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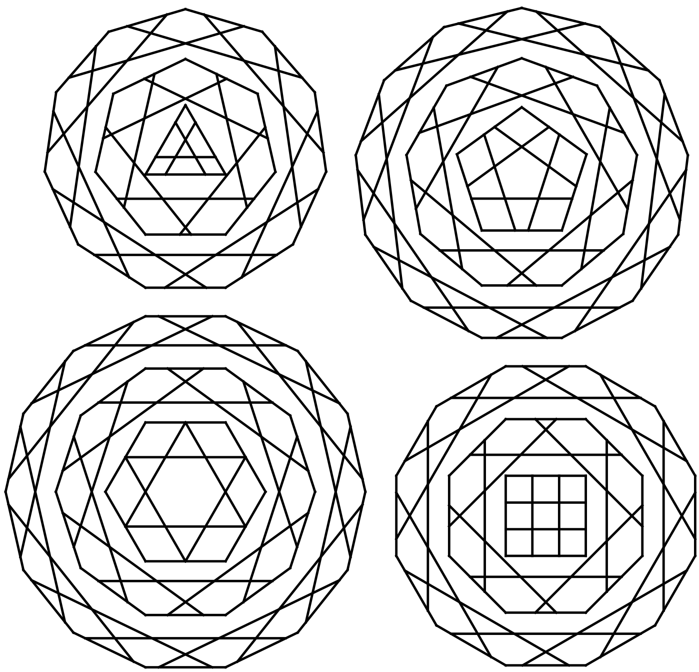
M500 188



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

The mathematics of bowls and marbles

Robin Marks

Having become the organizer of bowls competitions at Greenroyd Bowling Club in Halifax, I was wondering whether it was entirely fair to use our method of selecting partners for doubles bowling events.

The method we use is to pair the lowest handicap player with the highest handicap player, then repeat the process among the remaining players until all are paired off. Another practical problem that arises is that Club members wish to know when the competition will finish. It takes about 4 minutes to bowl an 'end' in a doubles competition. A score of 21 or more is required to win a game. How long will a game last—minimum length, average length, maximum length?

First I explain the rules of bowls. A small ball called the *jack* is thrown first and becomes the target. Each player has two bowls. These are bowled so as to arrive as close as possible to the jack. There may be one player on each side (singles) or two players on each side (doubles). Call one side 'A' and the other 'B'. Each player in turn throws a bowl until all the bowls have been thrown. Each session of throwing all the bowls is called an 'end'. The score for the end is determined as follows.

(1) Count the number of side A's bowls that have beaten the best bowl of side B. This is the score for side A.

(2) Count the number of side B's bowls that have beaten the best bowl of A's side. This is the score for side B.

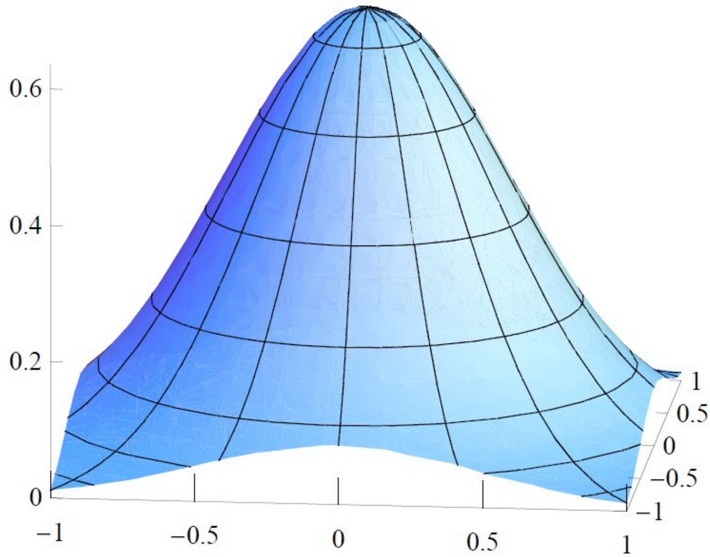
In a game the various bowls end up around the jack, with probabilities that could be reasonably modelled by a radial Gaussian distribution, probability equal to e^{-sr^2}, where r is the distance of a bowl from the jack, and s is a positive number that I will call the 'skill factor'. A higher value of s represents tighter grouping around the jack and therefore greater skill. To model the distribution on a 2-dimensional surface, we need a parameter θ, the direction of the bowl as measured from the jack. We need a total probability of 1, for a bowl to end up anywhere on the playing surface, that is, between $r = 0$ and $r = \infty$, and between $\theta = 0$ and $\theta = 2\pi$.

Now

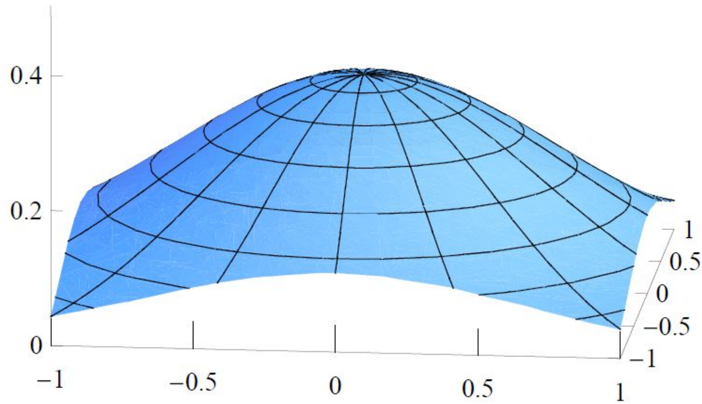
$$\int_0^{2\pi} \int_0^{\infty} e^{-sr^2} r \, dr \, d\theta \;=\; \frac{\pi}{s}.$$

The normalized Gaussian function is therefore $s/\pi e^{-sr^2}$. The graphs on the next page show what this function looks like when $s = 2$ and when $s = 1$.

Player A has skill factor 2



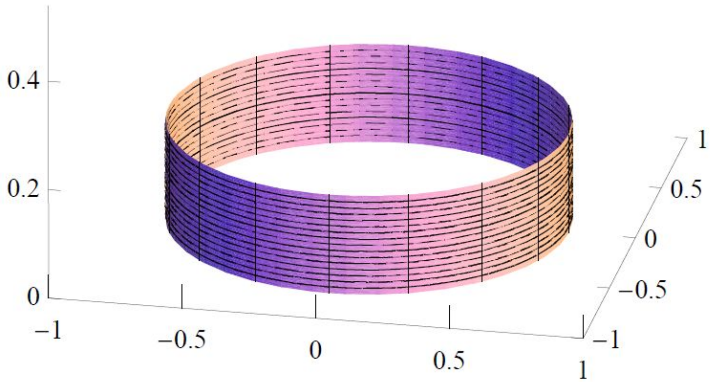
Player K has skill factor 1



To make things simple to begin with, let each side have one bowl each. What is the chance that A will beat B? Well, the probability of B's bowl arriving at a radial distance of between r and $r+\delta r$ from the jack is approximately equal to the sum of the volumes of elements in a hollow cylinder radius r, height $b/\pi e^{-br^2}$ and thickness δr, that is, the volume of an element is $b/\pi e^{-br^2}\delta r\, r\, \delta\theta$. Adding these together, in the limit as $\delta\theta \to 0$ this becomes

$$\int_0^{2pi} \frac{b}{\pi} e^{-br^2} r\, d\theta\, \delta r \;=\; 2be^{-br^2} r\, \delta r.$$

Bowl J (skill factor 1) arrives at radius 0.75



The probability of A winning, that is, the probability of A's bowl coming to rest inside this radius r, is

$$\int_0^{2\pi} \int_0^{r} \frac{a}{\pi} e^{-ar^2} r_A dr_A d\theta_A \;=\; 2a \int_0^{r} e^{-ar^2} r_A dr_A.$$

So the joint probability of B's bowl arriving at between r and $r + \delta r$, and A's bowl beating this is therefore

$$2be^{-br^2} r \delta r 2a \int_0^{r} e^{-ar^2} r_A dr_A,$$

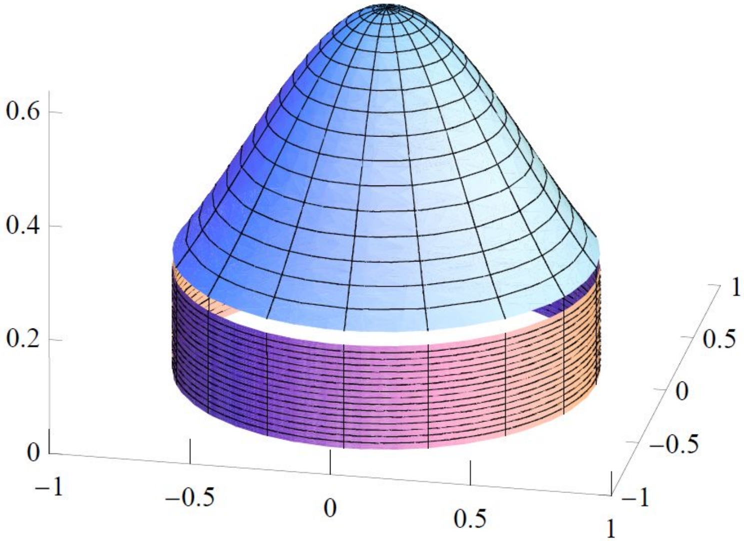
as shown on the next page.

