*

ISSN 1350-8539

M500 189

33267206735443197356651465091271496840063761409618
75870891419221266902387143114128623035533789457982
75752037706697836859777596374478310646941298993893
91530356694718313599175941296123697178103802627343
03909720226166725134828429710196830212759004184171
44903573153569238136751555145176613075183766389335
98297473106916527164722688613058992159137955039225
33169221442419625863660398048641683966468214647409
91150778758840182385169420861094525879108561714153
36162082354196692347160857298741154912372749922393
40413374225364981931382305551444489434539056120741
87346250066683823610863785377926260875147605262304
98279952714131841896881021825644372095309497933764
59534083606134443919612619864247647773483402233339
41876583714212779617441349648225550097247158488548
54923216377958376759658610489971853146340227018039
45222309611809596121641701618545779342557738521950
41741189592730186426255705998985505515170921935737
50106037900368072523069288557346559075362116934081
32546240991220234257748688929696079015903960518025
9427890928720033411211993331280941

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

Solution 181.4 – Four points

> Choose two points inside a given circle and draw the line segment joining them. Then randomly select another two points inside the same circle and draw the line segment joining these two points.
>
> What is the probability that the two line segments intersect?

Robin Marks

First we prove that a quadrilateral drawn at random inside a square has probability 25/36 of being convex.

This problem was posed by J. J. Sylvester in the middle of the 19th century. He stated the problem as follows.

> Find the probability that four points P_0, P_1, P_2, P_3 chosen at random inside a convex set K form a convex quadrilateral; that is, that none of the points is inside the triangle made by the other three.

To solve this, consider the complementary probability that the quadrilateral is not convex. This can occur in four different ways, according to which of the points P_i occurs inside the triangle made by the other three.

The measure of the set of quadrilaterals such that the specific point P_3 lies inside the triangle made by the specific points P_0, P_1 and P_2 is

$$T_2 \;=\; \int_{P_i \in K} T(0,1,2)\, dP_0 \wedge dP_1 \wedge dP_2,$$

where $T(0,1,2)$ denotes the area of the triangle made by the points P_0, P_1 and P_2. The '2' in T_2 denotes that we are working in two dimensions. The wedge signs ($\wedge$) denote what is known as a wedge product (or outer product or exterior product) of differential forms—this is too complicated to explain here. It means that we are finding the integral over all possible cases of the positions of the three vertices within K.

So, for example, in the case where K is a square of side 1 we get

$$T_2(\text{square}) \;=\; \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 T(0,1,2)\, dy_2\, dy_1\, dy_0\, dx_2\, dx_1\, dx_0,$$

where the co-ordinates of the points are (x_0, y_0), (x_1, y_1), and (x_2, y_2).

This expression is equal to the *probability* that the point P_3 lies within the triangle lying inside the square. (Note that the expression also gives the average area of a random triangle within a unit square.)

So the total probability that any one of P_0, P_1, P_2 and P_3 lies within the triangle made by the other three is $T_2 + T_2 + T_2 + T_2 = 4T_2$. Hence the probability that none of P_0, P_1, P_2 and P_3 lies within the triangle made by the others is $1 - 4T_2$.

The calculation of the value of the expression for T_2 turns out to be very difficult! It took many years before the answer was worked out. This was done by breaking the 6-dimensional integration into hundreds of individual parts and calculating each separately. The individual parts are typically very complicated, involving logarithms and many other functions. The main part of the calculation, repeated recently by Michael Trott on a Pentium 200MHz computer, using the program MATHEMATICA, took over 5 hours. Setting the problem up on the computer requires a high level of mathematical sophistication.

The answer, when K is a unit square, is $T_2(\text{square}) = 11/144$. This corresponds to a probability of

$$1 \;=\; 4T_2(\text{square}) \;=\; \frac{25}{36}$$

that a random quadrilateral inside a unit square is convex.

As mentioned in M500 **185** page 11, we can calculate the probability that pairs of lines drawn between random points in a unit square intersect. This is one third of the ways of choosing a pair of points from the four in a convex quadrilateral, that is,

$$\frac{1 - 4T_2(\text{square})}{3} \;=\; \frac{25}{108}.$$

Values for T_2 are known for all regular polygons, and also for the circle. The general formula for regular polygons was calculated by H. A. Alikoski in 1939. The formula is

$$\frac{9(\cos 2\pi/n)^2 + 52\cos 2\pi/n + 44}{36n^2(\sin 2\pi/n)^2}.$$

Some selected values are shown in the table on the next page.

