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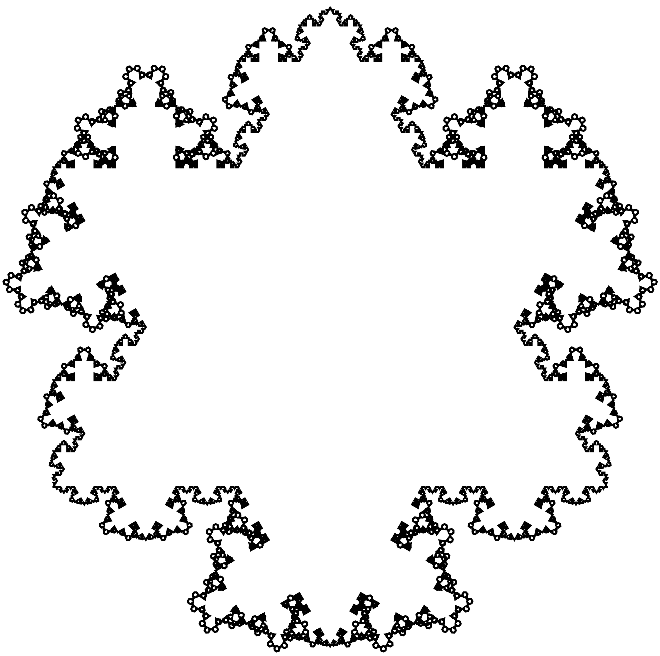
M500 181



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

John Fauvel

Costel Harnasz

Participants at this year's M500 Winter Week-end at Nottingham University might have noticed that John Fauvel looked a little off colour, and that he wasn't present at the plenary session on the Sunday. John was in fact seriously ill.

He died on May 12th, of a longstanding liver and kidney condition.

For those who didn't know John, he worked for the OU co-authoring the set book for MA290, *Topics in the History of Mathematics*. He was President of the British Society for the History of Mathematics, and editor of its newsletter, and there some of the qualities evident even during the couple of days with us at the beginning of this year had their full fruition. He helped make the BSHM into an internationally respected institution, and anyone who has seen its newsletter will see why—beautifully produced, with something for everyone.

He introduced HIMED—History in Maths Education conferences—to the world of teaching and these were a joy to be part of. You see, John would talk to everybody, and introduce everybody to everybody else with this special knack such that schoolteacher would be quite at ease with emeritus professor. John didn't make distinctions.

He was a rare human being, generous in many ways, never complaining. He stood in at short notice when the main speaker couldn't make it at the M500 Revision Week-end in 1991. Those of you who were there will remember the delightful lecture he gave on the mathematical work of Charles Babbage. And when he asked me to apologize on his behalf for his early departure on that Sunday afternoon in Nottingham, he disguised the true severity of his illness by excusing himself in the note he'd written: '… on account of my hip playing up'.

Those wishing to find out more about the BSHM, and to read the extensive worldwide tributes to this remarkable man, can go to www.dcs.warwick.ac.uk/bshm.

Sidney Silverstone

John was a supporter of M500 from its early days (in the 1970s). Not all of the members of the maths faculty supported the organization then; indeed, some of them were quite anti. He was one of our early tutors at the M500 Week-end in the History of Mathematics course, AM289.

He played a significant role in creating the high regard that the OU is given internationally for its research into the history of our subject.

Peter Baxandall

I am very sorry to have to share with you some of the sadness that I feel, but on the other hand I am sure that you would want to know that John Fauvel (Course Team Chair of MA 290 and joint author with Jeremy Gray of the associated book *The History of Mathematics – A Reader*) died on Saturday 12th May. He was only 53 years old.

Some of John's characteristic kindness and enthusiasm and energy comes through the Units and particularly the Reader. Everyone who knew him (even slightly) cannot fail to have been influenced by all three. There is ample evidence of that in the tributes that you will find from his many colleagues and friends in this country and from all over the world in the BSHM (British Society for the History of Mathematics) website http://www.dcs.warwick.ac.uk/bshm.

You will find an account of how much John did (and get some feeling of how much more there was in him had there been time) by looking at the OU Pure Mathematics website at http://mcs.open.ac.uk/puremaths.

Forty-eight cubes

Tony Forbes

Forty-eight cubes have a pair of opposite faces marked with 'X' and 'O'. They are placed in a box in configuration A. Get them into configuration B. The only legal move is to roll a cube into the vacant space (leaving a hole behind it for the next move).

Having been successful with the 3×3, 4×4 and 5×5 versions of this problem [M500 **176** 20–21], I thought I would try something a little more difficult. We bypass the 6×6 case and tackle the 7×7 instead. It has the pleasing 'hole in the middle' symmetry and, as we shall see, there are two promising strategies.

A

X	X	X	X	X	X	X
X	X	X	X	X	X	X
X	X	X	X	X	X	X
X	X	X		X	X	X
X	X	X	X	X	X	X
X	X	X	X	X	X	X
X	X	X	X	X	X	X

$\longrightarrow$

B

O	O	O	O	O	O	O
O	O	O	O	O	O	O
O	O	O	O	O	O	O
O	O	O		O	O	O
O	O	O	O	O	O	O
O	O	O	O	O	O	O
O	O	O	O	O	O	O

Split into two halves. Select the cubes in the lower half of the array; specifically, the 21 cubes in rows 5, 6 and 7, and the last three cubes of row 4. Search for a solution, S, that inverts just these 24 cubes, with the hole starting and finishing in the centre of the array. Then rotate the whole array through 180 degrees and apply S again to invert the other 24 cubes. For example, this 74-move solution to the lower half,

LLUU LDRR ULUL DRDD LUUR DRUR ULLD RRRR ULDR
RULD DLLL DRUL DRUL DLUR URUR RDDR ULLD RULL DD,

yields a 148-move solution to the 7×7 problem. (We use the letters L, R, U, D to denote roll a cube left, right, up, down respectively.) On the other hand, it can be shown (by testing all possible sequences) that the lower half of the array cannot be solved in 62 moves or less. Hence this method will require at least 128 moves for all 48 cubes. It turns out that we can do better with our second strategy.

Expand a 5×5 *solution.* Take a known solution to the 5×5 problem and 'fatten it up' so that it applies to the whole 7×7 array. This is done by repeating a move iff either (i) L or R rolls a cube into column 3 or 5, or (ii) U or D rolls a cube into row 3 or 5. Then use the new sequence as a starting-point of a search for 7×7 solutions. For example, this 68-move 5×5 solution,

LURR DDDL LULU RUUR RDRD LUUR DDDD LULL DLUU
UURD DRDL ULUR URRD DRDL DLLU UURR DDLU,

expands to

LLUU RRRR DDDD DLLL LULU URUU URRR RDRD DLUU
URDD DDDD LULL LLDL UUUU UURD DDRR DDLL UULU
URUR RRRD DDRD DLDL LLLU UUUU RRRR DDDD LLUU

(108 moves), which inverts all the cubes except for a 4×3 block in the middle of the array. We truncate it to 63 moves and finish the process with

D RUUR DRRD DLDL LLUL UULU URUR RRRD DLDD
LUUL DLUU RRRR DDDD LLLD LUUR URUR DL

to obtain this 126-move solution to the 7×7 problem:

LLUU RRRR DDDD DLLL LULU URUU URRR RDRD DLUU
URDD DDDD LULL LLDL UUUU UURD DDRD RUUR DRRD
DLDL LLUL UULU URUR RRRD DLDD LUUL DLUU RRRR
DDDD LLLD LUUR URUR DL.