The overall probability of A winning is now given by integrating the above expression over radial distances 0 to ∞ of B's bowl from the jack:

$$\begin{aligned} \int_0^{\infty} 2be^{-br^2} r 2a \int_0^{r} e^{-ar^2} r_A dr_A dr \;&=\; 2b \int_0^{\infty} be^{-br^2} (1 - e^{-ar^2}) r dr \\ &=\; 2b \left(\frac{1}{2b} - \frac{1}{2(a+b)} \right) \;=\; \frac{a}{a+b}. \end{aligned}$$

Thus the probability of A beating B is simply $a/(a + b)$. By an amazing stroke of luck we have arrived at an elegant and delightfully simple answer! If A's skill factor was $2x$, say, and B's was x, the chance of A beating B would be 2/3.

Bowl J at radius 0.75, Bowl A at < 0.75



In a standard game of bowls the players have two bowls each. Let A's side consist of bowls $\{A, B\}$ and B's side $\{J, K\}$. Now compare each bowl's position against the position of bowl B. I will use the notation 'pattern $\{A\}J\{BK\}$' to mean the bowls have finished in the positions A before J, J before B and J before K.

To save space I introduce a compact notation for a certain double integral which is used on numerous occasions throughout the rest of the article. Let

$$\overset{y}{\underset{x}{I}}(z) \;=\; \int_0^{2\pi}\int_x^y \frac{z}{\pi} e^{-zr_Z^2} r_Z dr_Z d\theta_Z,$$

where z is the skill factor of the bowler of bowl Z, and the position of that bowl relative to the jack is represented by polar co-ordinates (r_Z, θ_Z).

Now let us look at an end where A's side scores no points, pattern $J\{EABK\}$. The joint probability of J beating A, J beating B and J beating K is

$$\int_0^\infty \int_0^{2\pi} \frac{j}{\pi} e^{-jr^2} r\, d\theta_J \overset{\infty}{\underset{r}{I}}(a) \overset{\infty}{\underset{r}{I}}(b) \overset{\infty}{\underset{r}{I}}(k)\, dr \;=\; 2j \int_0^\infty e^{-jr^2} e^{-ar^2} e^{-br^2} e^{-kr^2} r\, dr$$

$$= \; 2j \int_0^\infty e^{-(a+b+j+k)r^2} r\, dr \;=\; \frac{j}{a+b+j+k}.$$

Side A also scores nothing when we swap the position of bowl J with bowl K. So the total probability of A's side scoring zero on an end is $P/(a+b+P)$, where $P = j + k$. For example, if skills are such that $a + b = j + k$, then the probability of scoring nil on an end is 1/2. That is, B's side will win an end in 50 per cent of cases on average. Another example: if $a = b = 2$ and $j = k = 1$, the probability of A's side scoring nothing on an end is now reduced to 1/3.

Let us look at an end where A's side scores 1 point, pattern $\{A\}J\{BK\}$. The chance of A beating J, J beating B and J beating K is

$$\int_0^\infty \int_0^{2\pi} \frac{j}{\pi} e^{-jr^2} r \, d\theta_J \mathop{I}_0^r(a) \mathop{I}_r^\infty(b) \mathop{I}_r^\infty(k) \, dr$$

$$= 2j \int_0^\infty \left(e^{-(b+j+k)r^2} - e^{-(a+b+j+k)r^2} \right) r \, dr \;=\; \frac{j}{b+j+k} - \frac{j}{a+b+j+k}.$$

Side A also scores 1 when we swap the position of bowl J with bowl K, and also if we swap bowl A for bowl B. So the total probability of A's side scoring 1 is

$$\frac{j+k}{b+j+k} - \frac{j+k}{a+b+j+k} + \frac{j+k}{a+j+k} - \frac{j+k}{a+b+j+k}$$
$$= P \left(\frac{1}{a+P} + \frac{1}{b+P} - \frac{2}{a+b+P} \right).$$

For example, when $a = b = j = k = 1$ the probability is 1/3. Another example: if $a = b = 2$ and $j = k = 1$, the probability is, coincidentally, also 1/3.

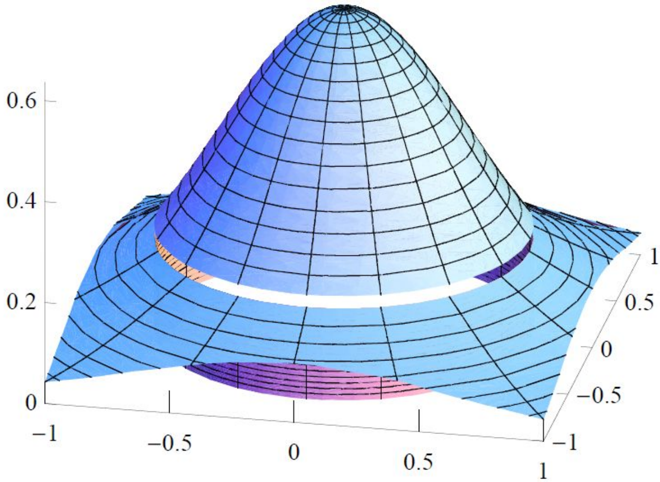
Now let us look at an end where As side scores 2 points. This is pattern $\{AB\}J\{K\}$. The chance of A beating J, B beating J and J beating K is

$$\int_0^\infty \int_0^{2\pi} \frac{j}{\pi} e^{-jr^2} r \, d\theta_J \mathop{I}_0^r(a) \mathop{I}_0^r(b) \mathop{I}_r^\infty(k) \, dr$$
$$= \frac{j}{j+k} - \frac{j}{a+j+k} - \frac{j}{b+j+k} + \frac{j}{a+b+j+k}.$$

A also scores 2 when we swap the position of bowl J with bowl K. So the total probability of A's side scoring 2 is

$$P \left(\frac{1}{P} - \frac{1}{a+P} - \frac{1}{b+P} + \frac{1}{a+b+P} \right),$$

Bowl A inside radius 0.75, J at 0.75, K at > 0.75



where, as before, $P = j + k$. For example, when $a = b = j = k = 1$ the probability is $1/6$. If $a = b = 2$ and $j = k = 1$, the probability of scoring 2 rises to $1/3$. To check, we add up the probabilities of side A scoring zero, 1 or 2 at an end,

$$P\left(\frac{1}{P} - \frac{1}{a+P} - \frac{1}{b+P} + \frac{1}{a+b+P}\right) + P\left(\frac{1}{a+P} + \frac{1}{b+P} - \frac{2}{a+b+P}\right) + \frac{P}{a+b+P} = 1.$$

Now I wish to consider a game of doubles. As well as finding out how many ends it takes to finish a game, I also want to look at how players of differing standards can be selected to make pairs that are approximately evenly matched. Sides now have four bowls each; side A has $\{A, B, C, D\}$ and side B has $\{J, K, L, M\}$. We write

$$Q = j + k + l + m.$$

First let us look at the case where A's side scores no points. The probability of pattern $J\{ABCDKLM\}$ is

$$\int_0^\infty \int_0^{2\pi} \frac{j}{\pi} e^{-jr^2} r \, d\theta_J \mathop{I}_r^\infty(a) \mathop{I}_r^\infty(b) \mathop{I}_r^\infty(c) \mathop{I}_r^\infty(d) \mathop{I}_r^\infty(k) \mathop{I}_r^\infty(l) \mathop{I}_r^\infty(m) \, dr = \frac{j}{a+b+c+d+j+k+l+m}.$$

Side A also scores zero when we swap the position of bowl J with each of the bowls K, L or M. So the total probability of A's side scoring no points is

$$\frac{Q}{a+b+c+d+Q}.$$

Let us look at the case where A's side scores 1 point. The probability of pattern $\{A\}J\{BCDKLM\}$ is

$$\int_0^\infty \int_0^{2\pi} \frac{j}{\pi} e^{-jr^2} r\, d\theta_J \overset{r}{\underset{0}{I}}(a) \overset{\infty}{\underset{r}{I}}(b) \overset{\infty}{\underset{r}{I}}(c) \overset{\infty}{\underset{r}{I}}(d) \overset{\infty}{\underset{r}{I}}(k) \overset{\infty}{\underset{r}{I}}(l) \overset{\infty}{\underset{r}{I}}(m)\, dr$$
$$= \frac{j}{b+c+d+Q} - \frac{j}{a+b+c+d+Q}.$$

Side A also scores 1 when we swap the position of bowl J with bowls K, L or M, and also if we replace A with any of the bowls B, C or D. So the total probability of A's side scoring 1 is

$$Q\left(\frac{1}{a+b+c+Q} + \frac{1}{a+b+d+Q} + \frac{1}{a+c+d+Q} \right.$$
$$\left. + \frac{1}{b+c+d+Q} - \frac{4}{a+b+c+d+Q}\right).$$

For example, when $a=b=c=d=j=k=l=m=1$ the probability is $2/7$. If $a=b=1$, $c=d=4$, $j=k=3$ and $l=m=2$, the probability is $23/76$, which is about 1.7 per cent more than $2/7$, whereas if $a=b=2$, $c=d=3$, $j=k=1$ and $l=m=4$, the probability is $44/153$, about 0.2 per cent more than $2/7$.

