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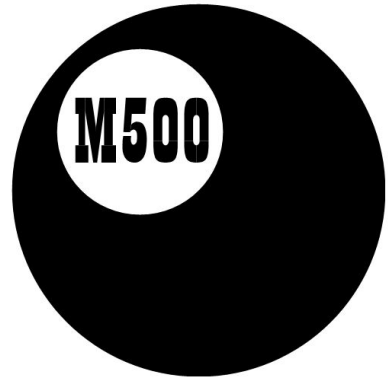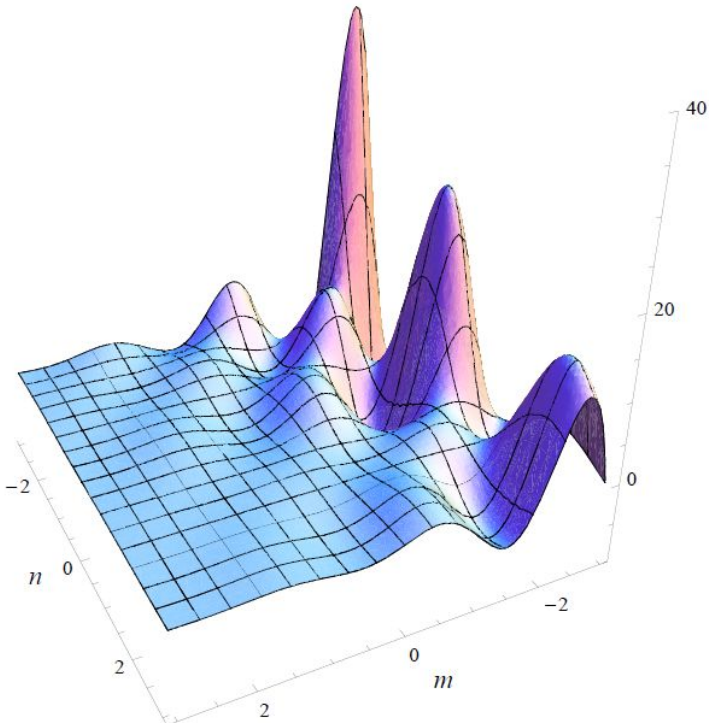
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The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

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The order 2×3 recurrence relation

Robin Marks

Consider the simultaneous order 3 recurrence relations

$$U(m,n) = pU(m-1,n) + qU(m-2,n) + sU(m-3,n), \qquad (1)$$
$$U(m,n) = tU(m,n-1) + vU(m,n-2) + wU(m,n-3). \qquad (2)$$

Try $U(m,n) = x^m y^n$ as a solution. From (1),

$$x^m y^n = px^{m-1}y^n + qx^{m-2}y^n + sx^{m-3}y^n.$$

Dividing by $x^{m-3}y^n$ gives $x^3 = px^2 + qx + s$. That is,

$$R = x^3 - px^2 - qx - s = 0. \qquad (3)$$

Similarly, from (2),

$$R = y^3 - ty^2 - vy - w = 0. \qquad (4)$$

Let the roots of (3) be x_1, x_2, x_3, and let the roots of (4) be y_1, y_2, y_3. We assume p, q are real and we now choose x_1 to be real. Thus $s = x_1^3 - px_1^2 - qx_1$. Separating out the factor $x - x_1$ in the left-hand side of (3) gives

$$R = (x - x_1)(x^2 + (x_1 - p)x + (x_1^2 - px_1 - q)) = 0. \qquad (5)$$

Solving the quadratic in (5) we get

$$x_2, x_3 = \frac{p - x_1}{2} \pm \frac{1}{2}\sqrt{(x_1 - p)^2 - 4(x_1^2 - px_1 - q)}.$$

Similarly, choosing y_1 be real and assuming t and v are real we have, from (4), $w = y_1^3 - ty_1^2 - vy_1$ and

$$y_2, y_3 = \frac{t - y_1}{2} \pm \frac{1}{2}\sqrt{(y_1 - t)^2 - 4(y_1^2 - ty_1 - v)}.$$

The solutions are $U(m,n) = x_i^m y_j^n$, $i = 1,2,3$, $j = 1,2,3$, which can be combined in any linear combination. We therefore have the general solution:

$$U(m,n) = \sum_{i=1}^{3}\sum_{j=1}^{3} k_{i,j} x_i^m y_j^n,$$

where the $k_{i,j}$ are complex constants.

If x_1 and y_1 are real, then the other pairs of roots (x_2, x_3) and (y_2, y_3) are either complex conjugate pairs or real pairs. We will choose to look at the case where (x_2, x_3) and (y_2, y_3) are both complex conjugate pairs.

To simplify this, let us consider first the solution in the m direction with two complex roots, then the solution in the n direction with two complex roots. The general solution in the m direction is $ax_1^m + bx_2^m + cx_3^m$; x_1 real; x_2, x_3 complex conjugates.

Consider $bx_2^m + cx_3^m$; b,c conjugate complex constants; let corresponding polar co-ordinates be $x_2 : (r_1, \theta_1)$, $x_3 : (r_1, -\theta_1)$. Then

$$
\begin{aligned}
bx_2^m + cx_3^m &= br_1^m(\cos\theta_1 + i\sin\theta_1)^m + cr_1^m(\cos-\theta_1 + i\sin-\theta_1)^m \\
&= br_1^m(\cos m\theta_1 + i\sin m\theta_1) + cr_1^m(\cos-m\theta_1 + i\sin-m\theta_1) \\
&\qquad\qquad\text{(by De Moivre's theorem)} \\
&= dr_1^m\cos m\theta_1 + er_1^m\sin m\theta_1,
\end{aligned}
$$

where $d = b + c$, $e = (b - c)i$; d, e will both be real if and only if b, c are conjugate complex constants. Hence the general real solution in the m direction is

$$
ax_1^m + dr_1^m\cos m\theta_1 + er_1^m\sin m\theta_1,
$$

where x_1 is real and a, d, e are real constants. Similarly, the general solution in the n direction is

$$
fy_1^n + gr_2^n\cos n\theta_2 + hr_2^n\sin n\theta_2,
$$

where y_1 is real and f, g, h are real constants.

Now multiplying the above expressions for the recurrence relations in the m and n directions gives

$$
\begin{aligned}
U(m, n) &= (ax_1^m + dr_1^m\cos m\theta_1 + er_1^m\sin m\theta_1) \\
&\qquad \cdot (fy_1^n + gr_2^n\cos n\theta_2 + hr_2^n\sin n\theta_2) \\
&= Ax_1^m y_1^n + Bx_1^m r_2^n\cos n\theta_2 + Cx_1^m r_2^n\sin n\theta_2 \\
&\quad + Dy_1^n r_1^m\cos m\theta_1 + Ey_1^n r_1^m\sin m\theta_1 \\
&\quad + Fr_1^m(\cos m\theta_1)r_2^n(\cos n\theta_2) \\
&\quad + Gr_1^m(\cos m\theta_1)r_2^n(\sin n\theta_2) \\
&\quad + Hr_1^m(\sin m\theta_1)r_2^n(\cos n\theta_2) \\
&\quad + Ir_1^m(\sin m\theta_1)r_2^n(\sin n\theta_2),
\end{aligned}
$$

where A, B, C, D, E, F, G, H, I are real constants. Hence, for example

$$
U(0,0) = A + B + D + F.
$$

Choosing nine different pairs of integers $(m_i,\ n_i)$, $i = 1, 2, \ldots, 9$, gives nine equations which can be solved for $A, B, \ldots, I$. Writing this as a matrix equation, we have $\mathbf{UA} = \mathbf{V}$, where $\mathbf{U}$ is the 9×9 matrix

$$
\begin{bmatrix}
x_1^{m_1} y_1^{n_1} & x_1^{m_1} r_2^{n_1} \cos n_1\theta_2 & \ldots & r_1^{m_1}(\sin m_1\theta_1)r_2^{n_1}(\sin n_1\theta_2) \\
x_1^{m_2} y_1^{n_2} & x_1^{m_2} r_2^{n_2} \cos n_2\theta_2 & \ldots & r_1^{m_2}(\sin m_2\theta_1)r_2^{n_2}(\sin n_2\theta_2) \\
\ldots & \ldots & \ldots & \ldots \\
x_1^{m_9} y_1^{n_9} & x_1^{m_9} r_2^{n_9} \cos n_9\theta_2 & \ldots & r_1^{m_9}(\sin m_9\theta_1)r_2^{n_9}(\sin n_9\theta_2)
\end{bmatrix},
$$

$\mathbf{A}$ is a column vector with elements $(A, B, \ldots, I)$ and V is a column vector of nine real numbers, the chosen initial values for the recurrence relation for each pair of integers $(m_i,\ n_i)$.