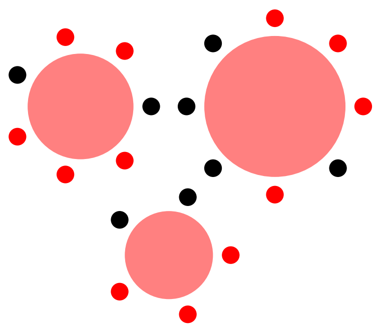
	Random triangle area: T_2	Pr(quadrilateral is convex)	Pr(two lines intersect)
Triangle	$\frac{1}{12}$	$\frac{2}{3}$	$\frac{2}{9}$
Square	$\frac{11}{144}$	$\frac{25}{36}$	$\frac{25}{108}$
Pentagon	$\frac{9+2\sqrt{5}}{180}$	$\frac{36-2\sqrt{5}}{45}$	$\frac{36-2\sqrt{5}}{135}$
Hexagon	$\frac{289}{3888}$	$\frac{683}{972}$	$\frac{683}{2916}$
Ellipse	$\frac{35}{48\pi^2}$	$1-\frac{35}{12\pi^2}$	$\frac{1}{3}-\frac{35}{36\pi^2}$

The values for the circle are the same as those for the ellipse.

ADF writes—Notice that the values of Pr(two lines intersect) for the square and the circle differ by about 1.4 per cent. So we can forgive ADF for suggesting in M500 **185** (p. 11) that they might be 'more or less exactly the same'.

Problem 189.1 – Neighbours

Tony Forbes



Some people are sitting at round tables in a restaurant, at least three to a table. Partition the diners into two sets, M and W, of m and w persons, respectively.

Show that the number of M–M neighbours minus the number of W–W neighbours is equal to $m - w$.

Notice that it doesn't work if you allow two- or single-seat tables unless you are willing to count M–M and W–W neighbours at such tables in a slightly bizarre manner.

Solution 184.7 – Deux nombres

> There are two Russian mathematicians, M1 and M2; M1 knows the product and M2 the sum of two integers between 2 and 100.
>
> S1: M1 says, 'I know the product but not the numbers.'
>
> S2: M2 says, 'I know the sum but not the numbers.'
>
> S3: M1 says, 'I know the numbers.'
>
> S4: M2 says, 'I know the numbers.'

Tony Forbes

I think I have arrived at a final but not entirely satisfactory solution. There are *two* possible candidates for the unknown numbers: $\{80, 85\}$ and $\{84, 88\}$. Here I am assuming that the numbers a and b satisfy $2 \le a < b \le 100$; in other words, the end-points of the interval are included and 'deux' is the cardinality of the set $\{a, b\}$. However, no other reasonable interpretation of the parameters gives rise to a unique answer.

David Kerr and **Stan Gondhawk** allowed the possibility of two equal numbers and obtained one of the solutions, $\{2, 6\}$. The only other solution that satisfies these conditions is $\{84, 88\}$. If we omit the end-points, $2 < a < b < 100$ leads to *three* solutions, $\{3, 8\}$, $\{72, 92\}$ and $\{75, 96\}$, the last of which was found by **Geoff Corris**. Observe the lack of continuity; a small change in the parameters results in a totally different answer! And if we allow equality, as in $2 < a \le b < 100$, we obtain $\{3, 8\}$, $\{72, 92\}$ and $\{72, 98\}$.

Chris Pile, who can only marvel at the instant enlightenment of the two Russians, informs me that the problem appeared in *IEE News* and there the answer was given as $\{4, 13\}$. I am unhappy because it is not included in any of the solutions given above. For instance, if we assume that $2 \le a < b \le 100$, M1 sees 52 and cannot decide between $2 \cdot 26$ and $4 \cdot 13$; this is consistent with statement S1. Then M2 sees 17 and cannot decide between $2 + 15$, $3 + 14$, $\ldots$, $8 + 9$; so S2 is true. But now M1 still cannot decide between $\{2, 26\}$ and $\{4, 13\}$; M1 cannot eliminate $\{2, 26\}$ because it, as well as $\{4, 13\}$, is consistent with both S1 and S2. Hence S3 is false.