Let us look at the case where A's side scores 2 points. The probability of pattern $\{AB\}J\{CDKLM\}$ is

$$\int_0^\infty \int_0^{2\pi} \frac{j}{\pi} e^{-jr^2} r\, d\theta_J \overset{r}{\underset{0}{I}}(a) \overset{r}{\underset{0}{I}}(b) \overset{\infty}{\underset{r}{I}}(c) \overset{\infty}{\underset{r}{I}}(d) \overset{\infty}{\underset{r}{I}}(k) \overset{\infty}{\underset{r}{I}}(l) \overset{\infty}{\underset{r}{I}}(m)\, dr$$
$$= \frac{j}{c+d+Q} - \frac{j}{a+c+d+Q} - \frac{j}{b+c+d+Q} + \frac{j}{a+b+c+d+Q}.$$

Side A also scores 2 when we swap the position of bowl J with bowls K, L or M, and also if we replace C and D with any of the other five pairs of bowls that can be chosen from A, B, C and D. So the total probability of A's side scoring 2 is

$$Q\left(\frac{1}{a+b+Q} + \frac{1}{a+c+Q} + \frac{1}{b+c+Q} + \frac{1}{a+d+Q} + \frac{1}{b+d+Q}\right.$$

$$+\frac{1}{c+d+Q}-\frac{3}{a+b+c+Q}-\frac{3}{a+b+d+Q}-\frac{3}{a+c+d+Q}$$
$$\left.-\frac{3}{b+c+d+Q}+\frac{6}{a+b+c+d+Q}\right).$$

For example, when $a=b=c=d=j=k=l=m=1$ the probability of A's side scoring 2 is $1/7$. Another example: if $a=b=1$, $c=d=4$, $j=k=3$ and $l=m=2$, the probability is $101/684$, which is about 0.5% more than $1/7$. Another example, if $a=b=2$, $c=d=3$, $j=k=1$ and $l=m=4$ the probability is $409/2856$, about 0.04% more than $1/7$.

Let us look at the case where A's side scores 3 points. The probability of pattern $\{ABC\}J\{DKLM\}$ is

$$\int_0^\infty \int_0^{2\pi} \frac{j}{\pi} e^{-jr^2} r\, d\theta_J \underset{0}{\overset{r}{I}}(a) \underset{0}{\overset{r}{I}}(b) \underset{0}{\overset{r}{I}}(c) \underset{r}{\overset{\infty}{I}}(d) \underset{r}{\overset{\infty}{I}}(k) \underset{r}{\overset{\infty}{I}}(l) \underset{r}{\overset{\infty}{I}}(m)\, dr$$
$$= \frac{j}{d+Q}-\frac{j}{a+d+Q}-\frac{j}{b+d+Q}-\frac{j}{c+d+Q}+\frac{j}{a+b+d+Q}$$
$$+\frac{j}{a+c+d+Q}+\frac{j}{b+c+d+Q}-\frac{j}{a+b+c+d+Q}.$$

Side A also scores 3 when we swap the position of bowl J with bowls K, L or M, and also if we swap D with A, B or C. So the total probability of A's side scoring 3 is

$$Q\left(\frac{1}{a+Q}+\frac{1}{b+Q}+\frac{1}{c+Q}+\frac{1}{d+Q}-\frac{2}{a+b+Q}-\frac{2}{a+c+Q}-\frac{2}{a+d+Q}\right.$$
$$-\frac{2}{b+c+Q}-\frac{2}{b+d+Q}-\frac{2}{c+d+Q}+\frac{3}{a+b+c+Q}+\frac{3}{a+b+d+Q}$$
$$\left.+\frac{3}{a+c+d+Q}+\frac{3}{b+c+d+Q}-\frac{4}{a+b+c+d+Q}\right).$$

For example, when $a=b=c=d=j=k=l=m=1$ the probability of A's side scoring 3 is $2/35$. Another example: if $a=b=1$, $c=d=4$, $j=k=3$ and $l=m=2$, the probability is $2293/52668$, about 1.4 per cent less than $2/35$, whereas if $a=b=2$, $c=d=3$, $j=k=1$ and $l=m=4$, the probability is $1039/18564$, about 0.1 per cent less than $2/35$.

Finally, let us look at the case where A's side scores 4 points. The probability of pattern $\{ABCD\}J\{KLM\}$ is

$$\int_0^\infty \int_0^{2\pi} \frac{j}{pi} e^{-jr^2} r\, d\theta_J \underset{0}{\overset{r}{I}}(a) \underset{0}{\overset{r}{I}}(b) \underset{0}{\overset{r}{I}}(c) \underset{0}{\overset{r}{I}}(d) \underset{r}{\overset{\infty}{I}}(k) \underset{r}{\overset{\infty}{I}}(l) \underset{r}{\overset{\infty}{I}}(m)\, dr$$

$$\begin{aligned}
= {} & \frac{j}{Q} - \frac{j}{a+Q} - \frac{j}{b+Q} - \frac{j}{c+Q} - \frac{j}{d+Q} + \frac{j}{a+b+Q} + \frac{j}{a+c+Q} \\
& + \frac{j}{a+d+Q} + \frac{j}{b+c+Q} + \frac{j}{b+d+Q} + \frac{j}{c+d+Q} \\
& - \frac{j}{a+b+c+Q} - \frac{j}{a+b+d+Q} - \frac{j}{a+c+d+Q} \\
& - \frac{j}{b+c+d+Q} + \frac{j}{a+b+c+d+Q}
\end{aligned}$$

Side A also scores 4 when we swap the position of bowl J with bowls K, L or M, so the total probability of A's side scoring 4 is

$$\begin{aligned}
Q\biggl(& \frac{1}{Q} - \frac{1}{a+Q} - \frac{1}{b+Q} - \frac{1}{c+Q} - \frac{1}{d+Q} + \frac{1}{a+b+Q} + \frac{1}{a+c+Q} \\
& + \frac{1}{a+d+Q} + \frac{1}{b+c+Q} + \frac{1}{b+d+Q} + \frac{1}{c+d+Q} \\
& - \frac{1}{a+b+c+Q} - \frac{1}{a+b+d+Q} - \frac{1}{a+c+d+Q} \\
& - \frac{1}{b+c+d+Q} + \frac{1}{a+b+c+d+Q}\biggr).
\end{aligned}$$

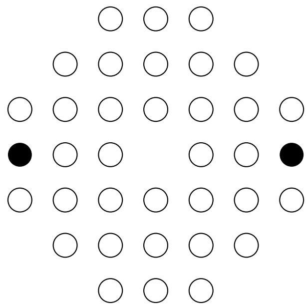
For example, when $a = b = c = d = j = k = l = m = 1$ the probability of A's side scoring 4 is 1/70. If $a = b = 1$, $c = d = 4$, $j = k = 3$ and $l = m = 2$, the probability is 325/52668, about 0.8 per cent less than 1/70, whereas if $a = b = 2$, $c = d = 3$, $j = k = 1$ and $l = m = 4$, the probability is 1475/111384, about 0.1 per cent less than 1/70.

In summary, then, if $a = b = c = d = j = k = l = m = 1$, the probabilities of A's side scoring 4, 3, 2, 1 and 0 are 1/70, 2/35, 1/7, 2/7, 1/2, respectively. (Note that they add up to 1.) The average score per end is 0.8. When $a = b = 1$, $c = d = 4$, $j = k = 3$ and $l = m = 2$, the probabilities are 325/52668, 2293/52668, 101/684, 23/76, 1/2, and the average score per end is 0.7532, whereas if $a = b = 2$, $c = d = 3$, $j = k = 1$ and $l = m = 4$, the probabilities are 1475/111384, 1039/18564, 409/2856, 44/153, 1/2 and the average score is 0.7948.