It is the best that I have so far. I also know that it is impossible to invert all 48 cubes in 100 moves or less. A complete answer to the problem, i.e. finding a solution of proven minimum length, therefore amounts to eliminating the huge gap between 100 and 126.

Fractal shapes

Sebastian Hayes

Barbara Lee (M500 **178** page 33) briefly mentions *fractal geometry* and states that 'most books tend to avoid actually defining *fractal*.'

The term was coined by Benoit Mandelbrot in his *The Fractal Geometry of Nature* (Freeman, 1975). This remarkable book includes many beautiful fractal designs along with philosophic discussion and much interesting mathematics—so there is something in it for everybody. Mandelbrot says (1977 edition, p. 4):

> I coined *fractal* from the Latin adjective *fractus*. The Latin verb *frangere* means 'to break': to create irregular fragments. ... How appropriate to our needs ... that in addition to 'fragmented' (as in *fraction* or *refraction*) *fractus* should also mean 'irregular', both meanings being preserved in *fragment*.

A more mathematical definition is 'a set in a metric space, for which the Hausdorff–Besicovitch dimension D exceeds the topological dimension D_T.'

The 'topological dimension' more or less corresponds to what we think of as a dimension—a curve, for example, has dimension 1 and a rectangle dimension 2. What the Hausdorff dimension is I have never managed to work out but it is not necessary to know.

Most of the simpler Mandelbrot fractals are built up iteratively using an *initiator* and a *generator*. An initiator is generally a well-known Euclidean shape such as a rectangle or octagon: in the case of the Koch 'curve' it is an equilateral triangle. We can forget it and concentrate on a single side.

A generator is 'an oriented broken line made up of N equal sides of length r.' At each stage you replace 'a straight line with a copy of the generator, reduced and displaced so as to have the same end points as those of the interval being displaced' (1977, p. 39).

In the case of the Koch curve, $N = 4$, and $r = 1/3$. If we start with a side of unit length we get an indefinitely extendable sequence

$$t_0 = 1, \quad t_1 = 4\cdot\frac{1}{3} = \frac{4}{3}, \quad t_2 = 4\cdot\frac{1}{3^2}\cdot 4 = \frac{4^2}{3^2} = \frac{16}{9}, \quad \ldots, \quad t_r = \frac{4^r}{3^r}.$$

The total perimeter of the figure will be three times this since we are considering one side of a triangle only.

Note that the sequence above is increasing without bound and so the perimeter can be made to exceed any stated value simply by taking r large enough even though the entire 'curve' remains inside a disc with radius 0.6 metres, or, for that matter, millimetres.

Is this a case of 'Infinity in the palm of your hand and Eternity in an hour' (Blake) as I have heard someone say during a public lecture? As far as I am concerned, most emphatically not. The majority of writers on the subject—and this includes a distressingly large number of so-called mathematicians—do not seem to realise that there is no 'Koch curve' as such but only a *family* of 'curves', each of which is strictly finite and indeed has derivatives at all points (except at the end points of each straight line section). Indeed, *the* 'Koch curve' is thus a limit-function to an indefinitely extendable set of self similar curves and like most limits it is not to be encountered except in the 'Platonic' world, wherever that is. In real life we rapidly attain the molecular and atomic levels beyond which measurement in the macroscopic sense ceases to have any meaning. All this is repeatedly stated by Mandelbrot himself, 'To obtain a Koch curve, the cascade of smaller and smaller promontories is pushed to infinity, but in Nature every cascade must stop or change character' (1983 edition, p. 39).

Mandelbrot defines the 'dimension' of such a self-similar ideal figure to be $(\log N)/(\log m)$, where $m = 1/r$, in this case $(\log 4)/(\log 3)$, or $1.2618\ldots$.

Why is this? According to the principle of construction, each little segment of a Koch 'curve' is a third of the previous one. Viewing the segment length as x and setting the original side at unity, we can envisage the length of each successive 'curve' as a function of x—but we must remember that x is not a proper variable since it can only take the values

$$1, \quad \frac{1}{3}, \quad \frac{1}{3^2}, \quad \frac{1}{3^3}, \quad \ldots.$$

The perimeter at any stage, $p(x)$, is then a function which must satisfy the requirements:

$$1.\ \ p(1) \ = \ 1, \qquad\qquad 2.\ \ p\left(\frac{x}{3}\right) \ = \ \frac{4}{3}\,p(x).$$

Now

$$\frac{(x/3)}{(x/3)^D} \ = \ \frac{4}{3}\cdot\frac{x}{x^D} \quad \text{implies that} \quad \frac{3^D}{3} \ = \ \frac{4}{3}$$

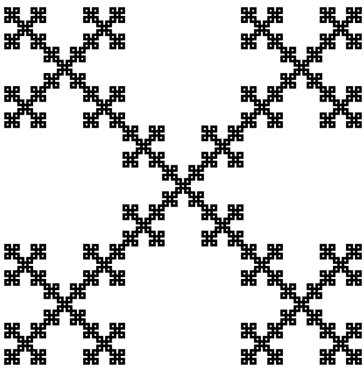
and so $D = \log_3 4$. This gives $(\log 4)/(\log 3)$ using the basic formula of logarithms:

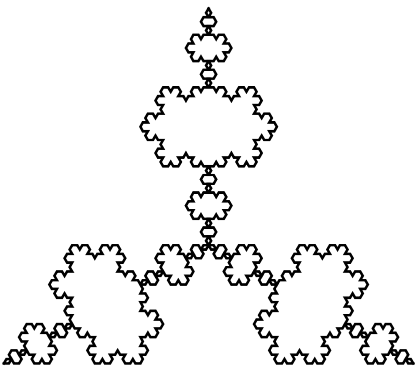
$$\frac{\log_a N}{\log_a M} \ = \ \log_M N.$$