Of course, it is intellectually satisfying to solve a mathematical puzzle with pure human brain-power, but here I have to admit defeat. After a number of false starts I concluded that the only reliable way to solve this problem is the sledgehammer approach. I examined all 4851 pairs of numbers $\{a, b\}$, $2 \le a < b \le 100$, and I rejected those which were inconsistent with the mathematicians' conversation. The two that remained, $\{80, 85\}$ and $\{84, 88\}$, must therefore be the only valid solutions. I give the details for $\{80, 85\}$ below and I gladly leave $\{84, 88\}$ for the reader to analyse.

During my investigations I discovered that for some sets of parameters the problem does have a unique answer. As we have seen, the solution depends sensitively on the limits. Let us fix the lower limit at 2 and for definiteness let us state that the two numbers must be different. Then there is a unique answer if the upper limit is one of the following:

15: $\{4, 5\}$,	21: $\{14, 18\}$,	26: $\{18, 20\}$,	27: $\{18, 21\}$,	28: $\{16, 27\}$,
29: $\{16, 27\}$,	32: $\{20, 27\}$,	33: $\{20, 27\}$,	34: $\{22, 27\}$,	35: $\{25, 28\}$,
38: $\{27, 32\}$,	45: $\{33, 40\}$,	46: $\{33, 40\}$,	47: $\{33, 40\}$,	48: $\{36, 40\}$,
49: $\{35, 42\}$,	50: $\{x, y\}$,	51: $\{33, 48\}$,	54: $\{42, 45\}$,	55: $\{44, 45\}$,
56: $\{40, 54\}$,	57: $\{36, 56\}$,	58: $\{36, 56\}$,	59: $\{36, 56\}$,	63: $\{48, 50\}$,
64: $\{48, 56\}$,	65: $\{48, 56\}$,	66: $\{48, 63\}$,	67: $\{48, 63\}$,	68: $\{48, 65\}$,
76: $\{54, 64\}$,	77: $\{54, 64\}$,	78: $\{65, 66\}$,	79: $\{65, 66\}$,	90: $\{75, 78\}$,
92: $\{72, 80\}$,	93: $\{72, 80\}$,	94: $\{72, 80\}$,	95: $\{72, 80\}$.	

To provide some form of ongoing entertainment I leave it for you to work out the numbers when the upper limit is 50.

Here is the reasoning for $\{80, 85\}$ in the original problem. To make it easier to follow, I have arranged that the indentation of a paragraph is proportional to the depth of thought of the mathematician under consideration. By *unique factorization*, I mean that the number in question, n, has only one possible factorization $n = a \cdot b$ with the restriction $2 \leq a < b \leq 100$.

We can assume that both mathematicians are male.

- M1 sees 6800 and thinks:
  - I see $6800 = 68 \cdot 100 = 80 \cdot 85$.
  - I cannot determine the numbers. Hence my statement, S1.
- M2 sees 165 and thinks:
  - I see $165 = 65 + 100 = 66 + 99 = \cdots = 81 + 84 = 82 + 83$.
  - Suppose the numbers are 65 and 100. Then M1 would have thought:
    - I see 6500 has unique factorization $65 \cdot 100$.
    - Therefore I can determine the numbers.
  - But he didn't. Hence I can reject $\{65, 100\}$.
  - Similarly, I can reject all pairs that sum to 165 except $\{69, 96\}$ and $\{80, 85\}$.
  - Suppose the numbers are 69 and 96. Then M1 would have thought:
    - I see $6624 = 69 \cdot 96 = 72 \cdot 92$.
    - I cannot determine the numbers. Hence S1.
  - Suppose the numbers are 80 and 85. Then M1 would have thought:

I see $6800 = 68 \cdot 100 = 80 \cdot 85$.
I cannot determine the numbers. Hence S1.
Hence the numbers could be $\{80, 85\}$ or $\{69, 96\}$.
I cannot determine the numbers. Hence S2.
M1 sees 6800 and thinks:
I see $6800 = 68 \cdot 100 = 80 \cdot 85$.
Suppose the numbers are 68 and 100. Then M2 would have thought:
I see $168 = 68 + 100 = 69 + 99 = \cdots = 82 + 86 = 83 + 85$.
Suppose the numbers are 68 and 100. Then M1 would have thought:
I see $6800 = 68 \cdot 100 = 80 \cdot 85$. I cannot determine the numbers. Hence S1.
Hence $\{68, 100\}$ is possible.
Suppose the numbers are 69 and 99. Then M1 would have thought:
I see $6831 = 69 \cdot 99$ has unique factorization.
I can determine the numbers.
But he didn't. Hence I can reject $\{69, 99\}$.
Similarly, I can reject all the other pairs that sum to 168.
Hence the only possibility is $\{68, 100\}$. I can determine the numbers.
But he didn't. Hence I can reject $\{68, 100\}$.
Suppose the numbers are 80 and 85. Then M2 would have thought:
I see $165 = 65 + 100 = 66 + 99 = \ldots = 81 + 84 = 82 + 83$.
Suppose the numbers are 65 and 100. Then M1 would have thought:
I see $6500 = 65 \cdot 100$ has unique factorization.
I can determine the numbers.
But he didn't. Hence I can reject $\{65, 100\}$.
Similarly, I can reject all pairs that sum to 165 except $\{69, 96\}$ and $\{80, 85\}$.
Suppose the numbers are 69 and 96. Then M1 would have thought:
I see $6624 = 69 \cdot 96 = 72 \cdot 92$.
I cannot determine the numbers. Hence S1.
Suppose the numbers are 80 and 85. Then M1 would have thought:
I see $6800 = 68 \cdot 100 = 80 \cdot 85$.
I cannot determine the numbers. Hence S1.
Hence the numbers could be $\{80, 85\}$ or $\{69, 96\}$.
I cannot determine the numbers. Hence S2.
Hence the numbers could be $\{80, 85\}$.
Hence the numbers must be $\{80, 85\}$. Hence S3
M2 sees 165 and thinks:
I see $165 = 65 + 100 = 66 + 99 = \cdots = 81 + 84 = 82 + 83$.
I have already rejected all pairs that sum to 165 except $\{69, 96\}$ and $\{80, 85\}$.
Suppose the numbers are 69 and 96. Then M1 would have thought:
I see $6624 = 69 \cdot 96 = 72 \cdot 92$.
Suppose the numbers are 69 and 96. Then M2 would have thought:
I see $165 = 65 + 100 = 66 + 99 = \cdots = 81 + 84 = 82 + 83$.
Suppose the numbers are 65 and 100. Then M1 would have thought:
I see $6500 = 65 \cdot 100$ has unique factorization.
I can determine the numbers.
But he didn't. Hence I can reject $\{65, 100\}$.

Similarly, I can reject pairs that sum to 165 except $\{69, 96\}$ and $\{80, 85\}$.
Suppose the numbers are 69 and 96. Then M1 would have thought:
I see $6624 = 69 \cdot 96 = 72 \cdot 92$.
I cannot determine the numbers. Hence S1.
Suppose the numbers are 80 and 85. Then M1 would have thought:
I see $6800 = 68 \cdot 100 = 80 \cdot 85$.
I cannot determine the numbers. Hence S1.
Hence the numbers could be $\{80, 85\}$ or $\{69, 96\}$.
I cannot determine the numbers. Hence S2.
Hence the numbers could be $\{69, 96\}$.
Suppose the numbers are 80 and 85. Then M2 would have thought:
I see $165 = 65 + 100 = 66 + 99 = \cdots = 81 + 84 = 82 + 83$.
I can reject all pairs that sum to 165 except $\{69, 96\}$ and $\{80, 85\}$.
I cannot determine the numbers. Hence S2.
Hence the numbers can be $\{80, 85\}$ or $\{69, 96\}$.
I cannot determine the numbers.
But he did determine the numbers.
Hence I can reject $\{69, 96\}$.
Therefore the numbers must be $\{80, 85\}$.
I can determine the numbers. Hence S4.

Problem 189.2 – Brown eyes

One day the elders of a village issued the following order:

> If you discover that you have brown eyes, you must take the 12:00 train on the next day and leave this village permanently.

Previously, nobody knew or cared about the colour of their eyes and, as you can imagine, after that terrible edict nobody wanted to know! People avoided mirrors and stopped communicating with each other.

Nothing happened for a few years. Then one evening a passing tourist announced to everybody that he had seen a brown-eyed inhabitant of the village. As a consequence of the elders' order, ten days later all the brown-eyed people left on the noon train.

Explain.

Errata

'The Fibonacci series' by Sebastian Hayes, M500 **187**, contains a few minor errors. Page 6: line -7 should read '... In this way we can define a whole family of solutions, with ϕ_n converging to 2 as n goes to infinity'. Page 7: change $F_{n+1} - 2$ on line 8 to $n - 1$, change F_{n+1} on line 9 to $n + 1$, and change $F_{n+1-r} - 2$ on line 13 to $n - 1 - r$.