The latter two cases show that a doubles side consisting of a player of skill 2 and a player of skill 3 has an advantage over a pair of players with skills 1 and 4. However the advantage is relatively slight at only 0.0416 points per end.

Now for the second question. How long will a game of doubles last? That is, how many ends does it last on average? The game finishes when one side achieves a score of 21 or more. The shortest possible game is one lasting six ends; for example when side A scores 4 on five consecutive ends, reaching a score of 20–nil, then wins the game on the 6th end. The longest possible game is when side A and side B each score 1 on 20 occasions, reaching a score of 20–20, the game finishing on the 41st end. Thus the game can take from 6 to 41 ends. Can M500 readers work out the average number of ends? Assume that the sides are evenly matched so that the chances of a side scoring 4, 3, 2, 1 and 0 are 1/70, 2/35, 1/7, 2/7 and 1/2, respectively.

Solution 185.5 – Two pegs



In the classical 37-hole French solitaire game, vacate the central hole, mark the two pegs at opposite ends of a centre line and play to leave just these two pegs on the board *having interchanged their initial positions.*

John Beasley

Suppose the columns are labelled a, b, …, g and the rows 1, 2, …, 7; then a possible solution is

$d2 \to d4$, $b2 \to d2$, $c4 \to c2$, $c1 \to c3$, $b3 \to d3$, $e3 \to c3$, $d1 \to d3 \to b3$, $a3 \to c3$, $e1 \to e3$, $e4 \to e2$, $f2 \to d2$, $g3 \to e3$, $e6 \to e4$, $g5 \to e5$, $e4 \to e6$, $e7 \to e5$, $d5 \to f5$, $c6 \to c4$, $a5 \to c5$, $c4 \to c6$, $c7 \to c5$, $d7 \to d5 \to b5$, $\boldsymbol{a4 \to c4 \to c2 \to e2}$, $b6 \to b4$, $\boldsymbol{g4 \to e4 \to c4 \to a4}$, $f6 \to f4$, $\boldsymbol{e2 \to e4 \to g4}$.

Our federal income tax law defines the tax y to be paid in terms of the income x; it does so in a clumsy enough way by pasting several linear functions together, each valid in another interval or bracket of income. An archaeologist who, five thousand years from now, shall unearth some of our income tax returns together with relics of engineering works and mathematical books, will probably date them a couple of centuries earlier, certainly before Galileo and Viète.

—Hermann Weyl

One-edge connections

Mike Warburton

Start with squared graph paper, shade a square somewhere in the middle of the sheet and index it as zero. Shade the four adjoining edge-connected squares. index with 1. *The rule*: You can add squares that connect with earlier squares only on one edge.

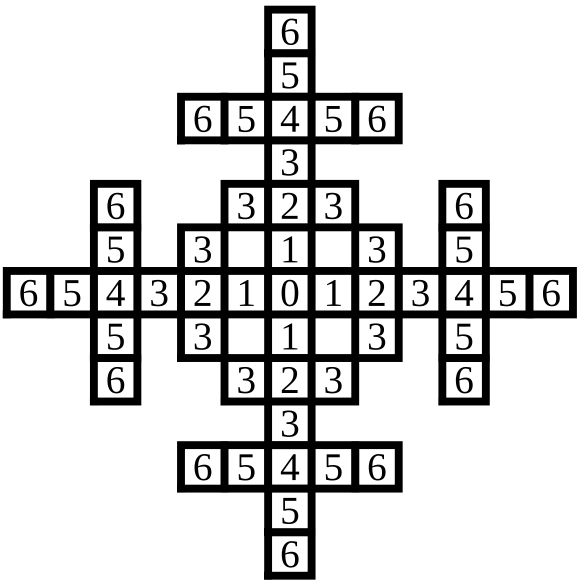
The question: What is the formula that describes the total number of shaded squares, $S(x)$, as a function of the index, x?

I have found an estimate that fits exactly when the x is an integer power of 2:

$$S(x) = \frac{4}{3}x^2 + \frac{11}{3}.$$

I sense a recursive (recursive . . .) expression. Consider replacing the square with an equilateral triangle. That 'looks' harder. Regular hexagons look easier but I'd given up by then.

I have data for squares and regular triangles up to an index of 1024, which make for some interesting plots, spiced with a hint of fractal, although I'm really not sure. The illustration below shows what happens when $x = 6$. Your thoughts are welcome.



Solution 186.5 – Horse

> A horse is tethered to the perimeter of a circular field with radius 1 kilometre. The tether allows the horse to graze all but one π-th the area of the field. How long is the tether?

Keith Drever

The rope is $\sqrt{2}$ kilometres long. If the rope is attached to the edge of the field at point A, the path that the horse will make when the rope is taut will be an arc of a larger circle which crosses the perimeter of the circular field at points B and C as shown.

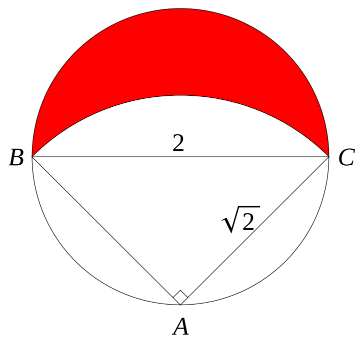
Suppose that BC is the diameter of the field. The shaded region represents the area of the field which the horse cannot reach. Since the area of the field is π square kilometres and the area of the shaded section is $1/\pi$ times the area of the field, the area of the shaded section that we want is 1 square kilometre. Since $AB = AC = \sqrt{2}$ and $\angle BAC = 90°$,

$$\begin{aligned}
&\text{area of shaded region}\\
&\quad = (\text{area of semicircle}) + (\text{area of triangle } ABC)\\
&\qquad\qquad - (\text{area of sector } ABC)\\
&\quad = \frac{\pi}{2} + 1 - \frac{90}{360}\,2\pi \;=\; 1,
\end{aligned}$$

as required.

ADF writes—We also had solutions from **David Kerr**, **Ralph Hancock** and **Jim James**. Ralph points out that a real horse tethered by a hind leg would trespass a few metres into the forbidden area. Jim was more concerned, on behalf of animal lovers, about the weight of over 1400 metres of rope which the poor horse has to drag around.

I don't know about you, but I think it's quite amazing to see the three most familiar numbers of increasing transcendence, 1, $\sqrt{2}$ and π, all occurring in one of the fundamental problems of the farming industry.



I wonder what happens if you replace π by a variable. I suspect the problem becomes unsolvable; it might be worth investigating.

Solution 186.4 – Sixteen coins

> One of 16 coins is identical to the others except for its weight. What is the minimum number of weighings that will guarantee to identify it? A *weighing* involves using a machine to tell you if the total weight of a set of coins is 'correct' or 'incorrect'. Sometimes—but no more than once in a sequence of 12 weighings—the machine gives a false answer.

Tony Forbes

Coding theorists might recognize the problem as a thinly disguised demand to construct a certain perfect 1-error-correcting code. I put it in M500 because I believed that a simple-minded solution might be possible, and for a while I thought I could find one; alas! it was not to be.

Label the coins 0, 1, ..., 15. Then do

$$W_1\text{: weigh } \{8, 9, 10, 11, 12, 13, 14, 15\},$$
$$W_2\text{: weigh } \{4, 5, 6, 7, 12, 13, 14, 15\},$$
$$W_3\text{: weigh } \{2, 3, 6, 7, 10, 11, 14, 15\},$$
$$W_4\text{: weigh } \{1, 3, 5, 7, 9, 11, 13, 15\}.$$

If we assume that these weighings are true, then they uniquely identify the bad coin. For example, if W_1 indicates 'incorrect' and the other three indicate 'correct', the coin number is narrowed down to {8, 9, 10, 11, 12, 13, 14, 15} by W_1, to {8, 9, 10, 11} by W_2, to {8, 9} by W_3 and finally to 8 by W_4. By performing this 'binary chop' three times and selecting the majority solution, it is clear that the coin can be found in 12 weighings. In the other direction, seven weighings are necessary, for otherwise we would be able to resolve 112 different possibilities—16 for the coin and 7 for the false weighing (or the absence thereof)—from only $2^6 = 64$ sets of observations.

We can get quite near to seven weighings with the following relatively straightforward solution (which I once thought would actually work for seven). Begin with W_1, W_2, W_3, W_4 as above. Let c be the coin identified by assuming W_1– W_4 are all true; let f_i be the coin identified on the assumption that W_i is the false weighing, $i = 1, 2, 3, 4$. Then do

$$W_5\text{: weigh } \{f_1,\ f_2\}, \qquad W_6\text{: weigh } \{f_3,\ f_4\},$$

and let f be the coin identified by W_5 and W_6 on the assumption that the bad coin is one of the f_i. So after six weighings we have reduced the problem to two cases:

(i) $W_1 - W_4$ are true and coin c is truly bad.