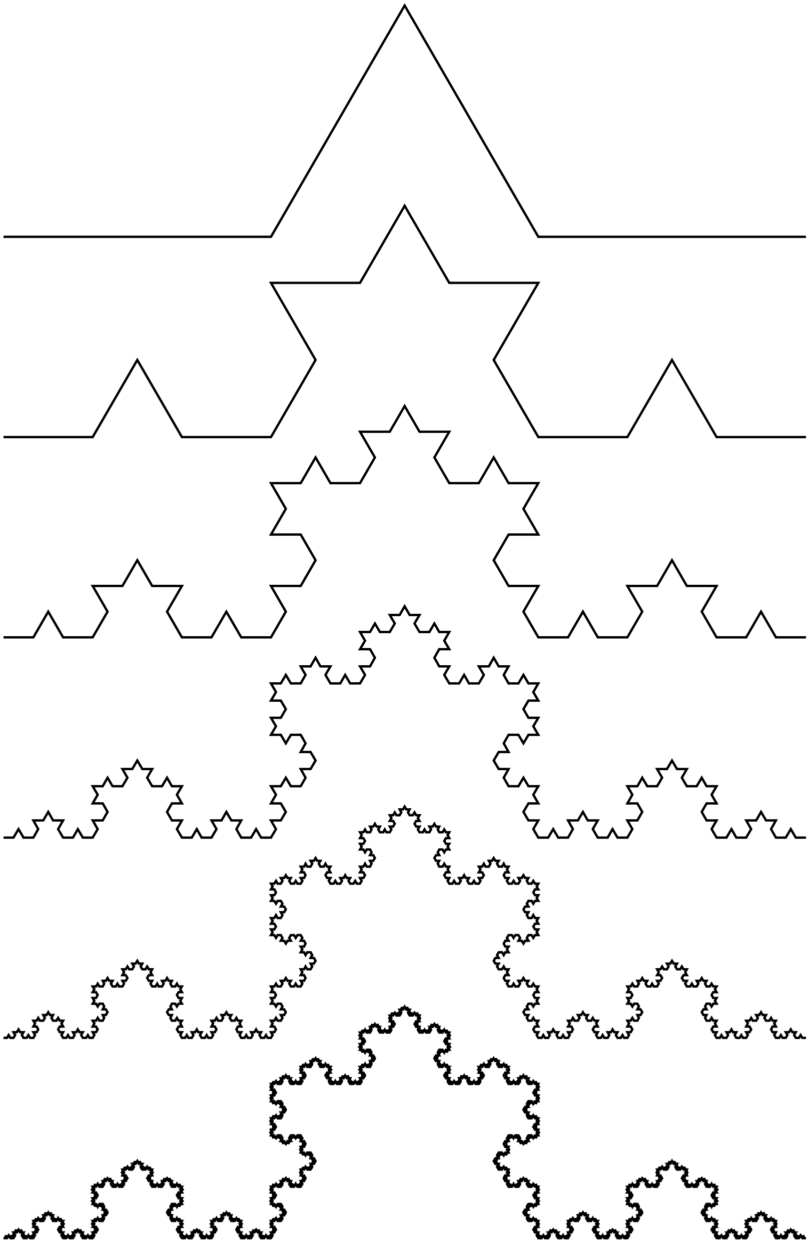
Now it transpires that this exponent D appears in the work of the forgotten physicist L. F. Richardson, who actually worked out lengths of frontiers and coastlines from known data using increasingly small scales. According to Mandelbrot, he found that if we approximate to a coastline by a broken line made up identical small segments of length s the total length increases with decreasing s according to Fs^{1-D}.

But Nature, most fortunately for most of us, does not seem to have taken lessons either from Euclid, Weierstrass or even Mandelbrot himself. Self-similar fractal shapes do seem in some cases to give a somewhat better picture of what actually goes on in nature than Euclidean geometry but you will look in vain around you for truly self-similar structures—can you see anything organic out of the window that is independent of change of scale? (The only case I know of is that of the *nautilus*, not found in our waters.) It is much to the credit of Mandelbrot that he recognizes this limitation of his own method and strives to remedy it by introducing chance—but not too much chance—into his computer simulations. Then and only then do we get fractal coastlines and cloud masses that really do look life-like.

The path to the fractal and beyond is a strange one. It commences in the twilight transfinite world of Georg Cantor (to which he eventually succumbed entirely), but ends up in river lengths and computer simulations of mountains that are so convincing they are extensively used today in films. At first the mathematical establishment did not take kindly to the monsters unleashed by analysis—'I recoil with fear and loathing', wrote Hermite, 'from that creature, a continuous curve without a derivative'—but when Mandelbrot showed the relevance of Cantor dusts, snowflake curves and all the rest of it to natural processes he was ostracized for taking too much of an interest in the real world.







Solution 179.5 – Subtract square root

Start with a large number, n. Replace n by $n - \sqrt{n}$. Repeat until you reach something less than 1. Approximately how many iterations are required? What about $n \to n - \sqrt{kn}$, where k is a given constant?

Jim James

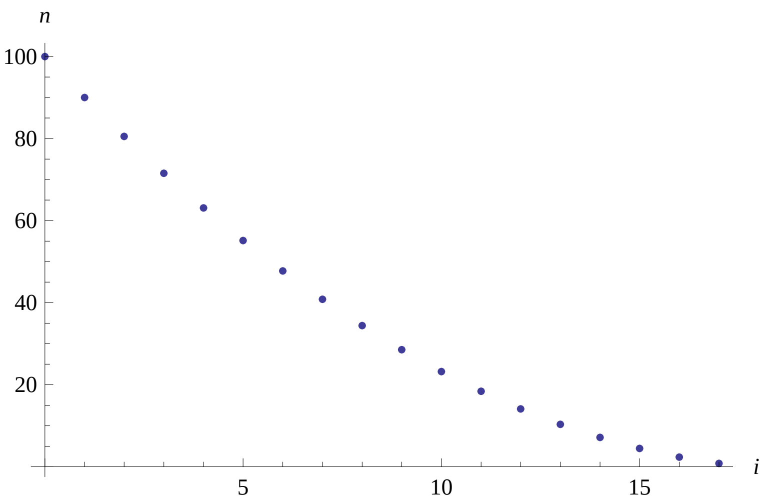
Figure 1 is a plot of n_i, the value of n following iteration i, for $n_0 = 100$. It is immediately apparent that the points lie on a fairly smooth curve and, as a first approximation, could be represented by a continuous real function. This would require treating i, temporarily, as a real number (and only for the purposes of deriving the approximating function.)

At any iteration point, i, the slope of the straight line connecting that point to the next, $i + 1$, is

$$\frac{n_i - n_{i+1}}{i + 1 - i} \;=\; n_i - n_{i+1} \;=\; -\sqrt{n_i}$$

(see Figure 2). This suggests considering the differentiable function mapping real i onto n having this slope over its whole domain of interest. The corresponding differential equation, $dn_i/di = -\sqrt{n_i}$, for i ranging from 0 to arbitrary real t, is easily solved, to give $t = 2(\sqrt{n_0} - \sqrt{n_t})$.

