6424} = 1.6832.$$

Hence

$$cr_3^n + dr_4^n = cr^n(\cos\theta + i\sin\theta)^n + dr^n(\cos\theta - i\sin\theta)^n,$$

where c, d are complex constants. Now define $C = c + d, D = ci - di$. Then

$$U(n) = Ar_1^n + Br_2^n + Cr^n \cos n\theta + Dr^n \sin n\theta.$$

For example,

$$\begin{aligned} U(2) &= Ar_1^2 + Br_2^2 + Cr^2 \cos 2\theta + Dr^2 \sin 2\theta \\ &= A + 0.3670B - 0.4023C - 0.0920D. \end{aligned}$$

Written as a matrix equation,

$$\begin{bmatrix} 1 & -1.65062919 & -0.1746854 & -1.54686889 \\ 1 & & 1 & & 0 \\ 1 & -0.60582959 & -0.07208521 & 0.63832674 \\ 1 & 0.36702949 & -0.40226474 & -0.09202783 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} U(-1) \\ U(0) \\ U(1) \\ U(2) \end{bmatrix}.$$

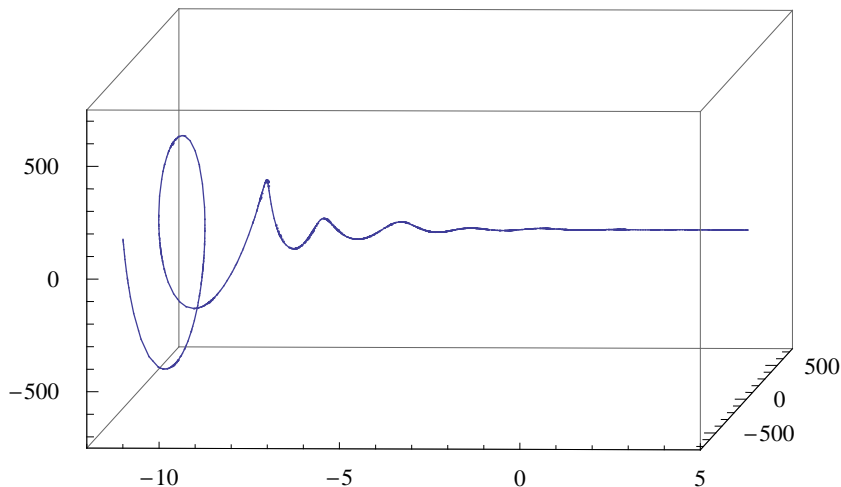
For initial values, let $U_{-1} = 10$, $U_0 = 0$, $U_1 = 0$, $U_2 = 0$. Then we can use the inverse matrix to determine A , B , C and D :

$$\begin{bmatrix} \frac{1}{10} & \frac{1}{5} & \frac{3}{10} & \frac{2}{5} \\ -0.22486329 & 0.14630262 & -0.46635467 & 0.54491535 \\ 0.12486329 & 0.65369738 & 0.166354675 & -0.94491535 \\ -0.3559745 & -0.10064351 & 0.672790633 & -0.21617233 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ = \begin{bmatrix} 1 \\ -2.24863294 \\ 1.24863294 \\ -3.559744994 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}.$$

With these values of A , B , C and D , the function

$$U(x) = Ar_1^x + Br_2^x + Cr^x \cos \theta x + Dr^x \sin \theta x$$

is complex for real x but if we plot $\Re U(x)$ horizontally (into the paper) and $\Im U(x)$ vertically, we obtain this spiral-shaped graph.



Order-5 recurrence relation

Consider the order-5 recurrence relation

$$U_n = \frac{1}{5}U_{n-1} + \frac{1}{5}U_{n-2} + \frac{1}{5}U_{n-3} + \frac{1}{5}U_{n-4} + \frac{1}{5}U_{n-5}.$$

The quintic auxiliary equation is

$$x^5 = \frac{1}{5}(x^4 + x^3 + x^2 + x + 1).$$

Let the roots be r_1, r_2, \dots, r_5 . The general solution is

$$U(n) = Ar_1^n + Br_2^n + cr_3^n + dr_4^n + er_5^n,$$

A, B, c, d, e constants. Solving the quintic equation gives a real root $r_1 = 1$ and two pairs of complex roots: $r_2 = -0.5378 + 0.3583i$, $r_3 = -0.5378 - 0.3583i$, and $r_4 = 0.1378 + 0.6782i$, $r_5 = 0.1378 - 0.6782i$.

| Root | r_1 | r_2 | r_3 | r_4 | r_5 |
|----------------------------------|-------|-------------|-------------|------------|-------------|
| Real part | 1 | -0.53783227 | -0.53783227 | 0.13783227 | 0.13783227 |
| Imaginary | 0 | 0.35828469 | -0.35828469 | 0.67815439 | -0.67815439 |
| Polar ρ | 1 | 0.64624413 | 0.64624413 | 0.69201959 | 0.69201959 |
| Polar θ | 0 | 2.55393789 | 2.55393789 | 1.37028142 | 1.37028142 |

Set $\rho_1 = 0.64624413$, $\theta_1 = 2.55393789$, $\rho_2 = 0.69201959$, $\theta_2 = 1.37028142$ and define $B = b + c$, $C = bi + ci$, $D = d + e$, $E = di + ei$. Then

$$U(n) = Ar_1^n + B\rho_1^n \cos n\theta_1 + C\rho_1^n \sin n\theta_1 \\ + D\rho_2^n \cos n\theta_2 + E\rho_2^n \sin n\theta_2.$$

For example,

$$U(2) = A + 0.1609B - 0.3854C - 0.4409D + 0.1869E.$$

Written as a matrix equation this becomes

$$\begin{bmatrix} 1 & -1.28781548 & -0.85789676 & 0.28781548 & -1.41609308 \\ 1 & & 1 & & 0 \\ 1 & -0.53783227 & 0.35828469 & 0.13783227 & 0.67815439 \\ 1 & 0.16089564 & -0.38539414 & -0.44089564 & 0.18694312 \\ 1 & 0.05154595 & 0.26492385 & -0.18754595 & -0.27322852 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} U(-1) \\ U(0) \\ U(1) \\ U(2) \\ U(3) \end{bmatrix}.$$

For initial values, let $U_{-1} = 15$, $U_0 = U_1 = U_2 = U_3 = 0$. Then we can use the inverse matrix to determine A, B, C, D and E :