(ii) One of $W_1 - W_4$ is false, c is good and f is bad.

Now we could continue with W_7: weigh c, W_8: weigh c, and then it would be easy to decide which of $\{c,\ f\}$ is the counterfeit. Unfortunately I cannot see how to do it with just one additional weighing.

Instead we present the 'coding theory' solution in seven weighings. If you want to follow the argument, I suggest get you get yourself a pencil and some large pieces of paper. Alternatively you may wish to use a computer and your favourite matrix-bashing software.

With the same numbering of the coins as above, perform the seven weighings:

W_1, W_2, W_3, W_4: as before;

W_5: weigh $\{1, 2, 4, 7, 9, 10, 12, 15\}$;

W_6: weigh $\{1, 2, 5, 6, 8, 11, 12, 15\}$;

W_7: weigh $\{1, 3, 4, 6, 8, 10, 13, 15\}$.

Let

$$L = \begin{bmatrix} 0&0&0&1&1&1&1 \\ 0&1&1&0&0&1&1 \\ 1&0&1&0&1&0&1 \end{bmatrix}, \quad S = \begin{bmatrix} 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1 \\ 0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1 \\ 0&1&0&1&1&0&1&0&1&0&1&0&0&1&0&1 \end{bmatrix},$$

$$\begin{aligned} E_0 &= (0,0,0,0,0,0,0), & E_4 &= (0,0,0,1,0,0,0), \\ E_1 &= (1,0,0,0,0,0,0), & E_5 &= (0,0,0,0,1,0,0), \\ E_2 &= (0,1,0,0,0,0,0), & E_6 &= (0,0,0,0,0,1,0), \\ E_3 &= (0,0,1,0,0,0,0), & E_7 &= (0,0,0,0,0,0,1), \end{aligned}$$

where we adopt the usual convention of writing column vectors horizontally but separating the elements by commas rather than blank spaces. Notice that (i) the columns of L are the binary digits of the numbers 1, 2, ..., 7, and (ii) the weighing instructions are represented by S.

Let $R = (R_1, R_2, \ldots, R_7)$ be a vector of zeros and ones which represents the results of the weighings. For $i = 1, 2, \ldots, 7$, $R_i = 1$ if the machine indicates that in W_i the weight of the coins is incorrect, $R_i = 0$ if the machine indicates that the weight is correct. The main purpose of what follows next is to identify the false weighing (if any). That is, we want to map result vectors R to numbers $k \in \{0, 1, \ldots, 7\}$, where either $k > 0$ and k is the index of the false result in R, or $k = 0$ and all the results are true.

Indeed, the function $F(R) = L \cdot R$ mod 2 provides this mapping. It so happens that $F(R) = (0,0,0)$ if there are no false weighings in result vector R and otherwise weighing number $F(R)$ (in binary) of R is false. (Here we are interpreting the 3-vector (x,y,z) as the integer $4x + 2y + z$.) Once the false weighing (if any) has been located the bad coin can be determined from the results of W_1–W_4, corrected if necessary.

To see why it works, first observe that for each coin c, the result vector S_c, column c of S, identifies c on the assumption that all the weighings are true. Check that $L \cdot S_c \equiv (0,0,0) \pmod 2$. In fact, you only need to check the vectors

$$B = \{(0,0,0,1,1,1,1),\ (0,0,1,0,1,1,0),\ (0,1,0,0,1,0,1),\ (1,0,0,0,0,1,1)\}$$

for, as linear algebraists would say, S is a four-dimensional vector space over GF(2), the finite field of order 2, and it is generated (by component-wise addition modulo 2) from B. Indeed, S is the kernel of the linear mapping $L : \mathrm{GF}(2)^7 \to \mathrm{GF}(2)^3$.

Now verify that every possible result vector R can be expressed uniquely in the form $S_c + E_k$ mod 2 for some coin number c and one of the vectors E_k. I'm sorry, I can't offer a quick and easy way of performing this task. Take eight copies of S, add E_k (mod 2), $k = 0, 1, \ldots, 7$, to each column vector and you should end up with 128 *different* 7-vectors. The numbers work out exactly: eight E_k, sixteen S_c and 128 zero-one 7-tuples; hence the perfectness of this arrangement. Thus, if R is any 7-vector of zeros and ones, we have

$$F(R) \equiv L \cdot R \equiv L \cdot (S_c + E_k) \equiv L \cdot E_k \pmod 2$$

for some $c \in \{0,1,\ldots,15\}$ and $k \in \{0,1,\ldots,7\}$. Hence $F(R) = L \cdot E_k$, which has the stated property.

For example, let the bad coin be 5 and suppose the third weighing is false. Then $S_5 = (0,1,0,1,0,1,0)$ and we get the result vector $(0,1,1,1,0,1,0)$. Thus

$$L \cdot (0,1,0,1,0,1,0) \equiv (0,0,0) \pmod 2,$$

and

$$L \cdot (0,1,1,1,0,1,0) \equiv L \cdot (0,0,1,0,0,0,0) \equiv (0,1,1) \pmod 2,$$

the binary representation of 3.

Question. What is the difference between British and American calculus?

Answer. The British derive on the left.

When two fives don't make ten

Puzzling over percentages

Dilwyn Edwards

The magazine *New Scientist* recently reported the experiences of a reader who was told by a salesperson that if he chose to pay both his gas and electricity bills by direct debit he could save 5 per cent off each, or '10 per cent in total'. When he tried to explain that this was wrong he only managed to convince the caller that he was clearly mad. What made him more worried was when he repeated this to his well-educated (but non-mathematical) colleagues. Unamused, they all just looked blank and were mystified as to the point of his story. This item reminded me once again of the apparent universal lack of proper understanding of simple percentages which I have noted consistently in 30 years' teaching in higher education. From whatever school in whatever part of the country (or even the world) they come, young people seem to me to have been let down by their school teachers when it comes to percentages. I think the basic fault stems from a failure to see percentage changes as multiplications (which they naturally are) rather than additions or subtractions. Many people, when told that 'adding on 17.5 per cent VAT' is the same as multiplying the original amount by 1.175 will look puzzled. (You don't believe me?—try it!) They will also be rather hazy about the fact that 1 per cent a month is not the same as 12 per cent a year. The astonishing fact is that the same often applies to professionals working in insurance, investment and banking who actually use percentages every day of their lives and in at least one case (I'll return to this in a moment) a distinguished and highly respected professor of economics.

Once the fact is grasped that an increase of x per cent is equivalent to multiplying by $1+x/100$, it all becomes so easy that it's hard to understand why children are not taught to think like this in schools. If you invest £1000 for 5 years and receive interest at 7 per cent a year, your money is multiplied by 1.07 every year so the final amount will be $£1000 \cdot (1.07)^5$. This is very easy to do with a pocket calculator and it is just as easy to see that 1 per cent a month accumulates to $(1.01)^{12} = 1.126825$ so about 12.7 per cent a year and the monthly equivalent of 12% a year is given by $(1.12)^{1/12} = 1.009489$ i.e. about 0.95 per cent a month. The mysteries of mortgage repayments, too, all become clear.

Ask an A-Level maths student to write down in symbols the statement 'Y is five per cent greater than X'. You are unlikely to get the correct answer $Y = 1.05X$; instead something nonsensical like $Y = X + 5$ per cent appears, again because they think of addition rather than multiplication. Which is the bigger price cut, 10 per cent followed by 20 per cent or two cuts of 15 per cent?

The first is $(0.9)(0.8) = 0.72$ while the second is $(0.85)^2 = 0.7225$ so the first cuts off more. What is the average annual percentage increase if three consecutive years show increases of 10 per cent, 22 per cent and 37 per cent? Not $69/3 = 23$ per cent! We need a value x such that multiplying by $1 + x/100$ three times is the same as $(1.1)(1.22)(1.37) = 1.83854$. The cube root of this is 1.225061 so x is 22.5.

Here is another common fallacy: 'inflation at 2 per cent a year (for example) means that today's £ will be worth only 98p in one year's time'— Wrong again! for the same reason. An annual inflation of 2 per cent means that you will need £1.02 to buy the same goods in one year's time. So today's £1 will be worth 1/1.02 of its present value in a year's time, which brings me back to the professor of economics. He was writing about the rate of increase (in real terms) of house prices at a time of inflation and his calculations were all based on the *difference* between the annual percentage rise in house prices and the annual rate of inflation. Again, it is the *ratio* not the difference that measures the change *in real terms*. If your house value goes up by 10 per cent while inflation is 2 per cent the increase in the value of your house in real terms is not $10 - 2 = 8$ per cent. It is given by the ratio $1.1/1.02 = 1.078431$ so about 7.8 per cent. Will we ever learn?