Solving for $\mathbf{A}$, $\mathbf{A} = \mathbf{U}^{-1}\mathbf{V}$, gives the values for $A, B, \ldots, I$ from which we can calculate values of the continuous function $U(m, n)$ for all real values of m and n.
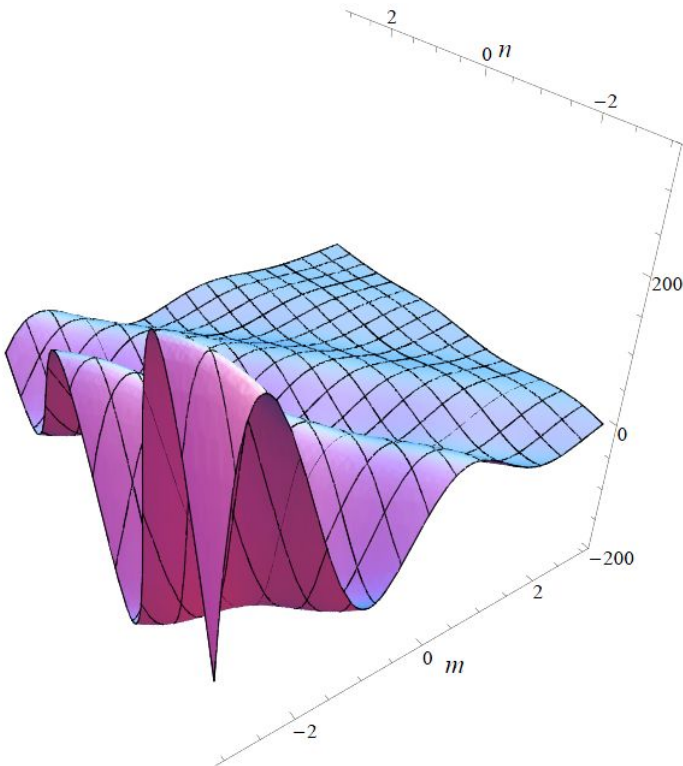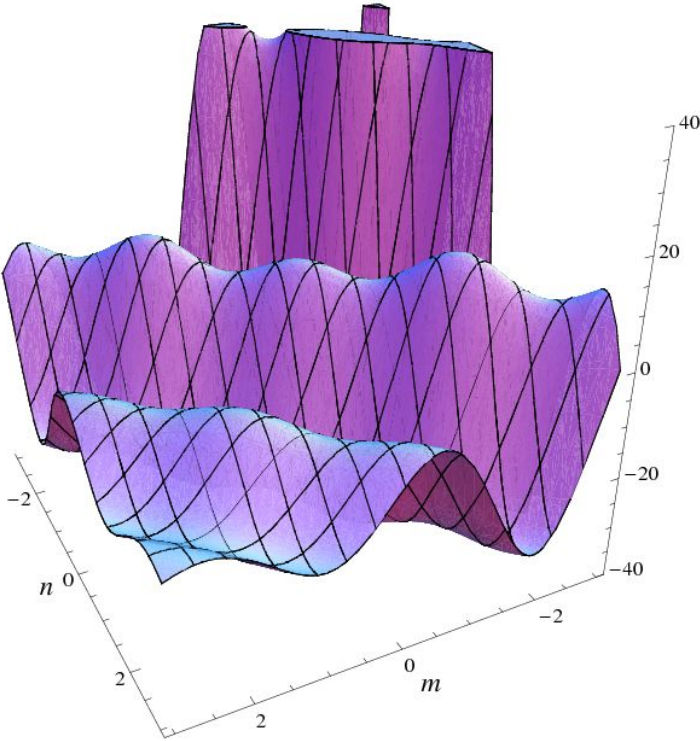
Here are some examples.

(i) Previous page

$x_1 = y_1 = 1$, $p = 0.4$, $q = 0.3$, $s = 0.3$, $t = 0.2$, $v = 0.25$, $w = 0.55$,

$U(4, -4) = 0, U(3, -3) = 0, U(2, -2) = 0, U(1, -1) = 0, U(0, 0) = 1,$

$U(-1, 1) = 0, U(-2, 2) = 0, U(-3, 3) = 0, U(-4, 4) = 0.$

(ii) This page

$x_1 = y_1 = 1$, $p = 0.3$, $q = 0.2$, $s = 0.5$, $t = 0.2$, $v = 0.25$, $w = 0.55$,

$U(4, 4) = 0, U(3, 3) = 0, U(2, 2) = 0, U(1, -1) = 0, U(0, 0) = 1,$

$U(-1, -1) = 0, U(-2, -2) = 0, U(-3, -3) = 0, U(-4, 4) = 0.$

(iii) Cover

$x_1 = y_1 = 1$, $p = 0.21$, $q = 0.63$, $s = 0.16$, $t = 0.2$, $v = 0.25$, $w = 0.55$,

$U(4, -4) = 0, U(3, -3) = 0, U(2, -2) = 0, U(1, -1) = 0, U(0, 0) = 1,$

$U(-1, 1) = 0, U(-2, 2) = 0, U(-3, 3) = 0, U(-4, 4) = 0.$

Solution 178.3 – Square-free integers

> An integer is *square-free* if it is not divisible by the square of a prime. Are there infinitely many positive integers n such that both n and $n+1$ are square-free?

ADF

Yes. There is a proof based on a result that is well known to number theorists. Let $Q(N)$ denote the number of square-free positive integers $\leq N$. Then

$$\frac{Q(N)}{N} \rightarrow \frac{6}{\pi^2} \quad \text{as} \quad N \rightarrow \infty. \tag{1}$$

Or, to put it more intuitively: if you choose a positive integer at random (whatever that might mean), the probability that it is not divisible by the square of a prime is $6/\pi^2$. Related problem: Confirm this by experiment.

Anyway, it is Theorem 333 in G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*. The proof is roughly as follows. Denote by $[x]$ the largest integer $\leq x$. Clearly, the number of positive integers $\leq N$ which are divisible by a positive integer m is $[N/m]$. By the inclusion-exclusion principle we therefore have

$$Q(N) = N - \sum_{p \text{ prime}} \left[\frac{N}{p^2}\right] + \sum_{p, q \text{ prime}} \left[\frac{N}{p^2 q^2}\right] - \sum_{p, q, r \text{ prime}} \left[\frac{N}{p^2 q^2 r^2}\right] + \cdots$$

$$= N - \sum_{p \text{ prime}} \frac{N}{p^2} + \sum_{p, q \text{ prime}} \frac{N}{p^2 q^2} - \sum_{p, q, r \text{ prime}} \frac{N}{p^2 q^2 r^2} + \cdots + E(N),$$

where $E(N)$ is the error introduced by removing the square brackets from each term. By combining the terms of the last expression into a product we obtain

$$Q(N) = N \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) + E(N).$$

Moreover,

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},$$

and (1) follows if we can show that $E(N)/N \rightarrow 0$ as $N \rightarrow \infty$. As this proof is supposed to be rough I shall omit the details.