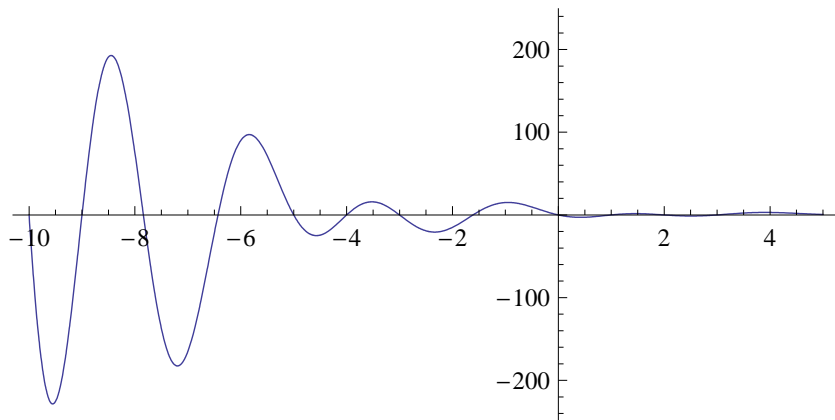
$$\begin{bmatrix} \frac{1}{15} & \frac{2}{15} & \frac{1}{5} & \frac{4}{15} & \frac{1}{3} \\ -0.26876479 & 0.31051264 & -0.53794689 & 0.26031768 & 0.23588136 \\ -0.27177859 & -0.15235036 & 0.19080835 & -0.97900742 & 1.21232802 \\ 0.20209812 & 0.55615403 & 0.33794689 & -0.52698435 & -0.56921470 \\ -0.20894735 & 0.01710405 & 0.58354731 & 0.43756909 & -0.82926711 \end{bmatrix} \begin{bmatrix} 15 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -4.031471847 \\ -4.076678797 \\ 3.031471847 \\ -3.134210713 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix}.$$

With these values of A, B, C, D and E , the function

$$\begin{aligned} U(x) &= A + B\rho_1^x \cos \theta_1 x + C\rho_1^x \sin \theta_1 x \\ &\quad + D\rho_2^x \cos \theta_2 x + E\rho_2^x \sin \theta_2 x \end{aligned}$$

is real for real x .



Order-6 recurrence relation

Consider the order-6 recurrence relation

$$U_n = \frac{1}{6}U_{n-1} + \frac{1}{6}U_{n-2} + \frac{1}{6}U_{n-3} + \frac{1}{6}U_{n-4} + \frac{1}{6}U_{n-5} + \frac{1}{6}U_{n-6}.$$

The sextic auxiliary equation is

$$x^6 = \frac{1}{6}(x^5 + x^4 + x^3 + x^2 + x + 1),$$

with roots r_1, r_2, \dots, r_6 . The general solution is

$$U(n) = Ar_1^n + br_2^n + cr_3^n + dr_4^n + er_5^n + Fr_6^n,$$

A, b, c, d, e, F constants. The equation has two real and two pairs of complex roots:

| Root | r_1 | r_2 | r_3 | r_4 | r_5 | r_6 |
|----------------------------------|-----------|-----------|-----------|----------|-----------|-------|
| Real part | -0.670332 | -0.375695 | -0.375695 | 0.294194 | 0.294194 | 1 |
| Imaginary | 0 | 0.570175 | -0.570175 | 0.668367 | -0.668367 | 0 |
| Polar ρ | 0.670332 | 0.682822 | 0.682822 | 0.730249 | 0.730249 | 1 |
| Polar θ | 0 | 2.153411 | 2.153411 | 1.156147 | 1.156147 | 0 |

Set $\rho_1 = 0.68282252$, $\theta_1 = 2.15341101$, $\rho_2 = 0.73024997$, $\theta_2 = 1.15614777$ and define $B = b + c$, $C = bi + ci$, $D = d + e$, $E = di + ei$. Then

$$U(n) = Ar_1^n + B\rho_1^n \cos n\theta_1 + C\rho_1^n \sin n\theta_1 + D\rho_2^n \cos n\theta_2 + E\rho_2^n \sin n\theta_2 + F.$$

For example,

$$U(2) = 0.4493A - 0.1840B - 0.4284C - 0.3602D + 0.3933E + F.$$

Written as a matrix equation,

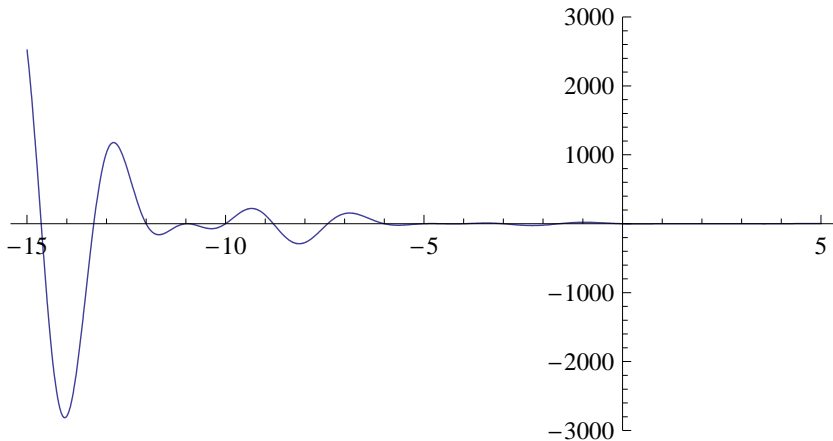
$$\begin{bmatrix} -1.491797 & -0.805786 & -1.222904 & 0.551685 & -1.253348 & 1 \\ & 1 & 1 & 0 & 1 & 0 \\ -0.670332 & -0.375695 & 0.570175 & 0.294194 & 0.668367 & 1 \\ 0.449345 & -0.183952 & -0.428424 & -0.360164 & 0.393259 & 1 \\ -0.301210 & 0.313386 & 0.056071 & -0.368800 & -0.125026 & 1 \\ 0.201910 & -0.149708 & 0.157619 & -0.024935 & -0.283276 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} U(-1) \\ U(0) \\ U(1) \\ U(2) \\ U(3) \\ U(4) \end{bmatrix}.$$