Solution 186.5 – Horse

A horse is tethered to the perimeter of a circular field with radius 1 kilometre. The tether allows the horse to graze all but one π-th the area of the field. How long is the tether?

What happens if you replace $1/\pi$ by a variable?

Simon Geard

I first solved the ordinary problem (given the length of the rope, find the area of the field) as a sixth-form A-level student back in 1976. Since then this type of problem has occupied a special place in my heart—so, armed with a pen, a pad of paper, my trusty HP15-C calculator and a fresh set of batteries, I came on holiday here to the Cotswolds ready to do some serious relaxation.

My first approach was to use my original 'blood & guts' method. In the diagram, the field has equation

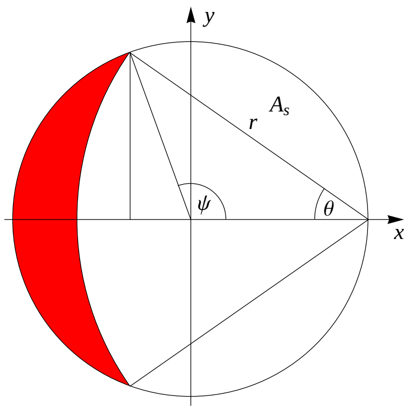
$$x^2 + y^2 \;=\; 1$$

and the area which the horse can access has equation

$$(x-1)^2 + y^2 \;=\; r^2.$$

Solving these shows that $x = 1 - r^2/2$ is the value of x at which the horse is standing on the perimeter of the field. We now use calculus to work out the grazeable area. Thus

$$\begin{aligned} A \;&=\; 2\int_{1-r}^{1-r^2/2} \sqrt{r^2-(x-1)^2}\,dx \\ &\quad +\; 2\int_{1-r^2/2}^{1} \sqrt{1-x^2}\,dx \\ &=\; 2\int_{-r}^{-r^2/2} \sqrt{r^2-t^2}\,dt \\ &\quad +\; 2\int_{1-r^2/2}^{1} \sqrt{1-x^2}\,dx \end{aligned}$$



$$= \left[t\sqrt{r^2-t^2}\right]_{-r}^{-r^2/2} + r^2\left[\sin^{-1}\frac{t}{r}\right]_{-r}^{-r^2/2} + \left[x\sqrt{1-x^2}\right]_{1-r^2/2}^{1} + \left[\sin^{-1}x\right]_{1-r^2/2}^{1},$$

which I admit does look horrendous; but with some careful algebra it reduces to

$$A = \frac{\pi}{2}(r^2+1) - r^2 \sin^{-1}\frac{r}{2} - \sin^{-1}\left(1-\frac{r^2}{2}\right) - r\sqrt{1-\left(\frac{r}{2}\right)^2}. \qquad (1)$$

For this particular problem the horse can eat all but $1/\pi$ of the area of the field. Thus we need to solve

$$\frac{\pi}{2}(r^2-1) + 1 - r^2 \sin^{-1}\frac{r}{2} - \sin^{-1}\left(1-\frac{r^2}{2}\right) - r\sqrt{1-\left(\frac{r}{2}\right)^2} = 0.$$

So I programmed the equation into my calculator, tried 2 and $\sqrt{2}$ as special values just to make sure that my programming was correct, and found that $\sqrt{2}$ is a solution. So I didn't need the SOLVE button after all!

The solution $r = \sqrt{2}$ corresponds to the case $\theta = \pi/4$ in the diagram. So I began to wonder if there was a simpler geometric solution. In fact, $A = \frac{1}{2}r^2 2\theta + 2A_s$, where $A_s = \frac{1}{2}(\psi - \sin\psi)$. Hence

$$A = r^2\theta + \psi - \sin\psi = (r^2-2)\theta + \pi - \sin 2\theta.$$

Simple trigonometry gives $r = 2\cos\theta$. Therefore

$$A = \pi + 2\theta\cos 2\theta - \sin 2\theta. \qquad (2)$$

For this particular problem, $A = \pi - 1$; thus we need to solve

$$1 + 2\theta\cos 2\theta - \sin 2\theta = 0.$$

It is, I think, easier to spot that $\theta = \pi/4$ is a solution to this than it was that $r = \sqrt{2}$ was a solution to the corresponding r equation.