Given (1), the solution to the original problem is quite straightforward. If there are at most a finite number of n such that n and $n+1$ are square-free then $Q(N)/N \rightarrow$ something $\leq 1/2$ as $N \rightarrow \infty$. But $6/\pi^2 > 1/2$.

Return of the lazy fat lady

Martin Cooke

'Lazy fat lady' is a sillier name for the lemniscate, the eight-like symbol that I have written about in **176** and **178**. In case I haven't explained my term 'incompletable' very well, the difference between an incompletable $\mathbb{N}$, which I will shorten to inc $\mathbb{N}$, and the standard one only actualizes within its endlessness, and then only ontologically, not quantitatively. Indeed, I shall assume an intrinsically well-founded, innate sense of both $\mathbb{N}$ and (three-dimensional, Euclidean) space, since these seem to be prerequisites for knowing almost anything else (and not just in maths).

The set inc $\mathbb{N}$ behaves like ω (and $\aleph_0$) in the contexts of ordinal (and cardinal) arithmetic, and I will show that we also use aspects of inc $\mathbb{N}$ that are definitively incompletable, e.g. when we use the definition of an infinite sum as the limit of the partial sums of the series (which I will hereafter shorten to lim sum) or require the derivations of infinitesimals within a Set Theory. Other evidence in defence of inc $\mathbb{N}$ includes these two references: [1], a 'popular science' book, which claims that (p. 245) '... the empirical data from neuropsychology seem to provide support for intuitionism ...' and [2], a Ph.D. thesis in maths education, which indicates that hyperreals (reals plus infinitesimals) and incompletable infinite sequences would be ideal for many children; and my ontology for hyperreals (geometric, rather than purely set-theoretic, see [3], or purely algebraic, derivations) and inc $\mathbb{N}$ (similarly derived, rather than requiring intuitionistic assumptions) could be useful in this context.

However, inc $\mathbb{N}$ may seem much like the usual $\mathbb{N}$, so perhaps (from that perspective) what I am challenging is the most common conception of the reduction of geometry to analysis (the replacement of the pre-twentieth-century geometric line with the real number line $\mathbb{R}$). A few thousand years ago, $\sqrt{2}$ was regarded as only a geometrical quantity, not a proper number, so in this context my $\infty = 0^{-1}$ and my other geometric infinities (reciprocals of the geometric infinitesimals, corresponding to points falling between inc $\mathbb{N}$ and the ∞ parts of the line, or to Robinson's ones in [3]) are like 'new irrationals' (to borrow Cantor's phrase). The vectorial argument below only uses $\mathbb{Q}$ (the rationals) but the geometrical aspect is whether or not $\mathbb{Q}$ exists in the manner of a spatial object, all at once, so to speak, or like the sequentially defined inc $\mathbb{N}$ (which is often modelled temporally).

The 'space' of which I write is not $\mathbb{R}^3$ but an innate template which we use to organize our earliest sensory experiences (probably because that is how the world is itself organized), and which consequently corresponds to our experience of objective reality (but may not be exactly how physical space is). Lines can be thought of as the conceptual edges to surfaces of

this space, and it is probably a line like this (with labelled points such as 1 and 0 on it) which we first think of when we are introduced to $\mathbb{R}$ (it being conceptually prior to thinking about the rather less precise edges of physical objects).

We may consider an infinite line in this geometric space, and label two arbitrary points 0 and 1, connected by a line $[0, 1]$. I will write this line minus the point 0 as $(0, 1]$, although usually, in what follows, this notation will refer to a specific set of points on this line. The connection between geometry and our number systems is through infinite divisibility (to get $\mathbb{Q}$) and the limit process (to get $\mathbb{R}$), so that there is a big 'spatial' aspect to $\mathbb{R}$ (corresponding to 'Platonistic' or 'combinatorial' foundations to $\mathbb{R}$ in the literature) which becomes a (desirably) direct correlation if we then use number systems like $\mathbb{R}$ to model actual spatial processes (e.g. in physics).

The following vectorial argument requires only $\mathbb{Q} \cap [0, 1]$ (a finite rational number line which I will write as $[0, 1]$ below) which is based on the usual $\mathbb{Q}$ as I shall aim for a conceptual contradiction. The vectors are the subsets $(1/(n + 1),\ 1/n]$, for $n \in \mathbb{N}$, and can be regarded as being in a vector space on a geometric line. The addition is defined to be $(1/(n + 1),\ 1/n] + (1/n,\ 1/(n-1)] = (1/(n+1),\ 1/(n-1)]$. And $\{1/n : n \in \mathbb{N}\}$ is a set of points on the geometric line $(0, 1]$ dividing it up into a set S of the vectors, each of magnitude $1/n - 1/(n+1)$, and they are therefore the points where pairwise vector addition can be considered to take place. Also, $r \in (0, 1] \Leftrightarrow r \in s$, for some $s \in S$, so the complete sum of the vectors of S is known to be $(0, 1]$, without having to apply the pairwise addition sequentially.

This is not the usual derivation of a vector sum, which is defined as 'lim sum', within a set-theoretical context which is devoid of geometry, but my argument is that it follows by legitimate steps from the spatial assumptions—e.g. all the pairwise additions can be considered to take place simultaneously because the labelled points exist 'all at once', and they only (individually) concern two vectors. Conversely, if we had started with inc $\mathbb{N}$ then lim sum would have been the only possibility, since to sum the entirety of S we would have to move through its endlessness without it being there all at once (note that this is a temporal way of speaking, but it is just the nature of any sequence which is involved, not time *per se*).

Originally lim sum was devised by mathematicians who held that $\mathbb{N}$ was incompletable and lines were geometrical, and the following equations (which produce the famous Grandi series, one of the series which led to the introduction of lim sum in the first place) show that the geometry which is to be reduced to an as yet unfinished set theory requires an inc $\mathbb{N}$.

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots \quad = \; 1 \quad \text{from the vectors}$$

$$\Rightarrow \quad 1 + 0 + 0 + \ldots \qquad\qquad = \; 1 \quad \#$$

$$\Rightarrow \quad 1 + (1 - 1) + (1 - 1) + \ldots \quad = \; 1 \quad \text{since } 0 = (1 - 1) \text{ for each } 0$$

$$\Rightarrow \quad 1 + 1 + (-1 + 1) + \ldots \qquad = \; 1 \quad *$$

$$\Rightarrow \quad 2 \qquad\qquad\qquad\qquad\quad = \; 1 \quad \text{from } \#$$

It is the absence of nested brackets, centred on the left of the infinite sums (which is what the lim sum definition would impose on these spatially prearranged vectors), which allows the contradiction to be obtained by step $*$. If applicable maths requires inc $\mathbb{N}$ then it is the incompletable structure of such collections which should underpin our conception of set.

When I use inc $\mathbb{N}$ to label points on this geometric line, this is the marriage of spatial and temporal which makes my inc $\mathbb{N}$ a 'Platonistic' concept, and it is this line which I earlier stated may have $+\infty$ points upon (any length of) it. A point can be anywhere on a line because it merely has reduced dimension, but it is part of a collection of points which is behaving like a line. The line on either side of it extends up to (and including?) zero distance from it, since it is next to it, which may be why only closed line segments can exist (basically because they are 'spatial' objects whereas open intervals of labelled points of it are 'quasi-temporal' objects) in any imaginable way.

For example, a (mental idea of a) material body moving through space contains a 'physical point' which moves through the space, as in fluid mechanics. If the space had an open edge within it, the point would move (fixed within the body) until it reached a point which was not there (when it would vanish, much as the point has vanished). Now, how could the physical point reach that position (or point) if there was no point there? Whereas if the space is closed, and a labelled point of this is then considered to be non-existent (in the sense of objects occupying it being non-existent), there is no such problem.