For initial values, let $U_{-1} = 21$, $U_0 = U_1 = U_2 = U_3 = U_4 = 0$. Then we can use the inverse matrix to determine A, B, C, D, E and F :

$$\begin{bmatrix} -0.175914 & 0.086514 & -0.304975 & 0.279048 & -0.592197 & 0.707525 \\ -0.072673 & 0.368001 & -0.186308 & -0.238152 & 1.034274 & -0.905140 \\ -0.312465 & -0.149558 & 0.258076 & -0.748257 & -0.000766 & 0.952970 \\ 0.200968 & 0.450246 & 0.348427 & -0.231372 & -0.680171 & -0.088099 \\ -0.110429 & 0.080532 & 0.498315 & 0.601185 & -0.068753 & -1.000851 \\ \frac{1}{21} & \frac{2}{21} & \frac{1}{7} & \frac{4}{21} & \frac{5}{21} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 21 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3.694197329 \\ -1.526149405 \\ -6.561770976 \\ 4.220346734 \\ -2.319010141 \\ 1 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix}.$$

With these values of A, B, C, D, E and F , we plot

$$\Re U(x) = Ar_1^x + B\rho_1^x \cos \theta_1 x + C\rho_1^x \sin \theta_1 x + D\rho_2^x \cos \theta_2 x + E\rho_2^x \sin \theta_2 x + F.$$



Dice

Chris Pile

Topics from the past occasionally resurface. I was reading *Innumeracy* by John Allen Paulos and came across a reference to ‘Efron dice’ on p. 100. It is stated that Bradley Efron discovered the four dice (as described by Bob Curling in M500 **32**) but the spotting is the same as I suggested in M500 **33**. As M500 **33** predates the first publication of the book by about 12 years, I wonder who spotted them first!

The four dice and their characteristics are as follows.

| | | | |
|----------|-------------|-------------------------------|-------------------------------|
| <i>A</i> | 4 4 4 4 0 0 | 16 spots, EV = $2\frac{2}{3}$ | <i>A</i> beats <i>B</i> 24:12 |
| <i>B</i> | 3 3 3 3 3 3 | 18 spots, EV = 3 | <i>B</i> beats <i>C</i> 24:12 |
| <i>C</i> | 6 6 2 2 2 2 | 20 spots, EV = $3\frac{1}{3}$ | <i>C</i> beats <i>D</i> 24:12 |
| <i>D</i> | 5 5 5 1 1 1 | 18 spots, EV = 3 | <i>D</i> beats <i>A</i> 24:12 |

I have now discovered that, with a minor modification, a similar non-transitive result can be obtained with three dice all having the same number of spots as a standard die (21), so each has the same expected value (EV) of $7/2$. All the faces are same as a face on a standard die (no blanks).

| | | |
|----------|-------------|-------------------------------|
| <i>A</i> | 6 3 3 3 3 3 | <i>A</i> beats <i>B</i> 21:15 |
| <i>B</i> | 5 5 5 2 2 2 | <i>B</i> beats <i>C</i> 21:15 |
| <i>C</i> | 4 4 4 4 4 1 | <i>C</i> beats <i>A</i> 25:11 |

The three dice all perform the same when rolled against a standard die, being evenly matched (win 15, lose 15, equal score 6). However, against a theoretical ‘normalized’ die with $7/2$ spots on each face *A* loses 6:30, *C* wins 30:6 and *B* is evenly matched (18:18). If the three dice are rolled together (216 outcomes), then *B* beats *C* beats *A* by 90:75:51.

There are 32 ways in which a die can be spotted with standard faces (i.e. 1, 2, ..., 6) so that the total number of spots is 21 (EV = $7/2$) but only *A* and *C*, above, record the extreme results of 30:6 or 6:30 against a ‘normalized’ die.

Other non-transitive spottings are possible. For example,

| | | |
|----------|-------------|---|
| <i>X</i> | 6 5 4 4 1 1 | <i>X</i> beats <i>Y</i> 18:12 (6 equal) |
| <i>Y</i> | 4 4 4 3 3 3 | <i>Y</i> beats <i>Z</i> 21:12 (3 equal) |
| <i>Z</i> | 6 6 3 2 2 2 | <i>Z</i> beats <i>X</i> 18:16 (2 equal) |

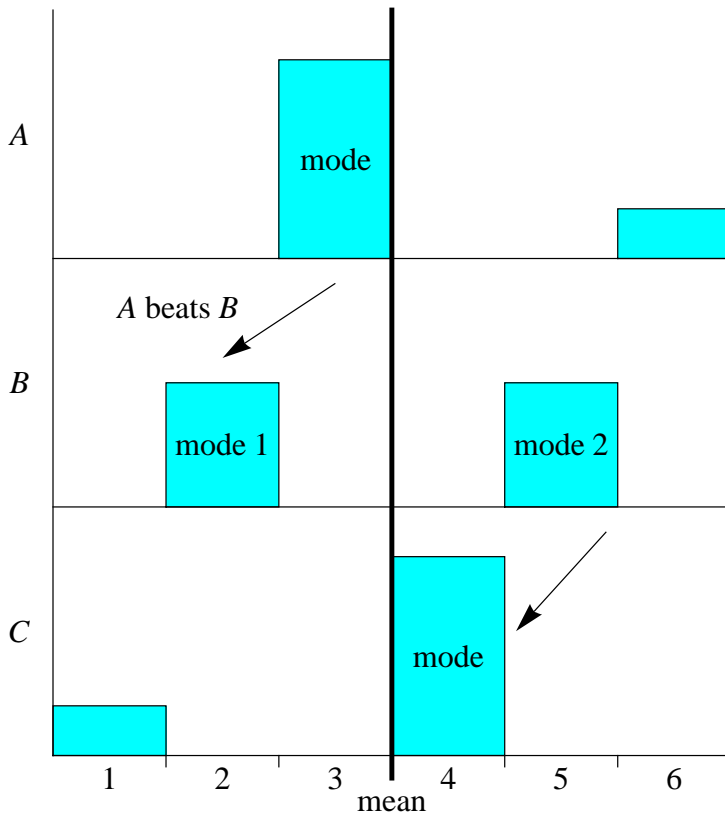
If every possible pair of dice from the 32 arrangements were played against each other (496 contests) I do not know how they would appear in the resulting league table—perhaps someone could investigate!