Figure 1

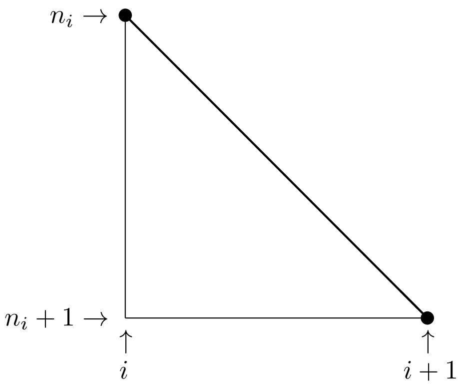


So for $n_t < 1$, as specified in the problem, we have

$$t \;>\; 2(\sqrt{n_0} - 1),$$

which implies that the number of iterations required for n to reach 'something less than 1' is the least integer greater than $2(\sqrt{n_0} - 1)$, approximately.

Figure 2



As in all such cases, we must check the approximation function against experimental data. This is done in Table 1, below, for n_0 in the range 10 to 10^{16}. Here, the 'actual' data have been computed using two different programs, both yielding the same result in all cases (tending to rule out possible errors introduced by their intrinsic square root functions).

The absolute error in using the estimation formula is seen to be quite small, slowly increasing as n_0 increases. This error could be further reduced. One way would be to add another term to the approximation function, giving

$$t \;=\; 2(\sqrt{n_0} - 1) - \log_{10}\sqrt{n_0},$$

say, to cover the range explored here. The relative error, however, declines rapidly as n_0 increases (and as the number of required iterations increases) so for many practical applications, and particularly as n_0 gets larger, the formula should be quite acceptable as originally derived.

A similar approximation may derived for $n \rightarrow n - \sqrt{kn}$. The minimum number of iterations required for n to reach less than 1 in this case is the least integer greater than $2(\sqrt{n_0} - 1)/\sqrt{k}$, so long as $0 < k < n_0$ (for $k \geq n_0$ one iteration will obviously suffice). Table 2 lists some results for k in the range 10^{-6} to 10^{10} ($n_0 = 10^{10}$ in all cases). Here the absolute error in using the approximation formula is similar to that of the base case, but the closeness of k to n_0 becomes a critical factor in the relative error (because the number of iterations required reduces as k increases).

Table 1 – Actual and estimated number of iterations for base case

n_0	Actual iterations	Estimated iterations	Absolute error	Relative error
10	4	5	1	0.25
10^2	17	18	1	0.058
10^3	60	62	2	0.033
10^4	196	198	2	0.010
10^5	628	631	3	0.0048
10^6	1995	1998	3	0.0015
10^7	6319	6323	4	0.00063
10^8	19994	19998	4	0.00020
10^9	63239	63244	5	0.000079
10^{10}	199993	199998	5	0.000025
10^{11}	632448	632454	6	0.0000095
10^{12}	1999992	1999998	6	0.0000030
10^{13}	6324546	6324554	8	0.0000013
10^{14}	19999990	19999998	8	0.00000040
10^{15}	63245543	63245551	8	0.00000013
10^{16}	199999990	199999998	8	0.000000040

Table 2 – Actual and estimated number of iterations for $n_0 = 10^{10}$ and variable k

n_0	Actual iterations	Estimated iterations	Absolute error	Relative error
10^{-6}	199997995	199998000	5	0.000000025
10^{-4}	19999795	19999800	5	0.0000025
10^{-2}	1999975	1999980	5	0.0000025
1	199993	199998	5	0.000025
10^2	19995	20000	5	0.00025
10^4	1996	2000	4	0.0020
10^6	197	200	3	0.015
10^8	18	20	2	0.11
10^{10}	1	2	1	1.00

Solution 177.6 – Factorial derivative

Prove that if n is a positive integer, $\left[\frac{d^n}{dx^n}(1-x^n)^{\frac{1}{n}}\right]_{x=0} = -(n-1)!$.

John Bull

Put $y = (1-x^n)^{1/n}$ and expand by the binomial theorem. This is permissible if $|x^n| < 1$, but this is no problem as we know that eventually we will be setting $x = 0$:

$$\begin{aligned} y &= 1 + \frac{1}{n}\frac{(-x^n)}{1!} + \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{(-x^n)^2}{2!} + \frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\frac{(-x^n)^3}{3!} + \ldots \\ &= 1 - \frac{x^n}{n} - \frac{(n-1)x^{2n}}{n^2\cdot 2!} - \frac{(2n-1)(n-1)x^{3n}}{n^3\cdot 3!} - \ldots. \end{aligned}$$

As this power series converges, we can differentiate term by term:

$$\frac{dy}{dx} = -x^{n-1} - \frac{(n-1)x^{2n-1}}{n\cdot 1!} - \frac{(2n-1)(n-1)x^{3n-1}}{n^2\cdot 2!} - \ldots. \qquad (1)$$

Now differentiate (1) a further $n-1$ times. Since $kn - 1 - (n-1) \geq 1$ for $k \geq 2$, all terms in the resulting series other than the first will have a positive power of x as a factor; hence they vanish when we set $x = 0$. Thus

$$\left[\frac{d^n y}{dx^n}\right]_{x=0} = \left[\frac{d^{n-1}}{dx^{n-1}}\left(-x^{n-1}\right)\right]_{x=0} = -(n-1)!.$$

Problem 181.1 – Find the centre

Tony Forbes

This was inspired by the discussion at the end of ‘Solution 178.2 – Construct another square’ by Dick Boardman (M500 **180** 15).

You have a ruler and a sheet of paper on which a circle has been drawn. The location of the centre is not known. Either

(i) devise a ruler-only construction to find the centre of the circle, or

(ii) prove that (i) is impossible.

If you opt for (i), your construction should use the ruler only for drawing straight lines in a manner that would have had Euclid’s approval. In particular, no using the ruler as a measuring device.

Solution 179.2 – Four tans

Show that $\tan 11^\circ = (\tan 19^\circ)(\tan 33^\circ)(\tan 41^\circ)$.

Miland Joshi

Put $\tan 11^\circ = a$, $\tan 19^\circ = b$, $\tan 33^\circ = c$, $\tan 41^\circ = d$. We will show that $a = bcd$ by expressing b, c, and d in terms of a and taking the product. We need to use the facts that

$$\tan 30^\circ = \frac{1}{\sqrt{3}} \quad \text{and} \quad \tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

(not too hard to prove!). Thus

$$\tan 30^\circ = \tan(11^\circ + 19^\circ) \Rightarrow \frac{1}{\sqrt{3}} = \frac{a+b}{1-ab} \Rightarrow b = \frac{1-\sqrt{3}a}{\sqrt{3}+a}.$$

Also

$$\tan 30^\circ = \tan(41^\circ - 11^\circ) \Rightarrow \frac{1}{\sqrt{3}} = \frac{d-a}{1+da} \Rightarrow d = \frac{1+\sqrt{3}a}{\sqrt{3}-a},$$

$$\tan 33^\circ = \tan(3\cdot 11^\circ) \Rightarrow c = \frac{3a-a^3}{1-3a^2}.$$

Hence

$$bcd = bd\cdot c = \frac{1-3a^2}{3-a^2}\cdot\frac{a(3-a^2)}{1-3a^2} = a.$$

Q.E.D.