It could be possible for a line to be made of $+\infty$ unlabelled (and hence totally 'spatial') points whilst the only conceptually clear ways to consider collections of those points are: (i) a complete, finite set of labelled points on top of (points acting exactly as) a line, (ii) incompletable sets of labelled points likewise, and (iii) lines going from one point to another (and higher-dimensional analogues). It is the way that there are $+\infty$ points on the line which stops this being a mere contradiction (for example, the collection of all the points of such a line which are zero distance from a given point may not be well defined by the usual combinatorial conceptions, e.g. since $0 \in 0\infty$ and although they are 'spatial', they are exactly so many as to be a line in the first place. Having the widest range of objects available has usually won out within set theory at the axiomatic level—see [4] for the

introduction of sets that are not unlike (points labelled as) $+\infty$ into ZFC (the usual set theory—obtained by appending the axiom of choice to the basic Zermelo-Fraenkel system).

Alternatively, you could think of these $+\infty$ points as being so packed together, when they form a line, that removing one of them does not change it at all (it would affect a collection of labelled points, of course, but removing one of them, in the context of that collection, would not leave bits of the line of either side of it)—zero is very small, after all. The continuity of the geometrical line, of $+\infty$ points, is conceptually prior to the definition of continuity which we use with $\mathbb{R}$, and the latter is basically a process (although a 'static' terminology has been preferred because of the assumed 'spatial' nature of the usual $\mathbb{N}$). With inc $\mathbb{N}$, these definitions (interpreted as processes) still apply, along with things like a geometric interpretation of the Dirac delta function etc. Or, having a point anywhere need not imply having points everywhere. For example, spheres are not space-filling, and the necessary gaps between distinct points may correspond to the overlaps which result if you try to fill space with spheres.

I am not thinking of the 'temporal' nature of inc $\mathbb{N}$ as some sort of time-dependence for $\mathbb{N}$ itself. It was once thought to be convenient to model $\mathbb{N}$ in this way, but this seemed to cause confusion for people who didn't understand inc $\mathbb{N}$. For example, it made them think that a maximum number, perhaps arbitrarily assigned, was involved at any given time. Actually we can look at inc $\mathbb{N}$ as a whole as long as we realize that it is a different sort of object, ontologically, from the equally primitive notion of a geometrical object. Its two defining rules generate a sort of feedback loop, as would the rules for any sequence, so that it is the nature of any sequence (even an atemporal, logical one) to be incompletable. To have got all the elements of the sequence out of the loop you would have had to go round it an infinite number of times, and if you go round such a loop an infinite number of times you generate an element just as you do if you go round it a finite number of times (unless it is a potential infinity).

If you are wondering what all this means in practice, consider how infinite numbers can't exist on the real number line $\mathbb{R}$. If they did then $\mathbb{N}$ would have to go all the way up to them, since it contains all the finite whole numbers in an equally spatial way, which would then imply that $\mathbb{N}$ had an end (at one less than an infinite number, i.e. it would contain infinite numbers) whereas it is endless (and purely finite). With inc $\mathbb{N}$ we can consider $\mathbb{N}$'s endlessness to apply only to the labelling process, so that infinite parts of the infinite geometrical line can actually exist as such. We do not need to know the detailed structure of those parts (e.g. whether it is ∞ or $(+\infty, -\infty)$) any more than we do now, but it is not consistent to deduce that they do not exist. A finer division of those parts may correspond

to Boffa's universe; see [4], however.

Platonistic inc $\mathbb{N}$ can also shed a new (old?) light on the computable reals (studied by Turing in the 1930s), $\mathbb{V}$, say, which is the collection of the real numbers which can be exactly described in a finite way (e.g. $1/4$, $\sqrt{2}$, π), or the points of the line so labelled which are therefore uniquely specified. By contrast, the points of $\mathbb{R} - \mathbb{V}$ are not uniquely specified (relative to the points 0 and 1) since they are given only by endless expressions (though they could, like 0 and 1, be arbitrarily assigned to a unique point in the appropriate partition of $\mathbb{R}$).

Take a point on the geometric line $[0, 1]$ and successively subdivide this line into tenths, noting where the point lies relative to this, so generating the decimal expansion of the point. A point that is only infinitesimally distinct from it would generate the same real 'number', and so an element of $\mathbb{R} - \mathbb{V}$ will not decide which of these points to label—whereas no infinitesimal amounts are introduced by the constructions of $\mathbb{V}$. This difference may help to explain the philosophical disputes over the introduction of $\mathbb{R}$ and its set theory (see [5]) and also relates to how ∞ (defined relative to 1 and 0 of course) specifies a huge part of the infinite line instead of the single 'point' in projective geometry.

Now, $\mathbb{V}$ is countable (since we can list its descriptions alphabetically) but with a correspondence $\mathbb{N} \to \mathbb{V}$ given by an element of $\mathbb{R} - \mathbb{V}$, thus avoiding Richard's paradox, see [5], and if there are doubts over the existential nature of points (their ontology) (e.g. some people like to think of arbitrarily small blobs instead) then using only two (or an arbitrary finite number) of them is advantageous, and $\mathbb{V}$ can capture applicable analysis, if applied within the right conceptual framework. So if you think that the hyperreals are unnecessary then consider that there are similar reasons to feel that way about most of the elements of $\mathbb{R}$ (not to mention the imaginary numbers).

Conversely, if it is spatial extension which allows the uncountability of $\mathbb{R}$ to develop from an inc $\mathbb{N}$, then there may be limitations (inherent in this spatial concept) which explain why the more fundamental (wrt labelling) incompletable property is exhibited by elements of totalities such as $\mathbb{On}$ (the class of all ordinals). The totality of the points of the line is 'spatial' but in a class of its own. Philosophers have called the idea that a line cannot be made of points because lots of zeroes is still zero the 'fallacy of composition', arguing that the 'lots' implies successive addition whereas $\mathbb{R}$ is uncountable. But perhaps they were themselves being fallacious in confusing geometric lines with the real number line? There is a physical concept of 'spatial' addition (e.g. gravity acts on each particle of a ball individually but simultaneously, whereas a kick will apply to part of it and then to neighbouring parts and so on in a wavelike manner) which could apply to the line—we know that $0 \in 0\infty$ is not an equality, whereas zero

times $\aleph_1$ might not be nonzero at all for all we know.

In conclusion, the ontological foundations of maths are even more interesting than they were when Aristotle first 'solved' Zeno's paradoxes, and perhaps our current set theoretical foundations (with their own philosophical problems) are just a least ontologically committed option. The ontological problems of the natural sciences are probably related (both in results and methodology) so this might also be a rewarding part of maths empirically. For a recent introduction to the philosophy of mathematics see [6] and for a classic overview of mathematics see [7].

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[2] J. D. Monaghan, 'Adolescents' understanding of limits and infinity' , Ph.D. thesis, University of Warwick 1986.

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[4] A. Rieger, 'An argument for Finsler–Aczel Set Theory', *Mind* **109** (April 2000).

[5] J. van Heijenoort (ed), *From Frege to Gödel*, Harvard University Press, 1967.

[6] S. Shapiro, *Thinking About Mathematics*, Oxford University Press, 2000.

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Solution 181.5 – Five digits

Find all solutions of

$$10^4 a + 10^3 b + 100c + 10d + e = f(10^4 e + 10^3 d + 100c + 10b + a)$$

in non-negative integers a, b, c, d, e, f subject to the usual constraints of decimal arithmetic, $0 \le b, c, d \le 9$ and $1 \le a, e \le 9$.

Paul Terry

Since a, b, c, d and e are all less than 10, the LHS of the equation is less than 100,000. Also, since $e > 0$ the RHS can only be less than 100,000 if $f < 10$.

Apart from the 900 'trivial' solutions where $a = e$, $b = d$ and $f = 1$ there are two which meet the given conditions:

$a = 8, b = 7, c = 9, d = 1, e = 2, f = 4$ from which $87912 = 4 \cdot 21978$;

$a = 9, b = 8, c = 9, d = 0, e = 1, f = 9$ from which $98901 = 9 \cdot 10989$.