| Number of ways of spotting a die (EV = 7/2) | | |
|---|-------------|-------------|
| | Spots | Performance |
| | 6 6 6 1 1 1 | E |
| | 6 6 5 2 1 1 | E |
| | 6 6 4 3 1 1 | E |
| | 6 6 4 2 2 1 | E |
| | 6 6 3 3 2 1 | 12:24 |
| | 6 6 3 2 2 2 | 12:24 |
| | 6 5 5 3 1 1 | E |
| | 6 5 5 2 2 1 | E |
| | 6 5 4 4 1 1 | 24:12 |
| Std. | 6 5 4 3 2 1 | E |
| | 6 5 4 2 2 2 | E |
| | 6 5 3 3 3 1 | 12:24 |
| | 6 5 3 3 2 2 | 12:24 |
| | 6 4 4 4 2 1 | 24:12 |
| | 6 4 4 3 3 1 | E |
| | 6 4 4 3 2 2 | E |
| | 6 4 3 3 3 2 | 12:24 |
| A | 6 3 3 3 3 3 | 6:30 |
| | 5 5 5 4 1 1 | 24:12 |
| | 5 5 5 3 2 1 | E |
| B | 5 5 5 2 2 2 | E |
| | 5 5 4 4 2 1 | 24:12 |
| | 5 5 4 3 3 1 | E |
| | 5 5 4 3 2 2 | E |
| | 5 5 3 3 3 2 | 12:24 |
| | 5 4 4 4 3 1 | 24:12 |
| | 5 4 4 4 2 2 | 24:12 |
| | 5 4 4 3 3 2 | E |
| | 5 4 3 3 3 3 | 12:24 |
| C | 4 4 4 4 4 1 | 30:6 |
| | 4 4 4 4 3 2 | 24:12 |
| | 4 4 4 3 3 3 | E |

In an attempt to analyse the non-transitive effect and identify a statistic that could predict this behaviour, it seems that Pearson's first coefficient of skew may be useful.

| Die | Mean EV | Median | Mode | Std. dev. | skew |
|----------|---------|--------|------|-----------|--------|
| <i>A</i> | 3.5 | 3 | 3 | 1.118 | 0.447 |
| <i>B</i> | 3.5 | 3.5 | 2,5 | 1.5 | 1, -1 |
| <i>C</i> | 3.5 | 4 | 4 | 1.118 | -0.447 |

From the distribution, the higher values of negative skew beat lower values and these beat positive skew. Thus 0.447 beats 1, -1 beats -0.447, -0.447 beats +0.447. This seems only partly plausible because the distribution for *B* is symmetrical (skew = 0). However, it seems that a set of three dice must have different skew values.



On ranks and cranks

Tony Forbes

A *partition* of a number n is a set of positive integers that add up to n . For example, $\{1, 1, 3, 6\}$ is a partition of 11 into four parts: 1, 1, 3 and 6.

The number of partitions of n is usually denoted by $p(n)$. Order is immaterial and we also count the number by itself as a partition, $\{n\}$. So $p(1) = 1$ and 1 has just one partition, namely $\{1\}$; $p(2) = 2$, the partitions being $\{1, 1\}$ and $\{2\}$; $p(3) = 3$: $\{1, 1, 1\}$, $\{1, 2\}$, $\{3\}$; $p(4) = 5$: $\{1, 1, 1, 1\}$, $\{1, 1, 2\}$, $\{1, 3\}$, $\{2, 2\}$, $\{4\}$; and so on.

The *rank*, $r(P)$, of a partition P is defined as the largest part minus the number of parts. Thus $r(\{1, 1, 3, 6\}) = 6 - 4 = 2$. Notice that the rank may be negative; for instance, a partition consisting of n ones, has rank $1 - n$.

In 1919, Srinivasa Ramanujan proved that the number $p(n)$ satisfies the two congruences $p(5k+4) \equiv 0 \pmod{5}$ and $p(7k+5) \equiv 0 \pmod{7}$, for $k = 0, 1, 2, \dots$. But what was lacking from his proofs was a simple property of the partitions of $5k+4$ and $7k+5$ that distributes them equally over the residue classes modulo 5 and 7, respectively.

Writing in the 1944 edition of *Eureka* (the journal of the Archimedean Society, a mathematical society for Cambridge University undergraduates), Freeman Dyson defined the rank and conjectured that it does just that—for $x = 0, 1, 2, 3, 4$, exactly $p(5k+4)/5$ partitions of $5k+4$ have rank $\equiv x \pmod{5}$, and for $x = 0, 1, \dots, 6$, exactly $p(7k+5)/7$ partitions of $7k+5$ have rank $\equiv x \pmod{7}$. Oliver Atkin and Peter Swinnerton-Dyer proved Dyson's conjecture in 1953.

Ramanujan proved a third congruence, $p(11k+6) \equiv 0 \pmod{11}$ for $k = 0, 1, 2, \dots$. However it turns out that in this case the rank function as defined above fails to work. It is not always true that for given $x \pmod{11}$, $p(11k+6)/11$ partitions of $11k+6$ have rank $\equiv x \pmod{11}$. In the same *Eureka* article, Dyson predicted the existence of another function c such that the $c(P)$ of the partitions P of $11k+6$ are distributed uniformly over the residue classes modulo 11. Dyson named this property the *crank* of a partition but at the time he was unable to define it. He expressed the hope that the crank would not have to suffer the same fate as the planet Vulcan and that one day somebody would discover the correct definition and thereby prove it does exist.

That had to wait until 1987, when George Andrews and Frank Garvan discovered the true definition of the crank—on the last day of the Ramanujan Centenary Conference at the University of Illinois.

Given a partition P , let $L(P)$ denote the largest part of P , let $M(P)$ denote the number of ones in P and let $N(P)$ denote the number of parts of P larger than $M(P)$. The crank, $c(P)$, is defined as follows: if $M(P) = 0$, then $c(P) = L(P)$, otherwise $c(P) = N(P) - M(P)$.

The tables on this page list the partitions of 4, 5 and 6. Observe that rank (mod m) is equally distributed over the partitions of 4 and 5, but not 6. In the table for $n = 4$ and $n = 5$, the 'rank mod m ' column contains the full complement of residues, $\{0, 1, 2, 3, 4\} \pmod{5}$ and $\{0, 1, 2, 3, 4, 5, 6\} \pmod{7}$ respectively, whereas for $n = 6$ there are two 1s, two 10s, no 4 and no 7.