Trigonometric Delights by Eli Maor

Barbara Lee

This book is written in the same style as *e*: *The Story of a Number*, reviewed in M500 **168**.

It traces the history of trigonometry from *Plimpton 322* (1800 BC) to Fourier, with plenty of mathematics such as infinite series and infinite products, epicycloids, the tangent function and the function $(\sin x)/x$. Everything you can think of in relation to trigonometry is here, always interesting, never dull.

The short items between the main chapters describe the lives and work of Regiomontanus, Viète and de Moivre, amongst others. There is also a section on Lissajous and his figures, and an excellent chapter on map-making and Mercator. The appendices contain trigonometrical formulae and special values of $\sin x$, some of which are related to nested roots and the regular pentagon.

This is one of the best mathematics books that I have come across in the last few years, and at £11 it is excellent value for money.

Problem 188.1 – Ones

Colin Davies

Throw n dice. The total score is s. What is the expected number of ones?

Interesting experiment

Tony Forbes

Try this scientific experiment. You will need a vacuum cleaner with the 'crevice tool' attached to the end of the hose, a spaghetti jar and a small amount of powdery material; pepper or household dust will do but you can experiment with other substances such as talcum powder and small sodium chloride crystals. For safety, you and anyone in your vicinity must wear appropriate protective clothing, hard hats, goggles, gloves, etc.

Put the powder in the spaghetti jar. Then insert the crevice tool of the vacuum cleaner into the top of the jar and hold it steady with one of your hands. You can use your other hand to vary the size of the air-gap between the vacuum cleaner and the rim of the jar. If you get the gap just right, you will see the powder at the bottom of the jar suddenly go into a rapid swirling motion, round and round, like a whirlwind. (The vacuum cleaner should be switched on.)

I am fascinated. What mathematical processes are at work? I would be very interested if someone can explain how the linear flow of air into the vacuum cleaner causes the powder in the jar to behave in a cyclical manner.

If you do not have a spaghetti jar, you can make do with a 500ml measuring cylinder of the kind used in a chemistry laboratory.

Problem 188.2 – Cylinder

Before you put the spaghetti jar away, here's something else you can do with it. **Colin Davies** submitted this interesting problem

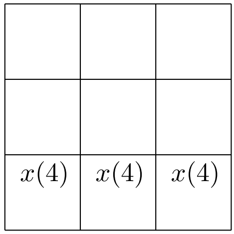
Take a cylindrical container. Gradually fill it with a liquid. It is clear that the centre of gravity of the system will start somewhere near the middle of the jar and then migrate downwards as the liquid pours in. At some point it must reach a lowest level and then move upwards because when the cylinder is full the centre of gravity is once again somewhere near the middle. *Question.* When is the centre of gravity at its lowest point?

Problem 188.3 – Window envelope

You have a window envelope of (internal) dimensions $1 \times \sqrt{2}/3$ with a window somewhere on the front of it. You also have a sheet of paper, $1 \times \sqrt{2}$, on which is printed an address in a region of the same size and shape as the window.

Is it always possible to fold the page such that it fits rigidly in the envelope, with the address viewable through the window? Of course, the size, shape and placement of the window and the address are not necessarily sensible.

Solution 186.1 – Polygon division



> Take a regular polygon of n sides and draw lines in its interior and parallel to the sides such that each line is divided into three equal segments of length $x(n)$ at the points where it intersects with the two adjacent lines. What is $x(n)$?

Ted Gore

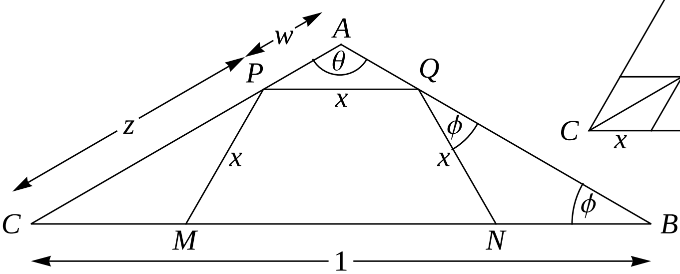
I use the triangle for illustration but the arguments apply to any regular polygon. First note that at each apex there is a rhombus of size x.

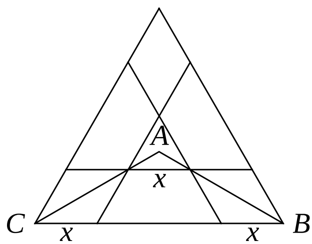
Working with the triangle ABC, $\theta = 2\pi/n$ and $\phi = (\pi - \theta)/2$; so $\sin\phi = \cos\theta/2$. Now $x = w/(w+z)$ (similar triangles APQ and ABC), $w/\sin\phi = x/\sin\theta$ (sine rule on triangle APQ) and $z/\sin\theta = x/\sin\phi$ (sine rule on triangle QBN). Therefore

$$x = \frac{\dfrac{x\sin\phi}{\sin\theta}}{\dfrac{s\sin\phi}{\sin\theta} + \dfrac{x\sin\theta}{\sin\phi}} = \frac{\sin^2\phi}{\sin^2\phi + \sin^2\theta}$$

$$= \frac{\cos^2\dfrac{\theta}{2}}{\cos^2\dfrac{\theta}{2} + 4\sin^2\dfrac{\theta}{2}\cos^2\dfrac{\theta}{2}} = \frac{1}{1 + 4\sin^2\dfrac{\theta}{2}};$$

$$x(n) = \frac{1}{1 + 4\sin^2\dfrac{\pi}{n}}.$$





Jim James

In this solution the term 'n-gon' implies any n-sided, regular convex polygon of side length s, and internal angle θ, where $n \geq 3$ and $0 < \theta < \pi$. To make the working easier, we shall use d, temporarily, in place of the given $x(n)$.

We shall also need two easily proved properties of n-gons, namely that $\theta = \pi(n-2)/n$ and their area is $ns^2(\tan\theta/2)/4$.

The sketches on the cover of this issue show the given construction for $n = 3, 4, 5, \ldots, 18$ at constant side length, s. We make the following observations.

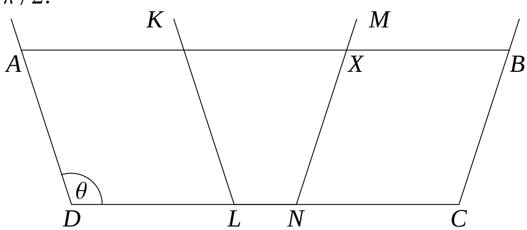
1) The length of each construction line is $3d$ and it connects points on next but one sides of the n-gon. As θ increases, so also does $3d$, but since $0 < \theta < \pi$, it follows that $3d$ is always less than $3s$, that is $d < s$, for all n-gons.

2) As a result of the construction lines being drawn parallel to the n-gon sides, their central third sections create a second n-gon, at the original n-gon's centre, having the same orientation but with side length d.

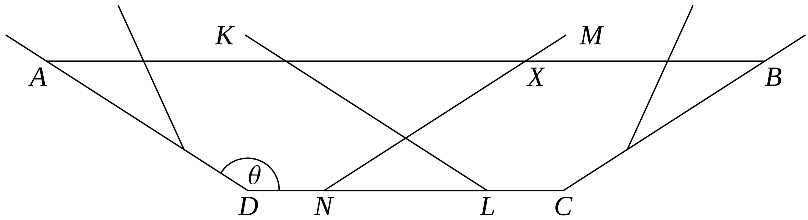
3) The region between the inner and outer n-gons is populated with a number of smaller polygons of various types and sizes. There is a geometric structure within this region, however, that is common to all n-gons. We examine the space between any one side of the outer n-gon and the construction line parallel to it. There are just two basic configurations possible, depending upon whether $d \leq s/2$ or $d > s/2$.

4) The sketch below illustrates the configuration for $d \leq s/2$, which occurs when $n \leq 6$. Lines KL and MN are the ends of the construction lines parallel to the two adjacent sides. The trapezium $AXND$ has parallel sides, AX and DN, of length $2d$ and $s-d$ respectively and the perpendicular distance between them is $d\sin\theta$. Its area is therefore $d(s+d)(\sin\theta)/2$.

By symmetry there are n such trapeziums and between them they fill the whole of the region between the inner and outer n-gons. Note that for the square, the trapeziums become rectangles simply because their interior angles are all $\pi/2$.