Solution 181.3 – Six-sided pencil

What is the function associated with the familiar curtain-like graph that appears at the border of the exposed wood when you sharpen a six-sided pencil?

John Bull

Consider just one example based on arbitrary dimensions. The equation of the cone of the pencil is

$$z^2 \;=\; x^2 - y^2.$$

The equations of the three pairs of opposite flat sides of the pencil are
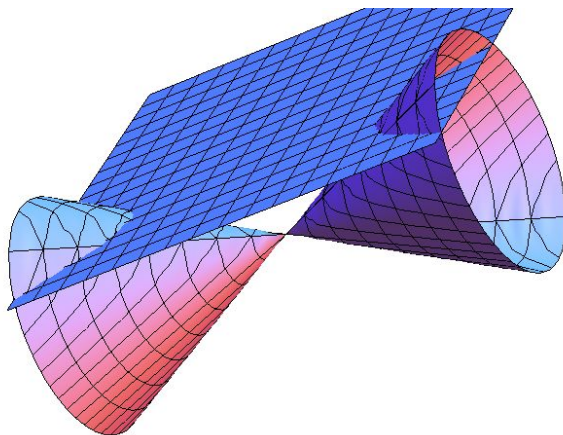
$$z \;=\; \pm\sqrt{3},$$
$$z \;=\; \pm(2+y)\sqrt{3}$$

and

$$z \;=\; \pm(2-y)\sqrt{3}.$$

Substituting $z = \sqrt{3}$ in the first equation gives

$$x^2 - y^2 \;=\; 3.$$

Thus the curve of each sharpened segment takes the form of a hyperbola.

Solution 181.2 – Six secs

Show that $\sec\dfrac{\pi}{7}\sec\dfrac{2\pi}{7}\sec\dfrac{3\pi}{7}\sec\dfrac{4\pi}{7}\sec\dfrac{5\pi}{7}\sec\dfrac{6\pi}{7} = -64$ and that $\sec\dfrac{\pi}{7}+\sec\dfrac{2\pi}{7}+\sec\dfrac{3\pi}{7}+\sec\dfrac{4\pi}{7}+\sec\dfrac{5\pi}{7}+\sec\dfrac{6\pi}{7}=0.$

Sue Bromley

Taking the second part of the question first: generally, $\sec\alpha = 1/(\cos\alpha) = -1/\cos(\pi - \alpha) = -\sec(\pi - \alpha)$, so

$$\sec\frac{\pi}{7} + \sec\frac{2\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{4\pi}{7} + \sec\frac{5\pi}{7} + \sec\frac{6\pi}{7}$$

$$= \sec\frac{\pi}{7} + \sec\frac{2\pi}{7} + \sec\frac{3\pi}{7} - \sec\frac{3\pi}{7} - \sec\frac{2\pi}{7} - \sec\frac{\pi}{7} = 0.$$

For the first part, working in cos seems safe here and also easier. Using $(\cos\alpha)(\cos\beta) = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$, we have

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{2}\left[\cos\frac{\pi}{7} + \cos\frac{3\pi}{7}\right]\cos\frac{3\pi}{7}$$

$$= \frac{1}{4}\left[\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} + 1\right]$$

and

$$\cos\frac{4\pi}{7}\cos\frac{5\pi}{7}\cos\frac{6\pi}{7} = -\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}.$$

Drawing a graph of cos from 0 to 2π helps us to see that

$$\cos 0 + \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{8\pi}{7} + \cos\frac{10\pi}{7} + \cos\frac{12\pi}{7} = 0,$$

$$2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right) = -1, \quad \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}.$$

Putting all this together,

$$\sec\frac{\pi}{7}\sec\frac{2\pi}{7}\sec\frac{3\pi}{7}\sec\frac{4\pi}{7}\sec\frac{5\pi}{7}\sec\frac{6\pi}{7} = \frac{-1}{\left(\cos\frac{2\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right)^2}$$

$$= -\left(\frac{1}{4}\left(-\frac{1}{2}+1\right)\right)^{-2} = -64.$$

Working this out certainly took longer than six secs!

Solution 181.7 – Five cots

Prove that

$$\cot\frac{\pi}{11}\cot\frac{2\pi}{11}\cot\frac{3\pi}{11}\cot\frac{4\pi}{11}\cot\frac{5\pi}{11} \;=\; \frac{1}{\sqrt{11}}.$$

John Bull

We use the identity $(\cos(n\pi/11)+i\sin(n\pi/11))^{11} = \left(e^{in\pi/11}\right)^{11} = (-1)^n$ with n an integer ≥ 1. For convenience, put $c=\cos(n\pi/11)$, $s=\sin(n\pi/11)$ and $p=\cot(n\pi/11)$. Then

$$\begin{aligned}
(c+is)^{11} \;=\;& c^{11} + 11c^{10}si - 55c^9s^2 - 165c^8s^3i \\
&+\; 330c^7s^4 + 462c^6s^5i - 462c^5s^6 - 330c^4s^7i \\
&+\; 165c^3s^8 + 55c^2s^9i - 11cs^{10} - s^{11}i \;=\; (-1)^n.
\end{aligned}$$

Equate imaginary parts and divide through by s^{11},

$$11p^{10} - 165p^8 + 462p^6 - 330p^4 + 55p^2 - 1 \;=\; 0.$$

We know that $\cot(\pi/11) = -\cot(10\pi/11)$, $\cot(2\pi/11) = -\cot(9\pi/11)$, etc. so the roots of this equation are $p^2 = \cot^2(\pi/11)$, $\cot^2(2\pi/11)$, ..., $\cot^2(5\pi/11)$. As a quintic in p^2, the product of the roots is given by the negation of the constant term divided by the leading coefficient. So we have

$$\cot^2\frac{\pi}{11}\cot^2\frac{2\pi}{11}\cot^2\frac{3\pi}{11}\cot^2\frac{4\pi}{11}\cot^2\frac{5\pi}{11} \;=\; \frac{1}{11}$$

and the required result follows.

The technique can be used to prove similar trigonometric identities, such as the other problem in M500 **181**, 'Six secs'.

Sue Bromley

I couldn't resist another '$\sqrt{11}$' problem but the thought of attempting 'Five cots' using trig. formulae was not very appealing. However, M500 **175**, p. 23, helpfully provides this equality:

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n} \;=\; \frac{n}{2^{n-1}}.$$

Having made the calculations for $n = 3, 5, 7, 9, 11$, I would put forward the additional theorems

$$\cos\frac{\pi}{n}\cos\frac{2\pi}{n}\ldots\cos\frac{(n-1)\pi}{n} \;=\; \frac{(-1)^{(n-1)/2}}{2^{n-1}}$$

and

$$\tan\frac{\pi}{n}\tan\frac{2\pi}{n}\ldots\tan\frac{(n-1)\pi}{n} \ = \ (-1)^{(n-1)/2}n,$$

where n is odd.

Using the formula for tan,

$$\tan\frac{\pi}{11}\tan\frac{2\pi}{11}\ldots\tan\frac{9\pi}{11}\tan\frac{10\pi}{11} \ = \ -11,$$

and since $\tan(\pi - \alpha) = -\tan\alpha$,

$$\tan\frac{\pi}{11}\tan\frac{2\pi}{11}\ldots\tan\frac{5\pi}{11} \ = \ \sqrt{11}$$

and

$$\tan\frac{6\pi}{11}\tan\frac{7\pi}{11}\ldots\tan\frac{10\pi}{11} \ = \ -\sqrt{11}.$$

Hence

$$\cot\frac{\pi}{11}\cot\frac{2\pi}{11}\cot\frac{3\pi}{11}\cot\frac{4\pi}{11}\cot\frac{5\pi}{11} \ = \ \frac{1}{\sqrt{11}}.$$

Note that the formula for cos gives a very quick solution to 'Six secs'. Since

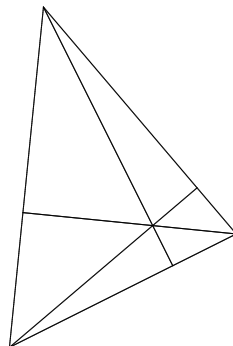$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\ldots\cos\frac{6\pi}{7} \ = \ \frac{(-1)^3}{2^6} \ = \ \frac{-1}{64},$$

$$\sec\frac{\pi}{7}\sec\frac{2\pi}{7}\ldots\sec\frac{6\pi}{7} \ = \ -64,$$

but I couldn't have arrived at this without first going through various thought processes and down many blind alleys trying to solve it.