On the other hand, crank (mod m) is equally distributed for 6 as well as 4 and 5. You verify by checking that each of the numbers 0, 1, 2, \dots , 10 appears exactly once as a crank (mod 11) when $n = 6$.

| n | m | partition P | rank | | | | | crank | |
|-----|-----|------------------|------|---------|--------|--------|--------|-------|---------|
| | | | rank | mod m | $L(P)$ | $M(P)$ | $N(P)$ | crank | mod m |
| 4 | 5 | 1 1 1 1 | -3 | 2 | 1 | 4 | 0 | -4 | 1 |
| | | 1 1 2 | -1 | 4 | 2 | 2 | 0 | -2 | 3 |
| | | 1 3 | 1 | 1 | 3 | 1 | 1 | 0 | 0 |
| | | 2 2 | 0 | 0 | 2 | 0 | 2 | 2 | 2 |
| | | 4 | 3 | 3 | 4 | 0 | 1 | 4 | 4 |
| 5 | 7 | 1 1 1 1 1 | -4 | 3 | 1 | 5 | 0 | -5 | 2 |
| | | 1 1 1 2 | -2 | 5 | 2 | 3 | 0 | -3 | 4 |
| | | 1 1 3 | 0 | 0 | 3 | 2 | 1 | -1 | 6 |
| | | 1 2 2 | -1 | 6 | 2 | 1 | 2 | 1 | 1 |
| | | 1 4 | 2 | 2 | 4 | 1 | 1 | 0 | 0 |
| | | 2 3 | 1 | 1 | 3 | 0 | 2 | 3 | 3 |
| | | 5 | 4 | 4 | 5 | 0 | 1 | 5 | 5 |
| 6 | 11 | 1 1 1 1 1 1 | -5 | 6 | 1 | 6 | 0 | -6 | 5 |
| | | 1 1 1 1 2 | -3 | 8 | 2 | 4 | 0 | -4 | 7 |
| | | 1 1 1 3 | -1 | 10 | 3 | 3 | 0 | -3 | 8 |
| | | 1 1 2 2 | -2 | 9 | 2 | 2 | 0 | -2 | 9 |
| | | 1 1 4 | 1 | 1 | 4 | 2 | 1 | -1 | 10 |
| | | 1 2 3 | 0 | 0 | 3 | 1 | 2 | 1 | 1 |
| | | 1 5 | 3 | 3 | 5 | 1 | 1 | 0 | 0 |
| | | 2 2 2 | -1 | 10 | 2 | 0 | 3 | 2 | 2 |
| | | 2 4 | 2 | 2 | 4 | 0 | 2 | 4 | 4 |
| | | 3 3 | 1 | 1 | 3 | 0 | 2 | 3 | 3 |
| | | 6 | 5 | 5 | 6 | 0 | 1 | 6 | 6 |

On the next page there is a more substantial example: the thirty partitions of 9, split into five classes of six by rank (mod 5) and also by crank (mod 5). Each of the numbers 0, 1, 2, 3 and 4 appears six times in both columns.

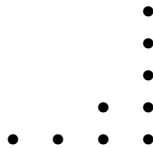
From this limited amount of evidence you may have noticed, as I did, the interesting fact that the ranks add up to zero. In other words

$$\sum_P r(P) = 0, \tag{1}$$

where P runs through the partitions of n . This is in fact quite easy to prove. Write a partition such as $9 = 1 + 1 + 1 + 2 + 4$ as an array of dots, where in this case the five parts are represented by columns.



Rotate the diagram through 90 degrees clockwise and reflect it in a vertical mirror. The pattern of dots now represents the conjugate partition $1 + 1 + 2 + 5$.



The largest part (4) and number of parts (5) of the original partition have become interchanged. The new partition has 4 parts; the largest part is 5.

More generally, start with a partition P of n into s parts, the largest of which is t . The rank of P is $t - s$. Now apply the rotation-and-reflection procedure to obtain a partition P' into t parts, the largest of which is s , and hence the rank of P' is $s - t$. Therefore $r(P) + r(P') = 0$ and, furthermore, P and P' will be different partitions of n , except possibly when $s = t$ in which case the rank is zero anyway. Thus (1) holds true.

Having shown that the ranks of the partitions of n sum to zero it occurred to me that the same result might be true of the crank, but stupidity and laziness have so far prevented me from finding a proof or a counterexample. I offer it to interested readers as a challenge. It is possible that a more sophisticated version of the dots argument might work.

The 30 partitions of 9

| partition | rank | rank (mod 5) | crank | crank (mod 5) |
|-------------------|------|--------------|-------|---------------|
| 1 1 1 1 1 1 1 1 1 | -8 | 2 | -9 | 1 |
| 1 1 1 1 1 1 1 2 | -6 | 4 | -7 | 3 |
| 1 1 1 1 1 1 3 | -4 | 1 | -6 | 4 |
| 1 1 1 1 1 2 2 | -5 | 0 | -5 | 0 |
| 1 1 1 1 1 4 | -2 | 3 | -5 | 0 |
| 1 1 1 1 2 3 | -3 | 2 | -4 | 1 |
| 1 1 1 1 5 | 0 | 0 | -3 | 2 |
| 1 1 1 2 2 2 | -4 | 1 | -3 | 2 |
| 1 1 1 2 4 | -1 | 4 | -2 | 3 |
| 1 1 1 3 3 | -2 | 3 | -3 | 2 |
| 1 1 1 6 | 2 | 2 | -2 | 3 |
| 1 1 2 2 3 | -2 | 3 | -1 | 4 |
| 1 1 2 5 | 1 | 1 | -1 | 4 |
| 1 1 3 4 | 0 | 0 | 0 | 0 |
| 1 1 7 | 4 | 4 | -1 | 4 |
| 1 2 2 2 2 | -3 | 2 | 3 | 3 |
| 1 2 2 4 | 0 | 0 | 2 | 2 |
| 1 2 3 3 | -1 | 4 | 2 | 2 |
| 1 2 6 | 3 | 3 | 1 | 1 |
| 1 3 5 | 2 | 2 | 1 | 1 |
| 1 4 4 | 1 | 1 | 1 | 1 |
| 1 8 | 6 | 1 | 0 | 0 |
| 2 2 2 3 | -1 | 4 | 3 | 3 |
| 2 2 5 | 2 | 2 | 5 | 0 |
| 2 3 4 | 1 | 1 | 4 | 4 |
| 2 7 | 5 | 0 | 7 | 2 |
| 3 3 3 | 0 | 0 | 3 | 3 |
| 3 6 | 4 | 4 | 6 | 1 |
| 4 5 | 3 | 3 | 5 | 0 |
| 9 | 8 | 3 | 9 | 4 |

No, it's not the home of *Star Trek's* Mr Spock. Dyson's Vulcan was a hypothetical member of the solar system invented by 19th century astronomers to account for deviations from the theoretical orbit of Mercury that could not be explained by gravitational effects of the other known planets. However, Vulcan became unnecessary when (in 1915) a young employee of the Swiss Patent Office developed a new theory of gravity which correctly predicted Mercury's erratic behaviour.