John Bull

As we are highly numerate we instantly spot that $33 - 11 = 41 - 19 = 2\cdot 11$, that $19 + 41 = 60$, and that $11 + 33 = 2\cdot(2\cdot 11)$. Following these clues we rewrite the equation as:

$$\frac{\tan 11^\circ}{\tan 33^\circ} = \tan 19^\circ \cdot \tan 41^\circ.$$

$$\begin{aligned} \text{LHS} &= \frac{\sin 11^\circ}{\cos 11^\circ}\cdot\frac{\cos 33^\circ}{\sin 33^\circ} = \frac{1/2(\sin 44^\circ - \sin 22^\circ)}{1/2(\sin 44^\circ + \sin 22^\circ)} \\ &= \frac{2\sin 22^\circ\cdot\cos 22^\circ - \sin 22^\circ}{2\sin 22^\circ\cdot\cos 22^\circ + \sin 22^\circ} = \frac{2\cos 22^\circ - 1}{2\cos 22^\circ + 1}, \end{aligned}$$

$$\text{RHS} = \frac{\sin 19^\circ}{\cos 19^\circ}\cdot\frac{\sin 41^\circ}{\cos 41^\circ} = \frac{1/2(\cos 22^\circ - \cos 60^\circ)}{1/2(\cos 22^\circ + \cos 60^\circ)} = \frac{2\cos 22^\circ - 1}{2\cos 22^\circ + 1} = \text{LHS}.$$

Q.E.D.

This suggests a generalization. Suppose $\tan a = \tan b\cdot\tan c\cdot\tan d$. Then we require that $b + d = 60^\circ$ and $c - a = d - b = 2a$. A little algebra gives $b = 30^\circ - a$, $c = 3a$, $d = 30^\circ + a$. So

$$\tan a = \tan(30^\circ - a)\cdot\tan 3a\cdot\tan(30^\circ + a).$$

Thus by substituting a from 1° to 89°, we can find a whole set of example equations.

A similar analysis was given by **Sue Bromley**.

Keith Drever

Problem 179.2 (Four tans) must be one of the easiest problems ever published in M500. We have $\tan 19^\circ = 0.344327613$, $\tan 33^\circ = 0.649407593$, and $\tan 41^\circ = 0.869286737$. Hence

$$(\tan 19^\circ)(\tan 33^\circ)(\tan 41^\circ) = 0.194380309;$$

$\arctan 0.194380309 = 11^\circ$, as expected.

ADF writes—We are of the opinion that this does not necessarily solve the problem. We cannot immediately convince ourselves that something = 11 to nine decimal places implies something = 11 exactly.

However, it is an approach that is worth further discussion. Let

$$\Theta = \tan 11^\circ - (\tan 19^\circ)(\tan 33^\circ)(\tan 41^\circ). \qquad (1)$$

Performing the calculations to 10010 decimal places, say, we obtain

$$-10^{-10000} < \Theta < 10^{-10000}.$$

Noting that the formula for Θ contains only eight decimal digits we invoke Einstein's principle, *God is subtle but He is not malicious*, to argue that there cannot be enough 'information' in (1) for Θ to have any value between -10^{-10000} and 10^{-10000} other than exactly zero.

The editor of this magazine would be very interested if it is possible to make this idea rigorous.

Solution 179.3 – Nine switches

> You are outside a room. You have nine switches. Inside are nine light-bulbs. How many trips must you make into the room to allocate bulbs to switches?

Barbara Lee

Four trips are needed. Start with all switches OFF. Label them 1, 2, ..., 9. Switch 1, 2 and 3 ON. Enter room. Label the three relevant bulbs A, B and C in any order. Exit room.

Switch 1 OFF. Switch 4, 5 and 6 ON. Enter room. Label next three relevant bulbs D, E and F in any order. Identify bulb for switch 1. Exit room.

Switch 2 and 4 OFF. Switch 7 and 8 ON. Enter room. Label next two relevant bulbs G and H in any order. Identify bulbs for switches 2, 3 and 4. Exit room.

Switch 5 and 7 OFF. Enter room. Identify bulbs for switches 5, 6, 7 and 8.

Switch 9 connects to remaining unlabelled bulb. Exit room. Make tea.

Also solved (in a similar manner) by **Miland Joshi** and **John Hudson**.

The Editors

Frankly, we were suspicious from the start about the appearance of a power of three in the statement of the problem. So we reasoned that it was necessary to contrive a three-way split which allows us to map bulbs to switches with only two trips into the room.

Start with all switches OFF. Switch 1, 2, 3, 4, 5, 6 ON. Wait 30 seconds. Switch 4, 5, 6 OFF. Enter room. Map lit bulbs $\to \{1, 2, 3\}$, unlit and warm $\to \{4, 5, 6\}$, unlit and cold $\to \{7, 8, 9\}$.

Switch 1, 2, 3 OFF. Wait 24 hours, say, for the bulbs to cool down. Switch 1, 2, 4, 5, 7, 8 ON. Wait 30 seconds. Switch 2, 5, 8 OFF. Enter room. You now have enough information to complete the task.

Problem 181.2 – Six secs

Show that

$$\sec\frac{\pi}{7}\sec\frac{2\pi}{7}\sec\frac{3\pi}{7}\sec\frac{4\pi}{7}\sec\frac{5\pi}{7}\sec\frac{6\pi}{7} \;=\; -64$$

and

$$\sec\frac{\pi}{7}+\sec\frac{2\pi}{7}+\sec\frac{3\pi}{7}+\sec\frac{4\pi}{7}+\sec\frac{5\pi}{7}+\sec\frac{6\pi}{7} \;=\; 0.$$

Solution 178.5 – Reward a friend

> In Chris Tarrant's popular television programme, *Who Wants to be a Millionaire?*, what is a fair reward for a person who helps the contestant when the 'phone a friend' lifeline is used?

Chris Pile

Merely some observations. The friend should not receive more than the contestant; i.e. if the friend gives the correct answer and the contestant subsequently falls back to a lower prize level there should be no payment to the friend.

If the friend correctly answers the question and the contestant accepts and wins the prize, the friend should receive 75% of the increment. (E.g. for a correct answer to the £8000 question, the contestant's winnings increase from £4000 to £8000, the 'expected' winnings being 25% of £4000 by pure guess. The friend should receive £3000.) If the contestant subsequently continues to a higher prize, the friend would receive a diminishing proportion of the increments. This seems to be a moral problem rather than a mathematical one!