Problem 183.1 – Three altitudes
John Bull

We all know that a triangle is uniquely specified by the lengths of three sides. Is a triangle also uniquely specified by the lengths of three altitudes?

General formula for sums of powers
Sebastian Hayes

Barry Lewis, in his two fascinating articles ('Hip, Hip, Array', M500 **162**, and 'Sums of powers – again', M500 **167**) shows how to obtain the coefficients for sums of arbitrary powers of natural numbers. A general formula can also be obtained without calculus by using the Method of Indeterminate Coefficients.

We want a function in n and r which will sum a^r for $a = 0, 1, 2, \ldots, n$ and $r = 0, 1, 2, \ldots$. By hypothesis there is a polynomial expression in n:

$$A_0 + A_1 n + A_2 n^2 + A_3 n^3 + \cdots + A_r n^r + A_{r+1} n^{r+1} + \ldots,$$

which is valid for any $r = 0, 1, 2, \ldots$. The A_k are, however, not constants as in a normal polynomial but functions of r, possibly constant functions:

$f(0, r)$, i.e., summing from 0 to 0, $= 0$ for all r and so $A_0 = 0$;

$f(1, r)$, i.e., summing from 0 to 1, $= 1$ for all r.

This leads to the extremely useful result that for all values of $r > 0$, the sum of the coefficients is 1.

Suppose $r = 3$. Since the formula we are looking for is (by hypothesis) valid for any n, we can take the difference between summing a^3 from 0 to $n+1$ and summing it from 0 to n. Using the binomial coefficients we obtain

$$A_1(1) + A_2(2n + 1) + A_3(3n^2 + 3n + 1) + A_4(4n^3 + 6n^2 + 4n + 1)$$
$$+ A_5(5n^4 + 10n^3 + 10n^2 + 5n + 1) + \text{higher powers.}$$

However, this difference is just $(1 + n)^3 = n^3 + 3n^2 + 3n + n^0$.

Equating coefficients for the same powers of n we can solve for A_1, A_2, A_3, A_4, ... All coefficients accompanying powers higher than the third must be zero, and there is no constant term so we obtain the well-known values $A_4 = 1/4$, $A_3 = 1/2$, $A_2 = 1/4$, $A_0 = 0$. Check: Sum of coefficients $= 1$.

This procedure can be made completely general and we can solve for arbitrary r in piecemeal fashion, starting with the coefficient of the $(r+1)$th power, A_{r+1}, and working backwards. (All coefficients for powers beyond the $(r + 1)$th will be zero.)

In effect we have to solve an indefinitely extendible set of simultaneous equations commencing with

$$
\begin{aligned}
{}^{r+1}C_1 A_{r+1} &= 1, \\
{}^{r+1}C_2 A_{r+1} + {}^{r}C_1 A_r &= {}^{r}C_1, \\
{}^{r+1}C_3 A_{r+1} + {}^{r}C_2 A_r + {}^{r-1}C_1 &= {}^{r}C_2.
\end{aligned}
$$

Once we have obtained a solution for one of the coefficients, we can feed this (functional) value back into the subsequent equation.

First, $A_{r+1} = 1/(r+1)$ starts the ball rolling; A_r turns out, surprisingly, to be independent of r and always $= 1/2$ *except* in the trivial case of $f(n, 0)$ when $1^0 + 2^0 + \cdots + n^0 = 1 + 1 + \cdots + 1 = n$ making $A_0 = 0$ in this case.

Feeding in these results, I found that

$$
\begin{aligned}
A_{r+1} &= \frac{1}{r+1}, \\
A_r &= \frac{1}{2} \quad (r > 0), \\
A_{r-1} &= \frac{r}{12}, \\
A_{r-2} &= 0, \\
A_{r-3} &= -\frac{r(r-1)(r-2)}{6!}, \\
A_{r-4} &= 0, \\
A_{r-5} &= -\frac{r(r-1)(r-2)(r-3)(r-4)}{6 \times 7!}.
\end{aligned}
$$

This led to the conjecture that all further coefficients A_{r-k} are either zero or begin with $r!/(r-k)!$. This is in fact exactly what is required if we are to have a single unchanging set of (functional) coefficients which must go to zero apart from the first $r+1$ coefficients for any r.

One could continue indefinitely in this fashion but the computation soon becomes tedious. In any case, what we really want is a formula for the general term expressed as $A_{r-k+1} n^{r-k+1}$, where k is fixed for a particular setting and r is variable. It is more useful to work with $r - k + 1$ than $r - k$ because k then gives the power we are dealing with—for example, the sum of the squares will have coefficients A_{r+1}, A_r, A_{r-1} or A_{r-k+1} with k ranging from 0 to r.

After a very short time I hit upon a general recursive formula

$$
A_{r-k+1} = (1 - (\text{coefficients obtained so far})) \frac{{}^{r}C_k}{r-k+1}.
$$

Here, r is a variable, k is fixed and it is to be understood that the coefficients within the brackets are previously obtained functions in r which have been given the *current* value of k. Thus

$$A_{r-k+1} = \left(1 - \sum_1^{k+1} r\right) \frac{{}^rC_k}{r-k+1}.$$

For powers higher than the $(r+1)$th, the coefficients are zero, so we commence with A_{r-0+1} $(k=0)$ which gives $(1-0)^rC_0/(r-0+1) = 1/(r+1)$.

For $k=1$, $A_{r-1+1} = \left(1 - \dfrac{1}{2}\right)\dfrac{r}{r} = \dfrac{1}{2}$.

For $k=2$, $A_{r-2+1} = \left(1 - \left(\dfrac{1}{3} + \dfrac{1}{2}\right)\right)\dfrac{{}^rC_2}{r-2+1} = \dfrac{r}{12}$.

So the formula seems to be working. However, I found this innocuous expression, so easily obtained, impossible to prove and, on Barry Lewis's suggestion, turned to the matrix representation to see if this approach would yield better results. In matrix terms, we have to solve the following equation for arbitrary r.

$$\begin{bmatrix} r+1 & 0 & 0 & \dots & 0 \\ \frac{(r+1)r}{2} & r & 0 & \dots & 0 \\ \frac{(r+1)r(r-1)}{3!} & \frac{r(r-1)}{2!} & r-1 & \dots & 0 \\ \frac{(r+1)r(r-1)(r-2)}{4!} & \frac{r(r-1)(r-2)}{3!} & \frac{(r-1)(r-2)}{2} & \dots & 0 \\ & & & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} A_{r+1} \\ A_r \\ A_{r-1} \\ A_{r-2} \\ \dots \\ A_1 \end{bmatrix} = \begin{bmatrix} 1 \\ r \\ \frac{r(r-1)}{2} \\ \frac{r(r-1)(r-2)}{6} \\ \dots \\ 1 \end{bmatrix}.$$

Since we do not require coefficients for powers beyond the $(r+1)$th—since they are all zero—finding the inverse of this matrix for arbitrary r will provide everything we need. The one piece of additional information we start with is that the coefficients must sum to 1 for any r.

At first sight it seems an impossible task to evaluate the inverse of this indefinitely extendible algebraic matrix. However, the determinant of this matrix is just $(r+1)!$. Why is this?