When does $2^\infty = 0$?

Martin Cooke

As well as the trivial solution ('when you've made a mistake, such as dividing by zero') I'd like to postulate the following:

Dividing by zero can be done consistently, as I did when constructing my infinite times tables (M500, issue **176**) so long as the usual problem is overcome, i.e. you don't derive $2 = 1$ from $2 \cdot 0 = 0 = 1 \cdot 0$. In **176** I used the result $0^0 = X$, where X is an extension of a field F . Then if $\infty \equiv 0^{-1}$, $\infty^0 = 0^{-0} = 0^0 = X$ too. This gives a mapping $X \rightarrow X$, where $x \rightarrow x^0$ for $x \in X$ since $x^0 = 1$ for non-zero elements of F .

When exponentiation is extended from the integers, the property $a^b a^c = a^{b+c}$ is taken as definitive. Since this would imply that $X = 0 \cdot X = 0^1 0^0 = 0^1 = 0$, the context of using this property definitively when extending exponentiation may be one situation where 0^0 should be regarded as only 1, rather than the whole of X , although it does hold in that part of X where distributivity also holds (i.e. in F) since $0 \cdot F = 0$. However, since $0/0 = X$ can be regarded as 0^0 in a fairly obvious way, and so used as the notation of the above mapping, it is also possible to take the inverse mapping (by merely reversing the mapping $x \rightarrow x^0$) $X \rightarrow X$, $x \rightarrow x^\infty$ as a mathematical object this way. Then $1^\infty = X$ and $x^\infty = (0, \infty)$ for non-unit x .

Note that this is a particular infinity with properties such as $\infty = -\infty$, and an exponentiation notation which is appropriate but not the only possibility. The equation of the title results from restricting this inverse mapping to $F \rightarrow F$ and looking at the value of 2 under it. The particular value of 2 was chosen because of the similarity of this equation to those of cardinal arithmetic, if ∞ is replaced by Ω , the size of the proper class of cardinals, whose power-set would be contradictory (if it were a set) and so belong in the empty set, of size 0. (Both Ω and ∞ are known as 'absolute' infinities; because of their conceptual contexts, nothing is bigger than they are.)

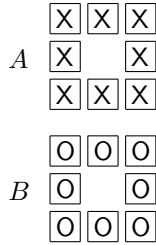
Problem 179.1 – Two cars

David Singmaster

Two cars are heading towards one another from 100 miles apart on a straight road. The first is going 60 m.p.h. and the second is going 40 m.p.h. A fly starts at the front bumper of the first car and flies to the second and then back to the first, then back to the second, etc. Eventually there is a god-awful crash and our fly is squashed. If the fly can fly 50 m.p.h., how far does he fly before the smash?

Solution 177.1 – Eight cubes

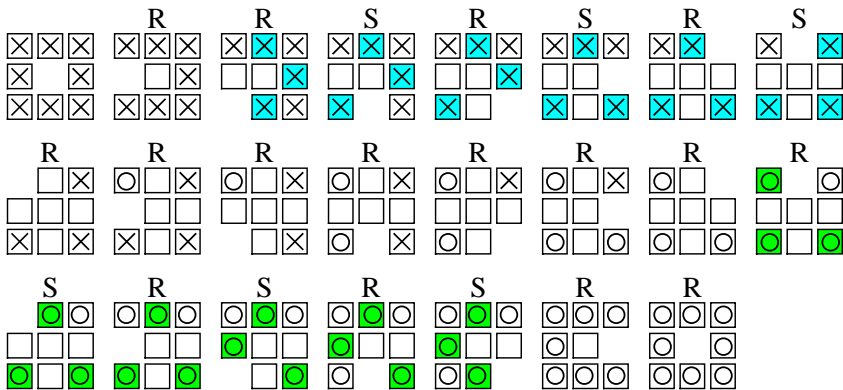
Eight cubes have a pair of opposite faces marked with ‘X’ and ‘O’. They are placed in a box in configuration *A*. Get them into configuration *B*. The only legal move is to roll or slide a cube into the vacant space. We asked you to show that 22 moves are necessary.



Malcolm Maclenan

Not a rigorous proof, more a simpleton’s approach. For eight cubes to be inverted, the configuration involving least ‘rolls’ would be a straight line, and 16 rolls would be needed. To achieve this within the constraints of the 3 × 3 box, three of the centre cubes (cyan/light grey) must be slid out to the edge and, finally, three must be slid back to the centre (green/dark grey). The fourth centre cube can be rolled across the centre.

A slide and two rolls is the same number of manoeuvres as a roll forward, or a roll sideways and a roll backwards. This latter manoeuvre does transpose a corner cube to the centre cube and a centre cube to a corner cube, but must always pass through the centre, which is usually occupied.



Norman Clark

Martyn Hennessy

We offer our sympathy to the relatives and friends of two of our members, Norman Clark and Martyn Hennessy, who died last year.

Solution 175.4 – The first prime

The prime numbers are arranged alphabetically. Which is the first?

Ralph Hancock

Greek: 2, duo, surely.

Latin: The best I can come up with is **100109** (centum milia novem).

Re: ‘centum bilia . . .’, a suggestion of Tony Forbes. It means ‘a hundred bilious women.’ The classical expression for a million is ‘decies centena milia’; my Polish-Latin dictionary, besides the latter also gives ‘mil(l)io’, with the assumed genitive in -onis, but such word is as valid for Latin as ‘instrumentum televisorium’, as you see, and perhaps was only given there to justify another vocable, ‘milionarius’, which of course has a synonym ‘homo divitissimus’. ‘Billion’ is a purely artificial word—see *AHD* or *Collins* for etymology. To treat it as a neuter noun with Greek ending and the plural in -a is inventive, but can hardly be taken as serious.

ADF

I found this in an item on the WWW:

American English: **8,808,808,889** (eight billion eight hundred eight million eight hundred eight thousand eight hundred eighty-nine)

British English: **8,808,808,808,851** (eight billion eight hundred eight milliard eight hundred eight million eight hundred eight thousand eight hundred fifty-one)

‘Milliard’? I have only ever seen it on German postage stamps. Anyway, the American version seems to answer the question if we agree to suppress the word ‘and’. Otherwise, Ralph Hancock’s **8,000,000,081** (eight billion and eighty one, M500 177 26) still stands.