ADF

Suggested answer for one typical scenario: Let $1 < a < b \leq 15$. Suppose the contestant, C, assisted by the studio audience but not the 50–50 option, can answer questions $1, \ldots, a-1, a+1, \ldots, b$ with 100% confidence and that C's optimum strategy after question b is to stop. Suppose also that the friend is 100% certain of the correct answer to question a but all four possibilities are equally likely as far as C is concerned. Then the friend's reward should be

$$\frac{1}{2}\left(R(b) - \max\left\{R(a-1),\ \frac{R(b)+W(a)}{2}\right\}\right).$$

The friend's help guarantees the correct answer to question a. Unaided, C's expected win is the greater of $\frac{1}{2}(R(b)+W(a))$ (using the 50–50 option) and $R(a-1)$. By some vague principle of fair play we split the difference between C and the friend.

For example, with $a = 11$ and $b = 13$ the reward works out at £54500 out of £250000. The functions R and W are defined by

$$R(n) = \begin{cases} £100 \cdot n & \text{if } 0 \leq n \leq 3 \\ £500 \cdot 2^{n-4} & \text{if } 4 \leq n \leq 11 \\ £125000 \cdot 2^{n-12} & \text{if } 12 \leq n \leq 15, \end{cases} \qquad W(n) = \begin{cases} R(0) & \text{if } 1 \leq n \leq 5 \\ R(5) & \text{if } 6 \leq n \leq 10 \\ R(10) & \text{if } 11 \leq n \leq 15. \end{cases}$$

There has to be an easier way

A novel solution to the problem of launched objects and calculating the height reached and the time taken.

Martin Wright

Sometimes when one is studying a course it is necessary to read other books and materials to help find methods to carry out calculations. Sometimes different methods are presented to you. This article is the result of that form of experimentation—'mucking about'—and shows it is possible for novices to mathematics to discover new methods of calculation. The reason why I have decided to show the new technique at the end of the article is to give the reader some idea of how this discovery came about. It is not groundbreaking, it will not solve the mystery of life, the universe and everything, but does show that sometimes, when a relationship between results is found, it is possible to devise new equations which present a more straightforward route to required results. Here is a typical 'rocket' problem:

> A rocket is launched vertically with an acceleration of $a = 8\,\text{ms}^{-2}$ for a period of $T = 10$ seconds. After that it is under the influence of gravity alone, an upward acceleration of $g = -10\,\text{ms}^{-2}$. Assume all this takes place in a frictionless atmosphere.

One is then asked to find its velocity, the height the rocket attains when the engines cut out, how long will the rocket take to attain its maximum height, what that height will be, how long will it take to fall back to earth, and how long from launch to landing.

The first two elements are its velocity, $v = 8 \cdot 10$ as $v = aT$, and its displacement at time T, $x = \int_0^T at\,dt = \frac{1}{2}aT^2 = 400$.

When the engines stop the equation changes to one where it is necessary to find the velocity under the influence of gravity ($v = -gt + c$), and from there find out how long it will be before the rocket reaches its apogee, but to do this we need to find the value of c. When $v = 80$ and $t = 10$, $80 = -10 \cdot 10 + c$, $c = 180$.

This gives the expression $v = -10t + 180$. The time when the object reached apogee can then by found by putting $v = 0$:

$$0 = -10t + 180, \qquad t = 18.$$

The displacement from the ground to the apogee can then by found by integrating $-10t+180$, which results in $-5t^2+180t+c$. As the initial value for x is known to be 400,

$$400 = -5 \cdot 10^2 + 180 \cdot 10 + c, \qquad c = -900,$$

$x = -5t^2+180t-900$ and $t = 18$ from which the maximum height is derived: $x = 720$.

The time it will take to reach the ground after take-off can be found by solving $-5t^2 + 180t - 900 = 0$, using the standard method for quadratic equations,

$$t = \frac{180 \pm 2\sqrt{180^2 - 4 \cdot 5 \cdot 900}}{2 \cdot 5} = 30 \text{ or } 6.$$

When one looks at the range and complexity of the equations that are used to derive the required results one can see that there are several points where error can easily creep in, and one also quickly loses sight of the original variables as they are worked and re-worked. There has to be a quicker and better way to find answers to these questions and reduce the possibility of error.

I decided to see if a relationship existed between the height when the engines stopped and the apogee. At the acceleration rate of $8\,\mathrm{ms}^{-2}$, I found that the factor was 1.8. When I varied the acceleration to $7\,\mathrm{ms}^{-2}$, it indicated the factor was 1.7.

I also varied the length of time that the acceleration rate was used. Again I found the relationship remained the same. From this I deduced that the relationship was dependent upon just two factors, the rate of acceleration and the value for g. I found that the height attained was directly related to the initial rate of acceleration divided by the value for gravity, and the apogee was always a multiple of $1 + a/g$. From these results I was able to formulate the equation

$$Z = \frac{1}{2}aT^2\left(1 + \frac{a}{g}\right).$$

This method removes several stages of calculation and keeps all the initial variables together. This also led to another method for finding the total time between take-off and landing. We take the first value for t, then add to it the time it would take for an object to fall from the height that is the difference between the apogee and the height when the engines stop, then add to that total the time it would take for an object to fall from the apogee to the ground:

$$T_a = T + \sqrt{\frac{2Z - aT^2}{g}} + \sqrt{\frac{2Z}{g}}.$$

This means, instead of using a large number of equations to answer the questions, it is now possible to use just three equations to find out most of the required elements. A further advantage of the above techniques is that it is also possible to carry out such calculations based on alternative gravitational strengths, such as other planets or moons.

Although I have called this approach 'novel', I would be interested to know if this connection has been made before—and if so where.

Solution 179.6 – Root 11 again

Prove that if p is an odd prime, the integer part of $(\sqrt{11}+3)^p - 2 \cdot 3^p$ is divisible by $66p$.

Sue Bromley

Since $2 \cdot 3^p$ is an integer, we know from the solution to the original $\sqrt{11}$ problem [M500 **176** 6] that the integer part of $(\sqrt{11}+3)^p - 2 \cdot 3^p$ must be

$$\begin{aligned}
&(\sqrt{11}+3)^p - (\sqrt{11}-3)^p - 2 \cdot 3^p \\
&\quad = 2\big({}^pC_1(\sqrt{11})^{p-1} \cdot 3 + {}^pC_3(\sqrt{11})^{p-3} \cdot 3^3 \\
&\qquad + \cdots + {}^pC_{p-2}(\sqrt{11})^2 \cdot 3^{p-2}6\big) + 2 \cdot {}^pC_p \cdot 3^p - 2 \cdot 3^p \\
&\quad = 2 \cdot 3 \cdot 11\big({}^pC_1(\sqrt{11})^{p-3} + {}^pC_3(\sqrt{11})^{p-5} \cdot 3^2 + \cdots + {}^pC_{p-2} \cdot 3^{p-3}6\big), \\
&\qquad\qquad \text{since } {}^pC_p = 1 \text{ and } p \geq 3, \\
&\quad = 66 \sum_{n=0}^{2n+1=p-2} {}^pC_{2n+1}(\sqrt{11})^{p-3-2n} \cdot 3^{2n}
\end{aligned}$$

(note that $p-3-2n$ is even). Generally, if p is prime then

$${}^pC_r = \frac{p!}{(r!(p-r)!)}$$

must be divisible by p, since both r and $p-r$ are less than p and cannot divide p. It follows that the integer part of $(\sqrt{11}+3)^p - 2 \cdot 3^p$ is divisible by $66p$.