By definition, the determinant of a single entry matrix is this entry, and of a 2×2 matrix it is the cross product $ad - bc$. For a larger square matrix you have to mentally block out the line and column of an entry in the top line and multiply this entry by the determinant of the remaining square block. However, since the first line is all zeros except for entry 1, $\det(r+1)$

matrix $= (r+1) \times \det(r \times r)$ matrix, i.e. the square minus the top row and first column, what the Greeks called the *gnomon*.

But the next square matrix turns out to have all zeros in the top line as well except for r in first place. Thus we go from square to inner square until we reach the final last entry which is 1 with determinant 1. Thus, rather pleasingly, the determinant of the $(r+1) \times (r+1)$ matrix is $(r+1)r(r-1)\ldots 1 = (r+1)!$.

It is in fact a basic theorem in linear algebra that if a square matrix A of order n is upper triangular, lower triangular or diagonal, then $\det A$ is the product of the entries on the main diagonal.

In principle, then, the *Power Matrix* has an inverse, since it has a non-zero determinant for any r.

Simply by observation we can fill in a few places of $(R+1)^{-1}$ which must, on multiplying matrix $R+1$, yield the unit matrix

$$
\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \quad \ldots
\end{array}
$$

Also, if we get out the inverses of some small matrices, we see a pattern emerging: the coefficients of the sums of rth powers (with one exception) appear as the bottom row of the inverse matrix. For $r=4$ the inverse of the 5×5 matrix is as follows.

$$
\begin{array}{ccccc}
\dfrac{1}{5} & 0 & 0 & 0 & 0 \\[2mm]
-\dfrac{1}{2} & \dfrac{1}{4} & 0 & 0 & 0 \\[2mm]
\dfrac{1}{3} & -\dfrac{1}{2} & \dfrac{1}{3} & 0 & 0 \\[2mm]
0 & \dfrac{1}{4} & -\dfrac{1}{2} & \dfrac{1}{2} & 0 \\[2mm]
-\dfrac{1}{30} & 0 & \dfrac{1}{6} & -\dfrac{1}{2} & 1
\end{array}
$$

If we apply this to the column matrix

$$
\begin{bmatrix} 1 \\ 4 \\ 6 \\ 4 \\ 1 \end{bmatrix},
$$

we obtain the coefficients of the sum of the fourth powers, namely

$$A_5 \;=\; \frac{1}{5}, \quad A_4 \;=\; \frac{1}{2}, \quad A_3 \;=\; \frac{1}{3}, \quad A_2 \;=\; 0, \quad A_1 \;=\; -\frac{1}{30}.$$

Check: Sum = 1.

It is necessary to prove that this will always be the case for any value of r. I did eventually establish this though in rather a long-winded way that I will not give here. This can be a test for the reader.

Model railways
ADF

Some time ago I went to see the model railway at New Romney, on the south Kent coast. It was probably typical of such things—rural landscape, two villages, each with station, houses, pub, etc. and, of course, lots of little trains running around on little tracks.

While I was watching the trains going about their business it occurred to me that the inhabitants of the model (all 30–40 of them) were blessed with a public transport system that real people could only dream about. Indeed, one could imagine the following timetable for about 11:00 on a typical day:

```
Village-A    d 11:00:00 11:00:03 11:00:07 11:00:10 11:00:12 11:00:15 ...
Village-B    a 11:00:05 11:00:08 11:00:12 11:00:15 11:00:17 11:00:20 ...

Village-B    d 11:00:01 11:00:04 11:00:06 11:00:09 11:00:13 11:00:16 ...
Village-A    a 11:00:06 11:00:09 11:00:11 11:00:14 11:00:18 11:00:21 ...
```

Clearly this is ridiculous, and certainly at odds with another model, the somewhat larger Romney, Hythe and Dymchurch Light Railway, where 15" gauge steam trains carry full-size people at (real) speeds of about 10 mph At peak times the service is half-hourly, I think.

So I ask: In a scale-model, what is the correct speed for objects to travel at? Or, to put it more simply, how should time and velocity scale with distance?

The weather is going to be much less predictable in future. That means there will be fewer white Christmases.—BBC News. [Sent by Peter Fletcher]

Seven a side

Eddie Kent

You know how it is when you open a box that hasn't been touched for years. It's like browsing through a junk shop with the added bonus that it isn't going to cost you anything, or add to the clutter in your house. That is how this puzzle turned up. It was clearly filched from some long forgotten publication, which we will be delighted to acknowledge if anyone recognizes it. The puzzle was cut out of cardboard and painted a different colour each side.

The rules are simple. You have one piece shaped like (a), ten like (b) and one like (c). Those who can count will notice that this gives a total of 49 little squares; and so the objective is to assemble them into one large square. No flipping is allowed, which is why the faces are differently coloured.

I don't recollect that anyone ever solved it, though I cannot believe that it is not possible. Surely someone can prove me right.

Good luck.

(a)   (b)   (c)
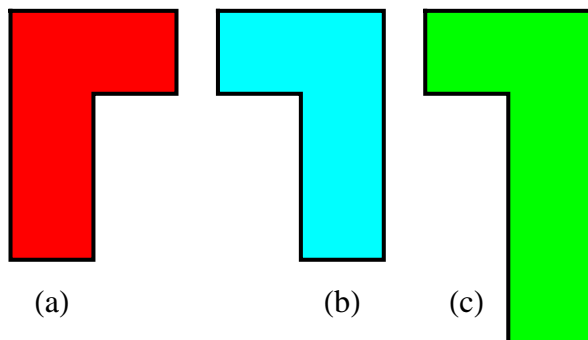
Problem 183.2 – Fifteen objects

There are fifteen objects to be painted red, yellow or blue. In each case the colour is chosen at random with probability 1/3. What's the probability of five red, five yellow, five blue?

The world's most powerful unclassified supercomputer is at Berkeley National Laboratory ... It has a top speed of 3.8 tetraflops, or 3.8 trillion calculations every second.—*The Big Issue*, September 17–23, 2001 [Spotted by JRH.]

Solution 179.2 – Four tans

Show that $\tan 11° = (\tan 19°)(\tan 33°)(\tan 41°)$.

ADF

I would like to resurrect the idea of solving this problem by an approximate numerical computation (M500 **181** 13). It can be done.

Let $\zeta = e^{\pi i/180}$. Then the formulae for the complex trigonometric functions look like this: $\cos x° = (\zeta^x + \zeta^{-x})/2$, $\sin x° = (\zeta^x - \zeta^{-x})/(2i)$ and $\tan x° = i\,(1 - \zeta^{2x})/(1 + \zeta^{2x})$. Solving the problem, therefore, is equivalent to showing that

$$\frac{1 - \zeta^{22}}{1 + \zeta^{22}} + \frac{1 - \zeta^{38}}{1 + \zeta^{38}} \cdot \frac{1 - \zeta^{66}}{1 + \zeta^{66}} \cdot \frac{1 - \zeta^{82}}{1 + \zeta^{82}} \;=\; 0,$$

or, multiplying out and simplifying, that $p(\zeta) = 0$, where $p(x)$ is the polynomial defined by

$$p(x) \;=\; x^{208} - x^{148} - x^{120} + x^{88} + x^{60} - 1.$$

Since ζ is an algebraic integer (it is a root of $x^{180} + 1$), so is $p(\zeta)$; therefore $\mathrm{N}(p(\zeta))$, the norm of $p(\zeta)$, is an ordinary integer. Furthermore

$$\mathrm{N}(p(\zeta)) \;=\; \prod_{1 \le k \le 359,\ \gcd(k,360)=1} p(\zeta^k).$$

The way ahead is clear. Pretending that we haven't spotted the factor $x^{120} - x^{60} + 1$ of $p(x)$, we use a pocket calculator, or whatever, to compute $\mathrm{N}(p(\zeta))$ with sufficient accuracy to verify that $-1 < \mathrm{N}(p(\zeta)) < 1$. Now we can conclude that $\mathrm{N}(p(\zeta)) = 0$ and hence $p(\zeta) = 0$.