5) The next sketch illustrates the configuration for $d > s/2$, which occurs when $n \geq 7$. The construction lines KL and MN now overlap and this causes the ends of two further construction lines to be introduced into the space under consideration. Despite this added complication, however, we still have trapezium $AXND$, with parallel sides $2d$ and $s - d$, distance $d \sin\theta$ apart, with area $d(s + d)(\sin\theta)/2$. Again, by symmetry there are n of them and they fill the whole of the region between the inner and outer n-gons. This feature, therefore, applies to all n-gons.



6) But the area of any n-gon is $ns^2(\tan\theta/2)/4$, so the area between the inner and outer n-gons must be

$$\frac{1}{4}\,n(s^2 - d^2)\tan\frac{\theta}{2} \;=\; \frac{1}{2}\,nd(s + d)\sin\theta.$$

This equation can be simplified by putting $\frac{1}{2}\sin\theta = (\sin\theta/2)(\cos\theta/2)$ and dividing both sides by $n(s + d)\sin\theta/2$, to give the general solution

$$d \;=\; \frac{s}{(1 + 4\cos^2\theta/2)},$$

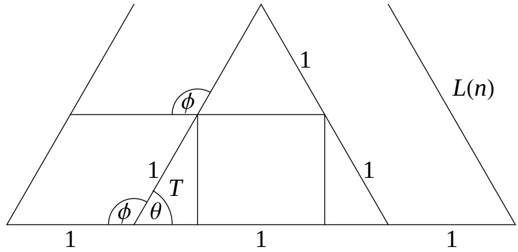
or, since $\cos\theta = 2\cos^2\theta/2 - 1$, $d = s/(3 + 2\cos\theta)$, whence

$$x(n) \;=\; \frac{s}{3 + 2\cos\dfrac{\pi(n-2)}{n}}.$$

Here are some calculated values of $x(n)$ with $s = 1$: $x(3) = 1/4$, $x(4) = 1/3$, $x(5) = 0.419821$, $x(6) = 1/2$, $x(8) = 0.630602$, $x(10) = 0.723607$, $x(100) = 0.996069$, $x(1000) = 0.999961$.

Also solved by **Basil Thompson** using a similar method.

Robin Marks



Let the length of each side of the inner polygon be 1. Drop two perpendiculars from two adjacent vertices of the inner polygon, as shown.

Consider the triangle T. Then $q(n) = \cos\theta = -\cos\phi$. Hence

$$L(n) \;=\; 1 + q(n) + 1 + q(n) + 1 \;=\; 3 - 2\cos\phi.$$

Also $\phi = 2\pi/n$; hence $L = 3 - 2\cos 2\pi/n$. Note that the length $q(n) = -\cos 2\pi/n$ is zero when $n = 4$ and negative for $n > 4$.

The problem asks for $x(n) = 1/L(n)$. Here is a table of $q(n)$, $L(n)$ and $x(n)$.

n	$q(n)$	$L(n)$	$x(n)$
3	$\frac{1}{2}$	4	0.25
4	0	3	0.333333
5	$-\frac{1}{4}\left(\sqrt{5}-1\right)$	$3-\frac{1}{2}\left(\sqrt{5}-1\right)$	0.419821
6	$-1/2$	2	0.5
8	$-\frac{1}{2}\sqrt{2}$	$3-\sqrt{2}$	0.630602
10	$-\frac{1}{4}\left(\sqrt{5}+1\right)$	$3-\frac{1}{2}\left(\sqrt{5}+1\right)$	0.723607
12	$-\frac{1}{2}\sqrt{3}$	$3-\sqrt{3}$	0.788675
16	$-\frac{1}{2}\sqrt{2+\sqrt{2}}$	$3-\sqrt{2+\sqrt{2}}$	0.867874
20	$-\frac{1}{2}\sqrt{\frac{1}{2}\left(\sqrt{5}+5\right)}$	$3-\sqrt{\frac{1}{2}\left(\sqrt{5}+5\right)}$	0.910841
24	$-\frac{\sqrt{3}+1}{2\sqrt{2}}$	$3-\frac{\sqrt{3}+1}{\sqrt{2}}$	0.936200

Letters to the Editors

Why does calculus work?

Dear Editor,

I was interested in Sebastian Hayes's piece 'Why does calculus work?' in M500 **185** because some of the issues he raises resonate with work I have been doing on the development of systems ideas in the 20th century.

In the course of my studies I came across the work of Jakob von Uexküll, a Danish vitalist, that is to say, someone from the tradition that believed life is an added ingredient to inorganic matter.

I have not yet had access to his work in the original German but he appears to have anticipated some lines of thought taken up by systems thinkers in the late 20th century.

Writing in the first decade of the 20th century, and thus well before the development of modern systems ideas, he argued that organisms have no access to an objective environment in the sense that every interaction is mediated through senses; rather the information received by the senses is processed in the *merkwelt* (or 'world of perception') to create, among other things, a subjective view of the environment.

Using the subjective view, the organism then proceeds to interact with its environment and to modify its perceptions and future interactions in the light of the feedback it receives through the senses.

His use of the word *umwelt* rather than *assenwelt* was to emphasize the subjective nature of our concept of the environment and, though his concept has gone into use across the world, this subtlety is lost in translation.

So, calculus works because it allows us to develop perceptions of our environment which, while not exact analogues, are sufficiently close for us to receive consistent feedback from our environment when we act on the basis of our perceptions.

There is plenty of evidence that organisms in many species communicate their perceptions and can develop quite sophisticated shared perceptions, along with some rather simplistic ones! The question is whether any number of shared perceptions tested repeatedly by many people would ever give us access to 'the true nature of the world we live in', whatever that is.

Forsaking Uexküll for a moment, let us suppose, as many people believe, that we have access to 'the true nature' of the world. What difference has it made? By what method and using what tests can we distinguish this 'knowledge' of an objective environment from what are merely shared perceptives based on interactions with a subjectively perceived environment?

Since we have good reason to believe that there will be no 'big bang' moving us from all states of shared perception to full knowledge of 'the true nature of the world we live in', there will presumably be a long period in which the boundaries between what is 'true knowledge' and what is just 'shared perception' will be shifted.

Yet in the meantime we will all continue to use the mathematics we have to solve problems in everyday life. Chemicals will be measured, machines will be designed, buildings will be erected and people will be tested using mathematical concepts which we use because the feedback from our interactions with our environment suggests that they are sufficiently close analogues to enable us to interact successfully on the basis of them. Isn't this what the scientist and engineer want—without getting into a debate about what is or is not a 'fact'.

No doubt we will continue to discover mathematical concepts which appear to be better bases for our interactions with our environment. No doubt mathematicians will continue to gain aesthetic satisfaction from developing and refining mathematical concepts. But as long as they work for us as organisms, does it really matter whether they are exact analogues of an objective environment? Our survival depends on their effectiveness for us as organisms interacting with our environment, not on their exact correspondence with anything in that environment.

Yours sincerely,

John Hudson

Calculus and irrational numbers

Dear Tony,

With regards to 'Why does calculus work?' by Sebastian Hayes, M500 **185**, I would like to raise a couple of points.

This article was wide in its scope in dealing with the historical issues around the calculus. It has often come to my attention that many scientists make the assumption that space is discrete, or there are no actual infinities, without stating it. To my mind we have two separate scientific models rather like separate Euclidean and non-Euclidean geometries. Nobody has proved that space is discrete experimentally. What they have proved is that there is an experimental limit to our knowledge of fine detail in space; and this is done by a formulation of the Heisenberg uncertainty principle.

It *is* possible that actual infinities exist and it is easy to see that no experiment can divide space for ever to prove it, but that does not disprove it. A well-used argument is that germs existed before there were microscopes to see them. The calculus is a good tool, as Mr Hayes says, and I am not convinced that the calculus is at fault when our models are not complicated enough to describe reality. It seems to me that this all points to a possible fundamental limit to a numerical understanding of the universe, but there is no need to blame the tools we have made so far.

It is also interesting to note that there are refutations of Zeno's paradoxes using the language of 'actual infinities'. And if there really are 'holes' in space to which a closing door does not transverse—then whereabouts are these holes located? To paraphrase Kant; it is impossible for the human mind to conceive of 'no space', it is an *a priori* concept.