Problem 181.3 – Six-sided pencil

Tony Forbes

What is the function associated with the familiar curtain-like graph that appears at the border of the exposed wood when you sharpen a six-sided pencil?

Problem 181.4 – Four points

Choose two points inside a given circle and draw the line segment joining them. Then randomly select another two points inside the same circle and draw the line segment joining these two points.

What is the probability that the two line segments intersect?

Fusion

A new physics society for OU students

Paul Ruffle

Fusion—The Open University Physics Society—has been formed by five students who met as a result of an appeal on the Open University FirstClass on-line conferencing system, and with help from the Institute of Physics, to which it is affiliated.

The Society is for all Open University students (including postgraduate and research), academic staff and tutors, plus anyone else who is interested in the world of physics and who would like to contribute to the aims of the Society.

There will be a quarterly Newsletter covering a variety of topics including feature articles, faculty news, course reviews and details of Fusion events. Members will also be able to air their views on various courses they have taken and any other aspects of physics that take their interest.

Members will also be able to get in touch with one another via a contact list and participate in the various Fusion events that will be organized.

Fusion will not duplicate the activities of the Institute of Physics, which runs a full programme of topical lectures, but will complement them, particularly in the summer months when other academic institutions are dormant. This will include visits to such places as the Science Museum in London, the Rutherford Appleton Laboratory in Oxfordshire, and CERN, the particle physics research centre at Geneva.

Fusion also aims to promote communication between the student body and the OU Physics Department at Milton Keynes. This will involve staff profiles, visits to the campus, lectures and seminars on current research programmes, and plans for the future. We hope this will allow students to influence the Physics Department in future developments in the curriculum.

The Fusion web site can be found at www.oufusion.org.uk and features information on how to join Fusion plus news, stories, reviews, events, feedback, constitution and committee member profiles, plus numerous physics related links. You can also look out for the Fusion desk at the M500 Weekend in Aston.

Problem 181.5 – Five digits

JRH

Find all solutions of

$$10000a + 1000b + 100c + 10d + e \;=\; f(10000e + 1000d + 100c + 10b + a)$$

in non-negative integers a, b, c, d, e, f subject to the usual constraints of decimal arithmetic, $0 \le b, c, d \le 9$ and $1 \le a, e \le 9$.

Letters to the Editors

Flies

Dear Jeremy,

I thought we had dealt with this question last time, but in fact it's different. If the first car is doing 60 m.p.h., but the fly can only do 50 m.p.h., surely the fly gets left behind. The smash will be after an hour, by which time the fly will have gone 50 miles and the smash will be 10 miles ahead of it. So why does the fly get squashed?

Or have I missed something somewhere?

Colin Davies

I am not a gifted mathematician, nor am I particularly intelligent, and I subscribe to M500 in the hope that some of the mathematical expertise might rub off on me.

As we're told that the fly starts at the front bumper of the first car, then it is impossible to move forward from that position in the air, since the car moves 10 m.p.h. faster than the fly. In that case, the fly would be permanently located against the bumper at of the first car, until the second car is reached, and the crash takes place. The fly would not get airborne at all. As David Singmaster points out, the answer in that instance is zero.

To take this a step further, if the fly started on the bumper of the second car, then it would be able to fly ahead of the 40 m.p.h. vehicle. In my estimation, when it met the vehicle coming from the other direction at 60 m.p.h., the oncoming car would have travelled 54.54 miles, and fly would have travelled 45.45 miles. When it turned to fly back in the opposite direction, it would be unable to do so for the reason that the oncoming car, travelled 10 m.p.h. faster than the fly. In that situation, the fly would have travelled a total distance of 45.45 miles as it would be resting on the bumper of the 60 m.p.h. car until the crash took place.

Jack Gibson

Courses

Dear Tony,

I HAVE NO MOUTH, AND I MUST SCREAM.

The above title is from science fiction, but in any case it has no origins in maths. However, from the content of this letter, I'm sure you will see the connection.

With another issue of the M500 magazine, I have noticed in MOUTHS that my list of courses studied is getting shorter. Surely I can't be retroactively unstudying courses? What happens when I have no more courses to unstudy?

To find out I rang Sue Barrass, who tells me that it is policy not to include previous course codes because the newer students would not understand them.

But wait a minute .... Students doing current study have a tutor, who is geared to the needs of particular courses and gets paid for the privilege. MOUTHS members act on a voluntary basis within their chosen speciality for the benefit of everyone. The constitution of M500 is a forum of like-minded people with a common interest in maths. Therefore, M500 is not here solely to look after the needs of new students. In any case, the course numbers provide a reasonable guide to the individual's specialist area.

So, my opinion is 'let the new students learn the meaning of the square brackets around those funny course numbers'. What do other members think?

Regards,

Ken Greatrix

Milliard

The terms 'billion', 'trillion', etc. have always been ambiguous (M500 **179** 22). The British idea of a million squared, cubed, to the fourth, etc., which was presumably based on the sequence mono, bi, tri, quad, etc., always seemed the most logical to me. However, the American system had the advantage of making big numbers look even bigger, and Americans like that sort of thing.

So years ago, one had a pretty good idea what a billion meant by deciding whether one was reading a British text (*The Times*, say) or an American text (*Life*, say). Unfortunately, this did not always work. The American publisher of Time Life Books used to publish a series of popular science books from a base in London for the UK market, and those books used the word 'billion' to mean ten to the twelfth. That helped to confuse the issue, but what would a publisher in a country where English was not normally used, mean by the word 'billion' when writing in English for the UK market?

When I was in the timber trade during the 1950s and 60s, the office regularly received a journal called *Finnish Paper and Timber*. This was published in Helsinki, but written in English, and distributed in the UK.

The publishers used the word 'billion' without explanation, so I wrote them a nice letter explaining that in the UK where their main readership was, a billion was one followed by twelve noughts, and was that what they really meant? They wrote back saying that by 'billion' they meant one followed by nine noughts. They said that they used the word 'billion' in preference to the word 'milliard' because, while 'billion' was familiar to all English speakers, the word 'milliard' was not. They had decided that a familiar word, although ambiguous, was preferable to a precise word that was less well known.

Colin Davies

Recurrence relations

Dear Tony,

Re: M500 **179**, 'Recurrence relations' by Robin Marks.