Problem 183.3 – Seven real numbers

Barbara Lee

Suppose a, b, c, d, e, f and g are seven non-negative real numbers that total 1. If M is the maximum of the five sums,

$$a + b + c, \quad b + c + d, \quad c + d + e, \quad d + e + f, \quad e + f + g,$$

what is the minimum value that M can possibly take?

Problem 183.4 – Two real numbers

John Bull

If x and t are real numbers, find values of t such that $\cosh x \le \exp(tx^2)$ for all x.

Letters to the Editors

Four points

Dear Jeremy,

Re: Problem 181.4 [Choose two points inside a circle and draw the line segment joining them. Then select another two points inside the circle and draw the line segment joining these two points. What is the probability that the two line segments intersect?]. Take four points randomly distributed on the circumference of a circle. Call one of them A, and the others B, C, D in that order. Then A can join to B, C, or D whence C must join D, A, or B. So there are three possibilities, in one of which the lines intersect. The probability of that happening is one third.

Then I reread the question, and saw that the points are inside the circle, not on the circumference. I thought it seemed too easy.

Take point A randomly inside the circle. It will be a point on an inscribed circle with the same centre. As a big circle will have more room for points than a little circle, it suggests that the random point is more likely to be on a big circle. But point A will also be on a diameter of the circle, and therefore has equal probability of being in the centre or at the edge, because all diameters are the same length. That seems to be a contradiction, or have I just shown an example of Cantor's argument that the number of points on all lines are equal? No it does not, because the diameter has two points at the circumference but only one at the centre.

Instinctively there does seem to be more room in the circle away from the centre, and that is where the four points are more likely to be. Such are my random thoughts.

Colin Davies

Chimps

CHIMP to WOMAN is a short journey (though not as short as APE to MAN!). CHIMP, CHAMP, CLAMP, CLAMS, CLAPS, CLOPS, COOPS, CORPS, CORES, COVES, COVEN, WOVEN, WOMEN, WOMAN (13).

Rob Williams

My best so far is 12 changes: CHIMP, CHAMP, CHAMS, CHAPS, CHOPS, COOPS, CORPS, CORES, COVES, COVEN, WOVEN, WOMEN, WOMAN.

Cham is in dictionaries, an old spelling of khan but still vaguely current because someone called Dr Johnson the Grand Cham of Literature.

Best wishes

Ralph Hancock

A small ditrigonal icosidodecahedron

Marion Stubbs

Reprinted from M500 38

Cut a regular 5-pointed star-shape template from card, also a similar card template for an equilateral triangle with sides equal in length to the length of the star sides. Place each template on the material to be used, such as computer cards, or even old wallpaper if strong enough, prick a mark on the material at each vertex of the template, and join up the marks with a scoring knife or pencil. Then cut round the shape, leaving about quarter inch tabs all round for gluing. Fold the tabs inwards along the scored lines.

You need 12 stars and 60 triangles. Start with one star, surrounded by ten triangles. It is easiest if the triangle pairs are glued together first and then glue them between star arms, as dihedral grooves. Then you can immediately add the next five stars, followed by the remaining pieces. As usual, the final star is the most difficult to insert. It is best done slowly, in stages, gluing one tab at a time, and using a suitable instrument as a probing needle to work in where needed.

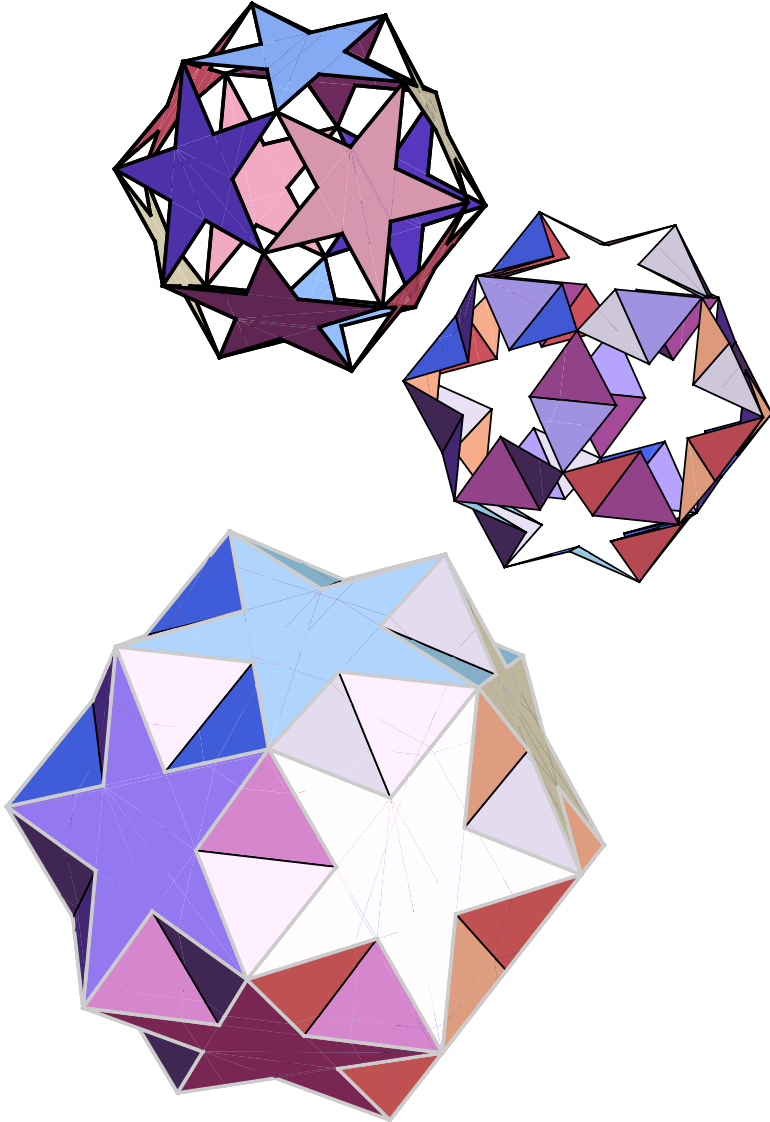The result is a pretty Christmas decoration . . . .

———————————

Tony Forbes writes—Here is the construction which I used for the illustrations.

Start with the point $e_1 = (1/(2\sin(\pi/5)),\ 0,\ 0)$. Scale e_1 by a factor of $\cos(2\pi/5)/\cos(\pi/5)$ and rotate through $\pi/5$ about the z-axis to create point f_1. Rotate both e_1 and f_1 by multiples of $2\pi/5$ about the z-axis to obtain eight new points, $e_2, f_2, \ldots, e_5, f_5$. Thus $s_0 = \{e_1,\ f_1,\ e_2,\ f_2,\ \ldots,\ e_5,\ f_5\}$ is a regular five-pointed star in the (x, y)-plane.

Lift s_0 up the z-axis by $(\sqrt{5}+1)^{5/2}2^{-7/2}5^{-1/4}$, the radius of the insphere of the regular dodecahedron, to get s_1, say. Rotate s_1 by $\arctan 2$ about the y-axis and then by $\pi/5$ about z to obtain another star, s_2, say. Rotate s_2 by multiples of $2\pi/5$ about z to create four more stars, s_3, s_4, s_5, s_6. Then $s_1, s_2, \ldots, s_6$ are the stars of the top half of the SDI. Rotate by π about y for the bottom half.

Now form triangle t with the points corresponding to e_1 and f_1 of star s_1 and the point corresponding to f_3 of star s_2. Rotate t by multiples of $2\pi/5$ about the z-axis to create a set of five triangles, t_1, say. Associate t_1 with star s_1 and propagate triangle sets exactly as we did with the stars. Interestingly, the 60 triangles generated in this manner appear in the right places and with no overlapping.

While we're on the subject, can anyone explain in an extremely convincing manner why joining together regular pentagons results in a three-dimensional figure that closes up exactly?

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