Further to these arguments, I would like to add two more comments in light of M500 **186**, 'the creation of irrational numbers'. The writer suggests that irrational numbers don't exist in the real world. I would contend the opposite for the same reasons in my earlier statements. This is again based on the assumption that space is discrete and therefore there are no irrational lengths. This is possible because we could be living in a topological space which is defined only on the rational numbers, but again, no one has proved this experimentally. For an amusing refutation of that argument, read *Conceptual Physics, Matter in Motion*, Ballif & Dibble, 1969. It is not quite watertight, because of the topological space without irrational number possibility, but is very convincing:

Most mathematicians are taught about the beauty and brilliance of Euclid's *Elements* of 2000 or so years ago. If you take the Proposition of Incommensurability (Book X), you have a Greek proof essentially stating something like a hypotenuse and a side of a triangle cannot be measured with the same measure (they did not like irrational lengths either). This leads us to the logical absurdity of never being able to draw a right-angled triangle. It is interesting that the scientists who refute infinite mathematics and continuity are quite happy to rely on the legacy of Euclid, which Newton used extensively in his *Principia*, may I add. It is possible to be as happy with the reality of the irrational numbers, and people do argue for the reality of complex numbers. After all, what is a number? You could argue that integers don't exist in reality; show me the number 3 in reality. All you can show me is 3 objects or a representation of this on paper. You can't actually point to the number 3. If you call it a distance, then you are still dodging the issue, because the number is representing the distance and it is clear it is far more than that, so point to it. As Piaget would argue, we gain our understanding of integers by being shown different sets, say of 3 objects, and extrapolate to this very abstract idea of number. Maybe the problem is also one of being unable to be explicit in our definition of what we mean by a number. The same goes for time (M500 **186**, 'Time'). We have immense difficulty pinning down what we mean by time. Again, as Kant said, time is another *a priori* concept, and I add that we happen to have a word for it, with which we play Wittgenstein type games.

Sheldon Attridge

Thirteen

Dear Eddie

Just got the answer to this, I think. [M500 **186**, 13]: Two French mathematicians. One knows the first letter and the other the last letter of the name of an integer between 1 and 22. The first mathematician says to the other, 'Je connais la lettre premiere du nombre mais pas le nombre.' The other replies, 'Je connais la lettre dernier du nombre mais pas le nombre.' The first replies, 'Ah! Je connais le nombre!' What number is it?]

TREIZE. The first mathematician has T, which can stand only for trois or treize. This is the only initial letter in the sequence that can stand for two numbers, all the rest being unique or standing for three or more. The second mathematician must have E, because trois is the only number ending in S, so if he had S he would know the number.

Best wishes,

Ralph Hancock

ADF—Interesting. That's the answer I got but I had to work harder. I listed the integers from *un* up to *vingt deux* and then I crossed out all those which were inconsistent with the conversation.

With regard to that other problem, the one about the two French-speaking Russians ('Deux nombres', M500 **185**, 25: There are two Russian mathematicians; one knows the product and the other the sum of two integers between 2 and 100. The first says to the other, "I know the product of the two numbers, but not the two numbers." The other replies, "I know the sum but not the two numbers." Then the first says, "I have just found them!" And the other replies, "Me too!"), I confess that I am puzzled. I have seen several attempts, including my own, and I am unable to arrive at a satisfactory answer. Is there anyone out there who can speak with authority? Perhaps something was lost in the translation. When I tried working the problem I found that could not avoid a non-unique answer. I shall continue to ponder.

By the way, I actually had someone question the interpretation of the word 'between'. Does 'between 2 and 100' mean [2, 100] or [3,99]? In an office where I once worked I would often preach to authors of management reports that there was never any serious ambiguity. Nor was it ever necessary to add the qualifier 'inclusive' to constructs such as 'from a to b'. If the space is discrete, surely it is perverse to omit the end-points from consideration; and in a continuous space it doesn't matter one way or the other because the set $\{a,\ b\}$ is null.

Solution 184.1 – Twelve boxes

> There are twelve closed boxes numbered 1, 2, ..., 12. On each turn you throw a pair of dice and you must open closed boxes whose numbers add up to the sum of the numbers shown by the dice. If this is impossible, the game stops and you lose. If you manage to open all the boxes, the game stops and you win. If neither, the game continues. What's the probability of winning?

Dick Boardman

It seems to me that the probability of winning depends on the strategy used by the player. If we slightly rearrange the example given in the statement of the problem, let the sequence of dice throws be

$$\{3,3\},\ \{6,6\},\ \{6,6\},\ \{6,6\},\ \{6,6\},\ \{6,6\},\ \{6,6\}.$$

If the player opens boxes 6, 12, {11, 1}, {10, 2}, {9, 3}, {8, 4} and {7, 5}, he wins. However, if he opens {4, 2}, 12, {11, 1}, {9, 3}, {7, 5}, he loses since the remaining closed boxes {6, 8, 10} cannot total 12.

I wrote a simulation with a strategy as follows:–

If possible, open exactly one box.
Else if possible open exactly 2 boxes.
Else if possible open exactly 3 boxes.
Else if possible open exactly 4 boxes.
Else lose unless all boxes open.

This strategy wins about 34 games per 10000. Can anyone devise a better one?

ADF

It is an improvement on a simulation of mine. When N is thrown I open boxes by invoking Openbox(N), where the procedure is defined by:

Openbox(X)
If box X is shut, open it and return(OK).
Otherwise for $I = 1, 2, \ldots, [(X-1)/2]$:
  Save the state of the boxes.
  If Openbox(I) = OK and Openbox($X - I$) = OK, return(OK).
  Restore the state of the boxes.
Return(FAIL)

Elegant it may be, but this strategy achieves only about 0.0030.

Ron Potkin

The probability of opening the twelve boxes is 0.003384 approximately. In order to win the game, three obstacles have to be overcome.

(1) Select those sequences that will open the boxes. We do this by ignoring the order in which the dice are thrown. For example, if we are given the complete sequence {12, 12, 12, 12, 12, 11, 7} it will be possible to open all of the boxes.

(2) If we obtain a sequence that will open all the boxes, we cannot be sure that a win can be achieved because boxes must be opened after each throw. A strategy has to be selected which will determine how the boxes should be opened in order to keep the odds as high as possible. The strategy selected is

(a) on each throw, open as few boxes as possible. So, if 10 is thrown and box 10 is still closed, open it;

(b) if the box is already opened then work inwards. For example, if box 10 is opened then open {9, 1}. If 9 or 1 is opened then open {8, 2}; if 8 or 2 is opened then try {7, 3}, and so on. Opening more than two boxes such as {1, 3, 6} or {1, 2, 3, 4} must be the last resort.

I don't think this strategy can be improved but I would be pleased to hear of a better one.

(3) Test every combination of throws that satisfy (1) against the strategy. The sequence {12, 12, 12, 12, 12, 11, 7} has 42 combinations of which only 14 are successful.

All sets from 2-sided dice and four boxes to 6-sided dice and 12 boxes were tested. The results are shown below.

The first two rows can be calculated with pencil and paper but for the last three one must resort to the computer. The last case required several hours of computing time.

Dice sides	Boxes	Number of successful throws	Probability of satisfying the strategy	Probability of throws
2	4	2	0.234375	0.234375
3	6	7	0.073972...	0.073972...
4	8	29	0.029616...	0.027699...
5	10	131	0.011188...	0.009674...
6	12	636	0.004453...	0.003384...

Problem 188.4 – Sixteen tarts

Tony Forbes

There are 16 indistinguishable jam tarts. Some jam has been removed from one and put back into another so that fourteen weigh the same, one weighs a bit more and one a bit less. Devise a scheme to find the light and heavy tarts in five weighings.

By a *weighing* we mean the process of selecting two sets of tarts, A and B, and determining whether A is lighter than B, A weighs the same as B, or A is heavier than B. Alternatively, you can imagine that the weighings are done with one of those simple two-pan-and-pointer devices which were so common a long time ago but are nowadays found only in the kitchens of those who can justify spending the exorbitant prices charged by up-market department stores. Of course, they do have the advantage over machines like the one in 'Sixteen coins' (p. 13)—they never make mistakes.

I claim no prize for originality; **Dick Boardman** sent us 'Nine tarts' for M500 **182** and a solution appeared in M500 **184**. However, I think 'Sixteen tarts' is of special interest because it seems to be the most difficult of the n-tart problems. When $n = 16$, there are $16 \cdot 15 = 240$ choices for the heavy and light tarts, and five weighings provide $3^5 = 243$ sets of observations. The system has a slack of only 3. As a further amusement, you might like to show that things never again get as tight.

Prove that if $w > 2$, the minimum non-negative value of $3^w - n(n-1)$ is 3, which occurs at $w = 5$, $n = 16$.

M500 Winter Weekend 2003

The twenty-second M500 Society WINTER WEEKEND will be held at **Nottingham University** from **Friday 3 to Sunday 5 January 2003**.

This is an annual residential weekend to dispel the withdrawal symptoms due to courses finishing in October and not starting again until February. It is an opportunity to get together with friends, old and new, and do some interesting mathematics. It promises to be as much fun as ever!

Cost: £155 for M500 members, £160 for non-members. This includes accommodation and all meals from dinner on Friday to lunch on Sunday. Please send a stamped, addressed envelope for a booking form to

Norma Rosier.

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