The trouble with computer programs is that they are good at processing and displaying data but not very good at analysis. Here we have a case in point. Robin actually solved the order-2 recurrence relation on page 2, but I think he missed the point of his argument. If $U_n = Ar_1^n + Br_2^n$ (for $r_1 = 1$ and $r_2 < 1$), then $U_n \to A$ as $n \to \infty$ (M203, null sequences).

This theme carries through with the order-3, -4, -5 and -6 recurrence relations in that one of the roots is 1 and all the others have a modulus which is less than 1.

Hint: this could be demonstrated in graph form if you take the x-axis from about -4 and reduce the y-axis to about ± 5. You will then see the tendency of long term conditions compared to the initial conditions (i.e. as n approaches infinity).

So, if it can be shown that this follows for all such recurrence relations, then the problem is solved. I will now attempt this:

In the auxiliary equation

$$kx^k = x^{k-1} + x^{k-2} + \cdots + x + 1$$

an obvious solution is $x = 1$ (since there are k terms on the RHS). If we divide by x^k, we get

$$k = \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^k}$$

and thus if $|x| \geq 1$ then we have no solution unless $x = 1$.

So now I show that the case where $x = 1$ is not a repeated root. Rearrange the auxiliary equation as a polynomial

$$kx^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = 0$$

and then divide by $x - 1$:

$$kx^{k-1} + (k-1)x^{k-2} + (k-2)x^{k-3} + \ldots 2x + 1 = 0.$$

If we now set $x = 1$, we get the sum of the first k numbers on the LHS and zero on the RHS—thus there are no repeated roots of $x = 1$.

The important point of this result is that if $|r| < 1$ then $r^n \to 0$ as $n \to \infty$ and thus $U_n \to A$ for all such recurrence relations.

Regards,

Ken Greatrix

Ten blocks

Dear Tony,

This puzzle [Problem 178.6] is fully described in Martin Gardner's *Sixth Book of Mathematical Games from Scientific American*. From a slightly different position a minimum 81-move solution is given. I made the puzzle many years ago after seeing it described in this book, and I was able to memorize the moves quite well after several attempts. I have seen it on sale recently in various guises (e.g. football, golf, fishing, etc.) and I consider it to be a classic among sliding block puzzles.

Chris Pile

Apes

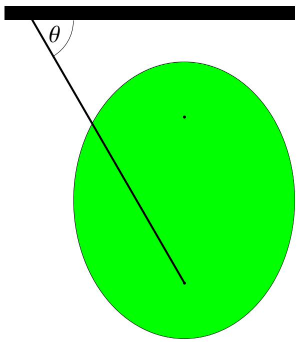
Dear Editors,

Re: Changing APE to MAN (M500 **179**, p. 25). APE, APT, EPT, EAT, MAT, MAN (Ept: Adroit, competent; appropriate, effective—*OED*).

John Hudson

[Alternatively: ..., OPT, OAT, ... Now try CHIMP to WOMAN. Also we would still like to see a proof of David Singmaster's assertion: CIRCLE to SQUARE is impossible.—**ADF**]

Problem 181.6 – Ellipse



An ellipse of eccentricity η is suspended by a string of length β attached at one end to a focus and at the other to a fixed point above the ellipse. The system is allowed to evolve under gravity from an initial position where the string is at angle θ to the horizontal and the major axis of the ellipse is vertical with the unattached focus uppermost.

Find the locus of the focus in terms of η, β and θ.

Problem 181.7 – Five cots

Prove that $\quad \cot\frac{\pi}{11}\cot\frac{2\pi}{11}\cot\frac{3\pi}{11}\cot\frac{4\pi}{11}\cot\frac{5\pi}{11} = \frac{1}{\sqrt{11}}.$

Hats

Eddie Kent

In *The Times* on 16 April there was a small piece on what is apparently the latest mathematical problem to obsess chat rooms around the globe. The puzzle is stated in what I would consider the worst-written prose I have seen in a very long time.

> Three people enter a room, and each tosses a coin. If it's heads, the person dons a blue hat; if tails, then a red hat. The coin tosses are independent of each other [Honestly—they are independent]; each person can see the other two players' hats but not her own [Why would she want to see her own?—she put it on].
>
> The trio get to share an imaginary [Not much use, then] jackpot of £2 million [How would they do that? Even if real it still won't divide into three] if, and only if, at least one player guesses the colour of her hat and none guesses incorrectly (a player can pass). The three must not communicate in the room, although they are allowed to develop strategy before they play.

It is possible to sort a problem out of the above. In which case, what would the best strategy be? You can look at

www.nytimes.com/2001/4/10/science/10MATH.html.

(I tried, but got nowhere. You are not allowed into the hallowed *New York Times* without registering and swearing allegiance to Old Glory so I didn't bother. Life is too short.)

JRH—The *NYT* page no longer exists. They delete them after a week, generally. I found these two below, which give a better statement of the problem, and some analysis.

http://www.comm.toronto.edu/ yymao/hatColor.html

http://www.ics.uci.edu/ ebert/coloredHatsSolution.html

ADF—If you are unhappy about the independence of the coin tossing mechanism used to determine the initial conditions of the problem, there is a simple alternative. Why not send the players to that party where each guest leaves wearing the wrong hat? See 'Hats' by Nick Pollock, M500 **178**, 22–24.

What did the bra say to the hat? 'You go on ahead while I give these two a lift.'

Twenty-five years ago

From M500 36

Rosemary Bailey—If you really want to annoy a statistician with a question about an unexplained often-used term, ask him/her what a 'degree of freedom' is.

Peter Weir—M351ers, currently suffering from linear programming, may be interested by some figures lifted from a 1970 Univac Users Association conference report. The solution method used is basically the same as that described in Unit 6 of M351, the product form of the simplex method, with the addition of some unexplained method of introducing the *two* most profitable variables into the basis at each iteration. The source of the problems was not given but was probably operations research for Sheel France.

Problem number	rows	columns	coefficients	time (minutes)	cost
1	1854	3708	16010	59	£590
2	2210	5500	22609	38	£380
3	1277	2911	14628	42	£420
4	2315	4591	18887	58	£580
5	3135	2738	31727	118	£1180

Imagine having to rerun the last one because one of the 31727 coefficients was wrong!

Marion Stubbs—The problem below is taken from *Games and Puzzles*, May 1974—before our member David Wells became puzzle editor. The solution is given quite simply and in one line as the most famous number you are likely to think of, perhaps. Being totally thick the only way I can devise to do it is by calculator. ...

Problem: Calculate the value of the expression shown here, not using pen, pencil or paper, and doing it within a minute.

$$\sqrt[196]{10^{59} \cdot \left(\frac{1025}{1024}\right)^5 \cdot \left(\frac{6560}{6561}\right)^3 \cdot \left(\frac{9801}{9800}\right)^4 \cdot \left(\frac{15624}{15625}\right)^8 \cdot \left(\frac{1048576}{1048575}\right)^8}$$

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