Problem 179.2 – Four tans

Dick Boardman

Show that

$$\tan 11^\circ = (\tan 19^\circ)(\tan 33^\circ)(\tan 41^\circ).$$

London Mathematical Society

Popular Lectures 2001

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LONDON Commences at 7.00 pm, refreshments at 8.00 pm, ends at 9.30 pm. Admission is free, with ticket. Apply by June 29th to Miss S. M. Oakes, London Mathematical Society, 57–58 Russell Square, London WC1B 4HS (e-mail: taylor@lms.ac.uk). A stamped addressed envelope would be appreciated.

Letters to the Editors

Two cars

Re: Problem 179.1, page 20 of this M500. How many people got 50 miles? Hands up! Good, I'm glad to see so many people know this problem. After all, the cars are approaching from 100 miles apart at a total speed of 100 m.p.h., so they'll collide in just one hour, during which time our fly has flown 50 miles (no flies on him!).

Sadly, you're all wrong! Since the first car is going at 60 m.p.h. and the poor fly can only do 50 m.p.h., he remains stuck fast on the front bumper of the first car, totally unable to do anything but stare at the oncoming disaster.

Moral: don't solve your problem until you've read it. (Some people claim that the fly would be able to head away from the first car at a total speed of 110 m.p.h., but air resistance would keep him from getting more than a negligible distance.)

David Singmaster

Find the missing terms

Re: M500 177, page 28. The piglets were named Millie, Christa, Alexis, Carrel and Dotcom. I found the rule of formation at

<http://www.toronto.globaltv.com/ont/technology/stories/technology-20000314-160918.html>.

'The names of the first cloned piglets each have their own significance. Millie was named for the millennium. Christa, Alexis and Carrel were named after Dr Christian Barnard, who performed the first human heart transplant, and Dr Alexis Carrel, who won the Nobel prize in 1912 for his work in the field of transplantation. And as for Dotcom ... "Any association with dotcoms right now seems to have a very positive influence on a company's valuation," said James.'

Nigel Mercier

ADF writes—**Ralph Hancock** treated it as that game in which you are given two words, W_0 , W_n , and you have to form a sequence, W_0, W_1, \dots, W_n . The rules are:

- (i) all the words have the same number of letters;
- (ii) word W_i differs from word W_{i-1} , $i = 1, 2, \dots, n$, in exactly one place;
- (iii) there exists an authority which confirms that W_0, W_1, \dots, W_n belong to the English language.

Ralph's answer: **Milly, Milty, Mitty, Ditty, Dotty**. (Milty: like the seed-filled reproductive gland of male fish—*COD*.)

Now that we have introduced the game, here a couple of interesting examples from **David Singmaster**:

(1) Turn APE into MAN. Find a shorter sequence than APE, APT, OPT, OUT, BUT, BUN, BAN, MAN.

(2) Change CIRCLE to SQUARE. David assures me that there is no valid solution. His reason: it is impossible to square the circle!

Problem 179.3 – Nine switches

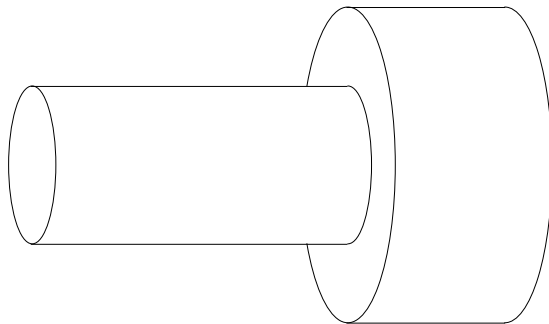
You are outside a room. You have nine switches. Inside are nine light-bulbs (which you can't see, of course). How many trips must you make into the room to allocate bulbs to switches?

Problem 179.4 – Two cylinders

Peter Bell

There are two cylinders, one can be thought of as short and fat, the other as long and thin. The short cylinder has radius R and length L , $R > r > 0$. The long cylinder has radius r and length l , $l > L > 0$. The two cylinders are touching such that their axes are co-linear. There exists a closed loop of string of length x .

Question: What is the smallest x (i.e. the length of the shortest loop of string) that can be passed over the fat cylinder onto the thin cylinder?



Solution 177.3 – Mahatma's triangle

Given x in the diagram, opposite, what is θ ?

Chris Pile

I have not seen this before and it has caused me some frustration because I cannot see a neat solution.

Make TQ perpendicular to AC and make BN perpendicular to TC , extending it to meet AC at M . Then

$$\frac{TB}{\sin \theta} = \frac{TR}{\sin(30^\circ - x)}.$$

Hence

$$\sin \theta = \frac{TB}{TR} \sin(30^\circ - x). \quad (1)$$

Triangle TBC is isosceles ($TB = BC$) as is $\triangle TMC$ ($TM = MC$); so $\angle CTM = 30^\circ = \angle MTQ$. Hence $TQ = TN$ ($= TM \sin 60^\circ$). Next, $\angle BRQ = 30^\circ + x$, so

$$\frac{TQ}{TR} = \sin(\theta + 30^\circ + x).$$

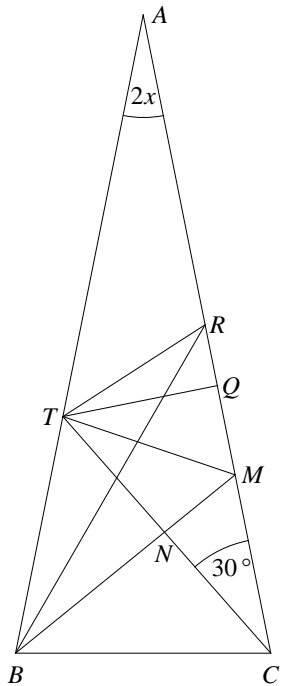
Also $\frac{TQ}{TB} = \frac{TN}{TB} = \sin(60^\circ - 2x)$, so $\frac{TB}{TR} = \frac{\sin(\theta + 30^\circ + x)}{\sin(60^\circ - 2x)}$. Thus from (1),

$$\begin{aligned} \sin \theta &= \frac{\sin(\theta + 30^\circ + x) \sin(30^\circ - x)}{\sin(60^\circ - 2x)} = \frac{\sin(30^\circ + \theta + x)}{2 \cos(30^\circ - x)} \\ &= \frac{(\sin \theta) \cos(30^\circ + x) + (\cos \theta) \sin(30^\circ + x)}{2 \cos(30^\circ - x)}. \end{aligned}$$

Hence $\tan \theta = \frac{(\tan \theta) \cos(30^\circ + x) + \sin(30^\circ + x)}{2 \cos(30^\circ - x)}$, which rearranges to give

$$\tan \theta = \frac{\sin(30^\circ + x)}{2 \cos(30^\circ - x) - \cos(30^\circ + x)} = \frac{\frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x}{\frac{\sqrt{3}}{2} \cos x + \frac{3}{2} \sin x} = \frac{1}{\sqrt{3}};$$

$\theta = 30$ degrees.



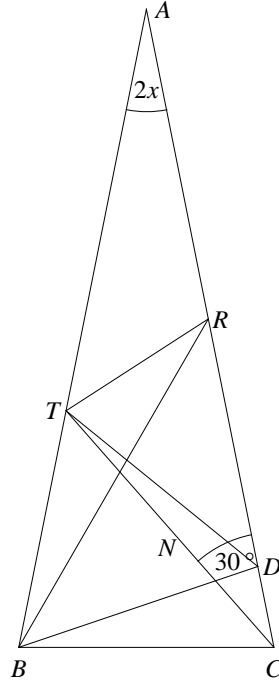
Elsie Page

Construct BD so that angle RBD is $30^\circ + x$ (i.e. $\angle TBD = 60^\circ$) and D lies on AC . Join DT .

First we prove that $\triangle TBD$ is equilateral. Triangle BCD is isosceles with $DB = CB$ since $\angle BDC = 2x + 60^\circ$. Triangle BTC is isosceles with $TB = CB$ since $\angle BTC = 2x + 30^\circ$. So $TB = DB$ and, since $\angle TBD = 60^\circ$, $\triangle TBD$ is equilateral.

Next, we prove that D is the centre of the circumcircle of $\triangle BTR$. Triangle RDB is isosceles with $DR = DB$ since $\angle BRD = 30^\circ + x$. Also $DT = DB$ (since $\triangle TBD$ is equilateral). So $DR = DB = DT$ and D is the centre of the circle through R , B and T . Hence

$$\angle TRB = \frac{1}{2}\angle TDB = 30^\circ.$$



Problem 179.5 – Subtract square root

ADF

Start with a large number, n . Replace n by $n - \sqrt{n}$. Repeat until you reach something less than 1. Approximately how many iterations are required?

What about $n \rightarrow n - \sqrt{kn}$, where k is a given constant?

Problem 179.6 – Root 11 again

Barry Lewis

This problem concerns an old friend, the integer part of the number $(\sqrt{11} + 3)^n$; cf. M500 174 26, 176 6.

Prove that if p is an odd prime, the integer part of $(\sqrt{11} + 3)^p - 2 \cdot 3^p$ is divisible by $66p$.

Solution 177.5 – 3 theta

Show that $\frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} = 2$.

Dick Boardman

Multiply both sides by $(\sin \theta)(\cos \theta)$. Then the left hand side is the expansion of $\sin(3\theta - \theta)$ and the right hand side is the standard expansion for $\sin 2\theta$. QED.

There are many trig. identities containing integers; e.g. if we take any of the standard expansion formulae and divide the LHS by the RHS we get an expression equal to 1. This problem has simply taken two standard formulae for $\sin 2\theta$ and divided one by the other.

Squawks

Eddie Kent

Squawks are problem listings that pilots leave for maintenance crews to fix before the next flight. Here's a selection of complaints submitted by US Air Force pilots and the replies from the crews.

Problem: Left inside main tyre almost needs replacement

Solution: Almost replaced left inside main tyre

Problem: Test flight OK, except autoland very rough

Solution: Autoland not installed on this aircraft

Problem: Something loose in cockpit

Solution: Something tightened in cockpit

Problem: Evidence of leak on right main landing gear

Solution: Evidence removed

Problem: DME volume unbelievably loud

Solution: Volume set to believable level

Problem: IFF inoperative

Solution: IFF always inoperative in OFF mode

Problem: Friction locks cause throttle levers to stick

Solution: That's what they're there for

Problem: Number three engine missing

Solution: Engine found on right wing after brief search

Twenty-five years ago

From M500 31 and 32

Richard Ahrens—*Mrs Read's knitting machine.* Graham Read's wife has a knitting machine that uses thread wound on special spools. Some knitting patterns call for the simultaneous use of several threads (k , say), which means having k spools with thread wound on them. Suppose we have n spools ($n \geq k$), s_1, s_2, \dots, s_n , containing various quantities of thread, t_1, t_2, \dots, t_n respectively.

i. What condition must the numbers t_1, t_2, \dots, t_n satisfy so that it is possible to knit all the wool without ever having to wind wool from one spool onto an empty spool? It is possible to change spools as often as you like (i.e. before they are empty). The pattern uses all k threads at the same rate. (Hint: try $k = 2, n = 3$ first.)

ii. Suppose we have the situation described above and the numbers t_1, t_2, \dots, t_n are such that it is possible to knit all the wool without rewinding a spool. Show that it is possible to knit all the wool with no more than $n - 1$ stops to change spools, but that in some cases $n - 1$ stops will be necessary.

iii. (Unsolved and looks difficult) Find an algorithm for devising the most efficient way of using all the wool (i.e. fewest stops to change spools) in those cases where the job can be done in fewer than $n - 1$ stops.

Coral Bytheway—In one mindless moment my spouse declared, "Since sec is just cos to the minus one, why don't we forget about secs?"

Are people suffering from secs-phobia turned on by cos's. (Cossie: Northern dialect for bathing costume.)

Eurocheques

Following a change in their policy, UK banks have withdrawn from the Eurocheque scheme as from the beginning of the year and will no longer accept Eurocheques in sterling at their face value.

As a consequence of the decision, any Eurocheques paid in by the Society to banks in the UK are now returned without presentation. The banks will not negotiate the cheques because the negotiation charges exceed the amount of the subscription.

We regret, therefore, that we are no longer able to accept Eurocheques as payment of subscriptions and must ask that overseas members find alternative methods for forwarding the funds to us. We realise that this is likely to incur extra expense for overseas members and do hope that they will not be deterred from renewing their subscriptions.

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