Equations (1) and (2) should be the same; but are they? If you make use of $r = 2\cos\theta$ and note that $\sin^{-1} r/2 = \pi/2 - \theta$ and $\sin^{-1}(1 - \frac{1}{2}r^2) = 2\theta - \pi/2$, then (1) becomes

$$\begin{aligned} A &= \frac{\pi}{2}(4\cos^2\theta + 1) - 4(\cos^2\theta)\left(\frac{\pi}{2} - \theta\right) - 2\theta + \frac{\pi}{2} - 2\sin\theta \\ &= \pi + \pi(1+\cos 2\theta) - 2(1+\cos 2\theta)\left(\frac{\pi}{2} - \theta\right) - 2\theta - \sin 2\theta \\ &= \pi + 2\theta\cos 2\theta - \sin 2\theta, \end{aligned}$$

as required. The circle is complete!

Do men and women design gardens differently?

Nick Pollock

On the BBC TV show *Gardeners' World*, Rachel claimed that she could tell whether a garden had been designed by a man or a woman, so Joe set up a test. He got two men and two women to plant a small garden each, and then asked Rachel to look at the gardens and say whether each had been designed by a man or a woman.

She got two out of four right. How significant a result is this?

Let the gardens be G_1–G_4. Then there are six ways, w_1–w_6, in which they could be designed.

	G_1	G_2	G_3	G_4
w_1	m	m	f	f
w_2	m	f	m	f
w_3	m	f	f	m
w_4	f	m	m	f
w_5	f	m	f	m
w_6	f	f	m	m

Suppose the gardens were actually designed in way w_1. Then the accuracy of each possible choice is given by the following table.

	w_1	w_2	w_3	w_4	w_5	w_6
Accuracy	100%	50%	50%	50%	50%	0%

The pattern is clearly similar however the gardens were actually designed.

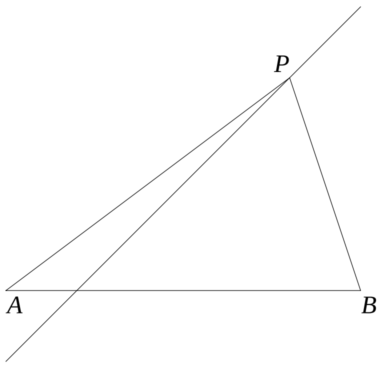
So by guessing whether each garden was designed by a man or a woman, there is an 83 per cent chance of being 50 per cent right or more!

... unplug the fax machine from the AC wall socket and telephone Jack before clearing'—instructions in a fax user guide. [Spotted by EK]

Moving point

Dilwyn Edwards

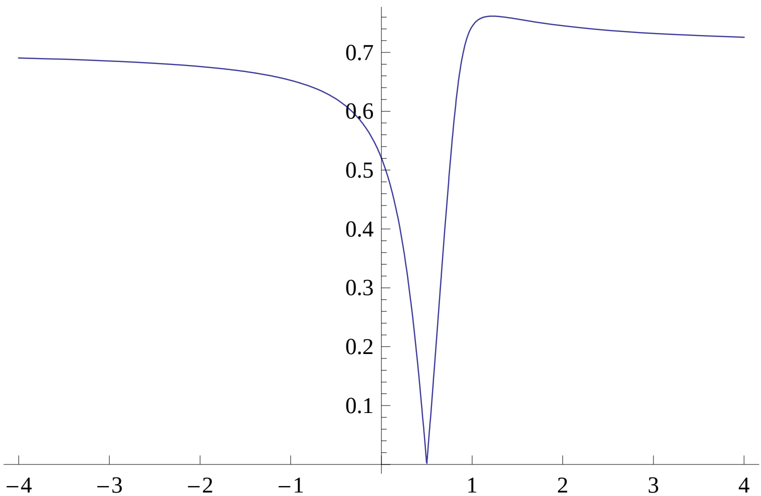
How is your mathematical intuition? You could check it out on the following simple problem.



A point P moves on a straight line which passes between two fixed points A and B. Think about the difference between the two distances PA, PB. Can you picture the graph of $d = |PA - PB|$ as P moves?

Personally I have to admit to getting it wrong. It is obvious that d falls to zero where the line cuts the perpendicular bisector of AB, so I imagined it as a roughly U-shaped curve. My faulty intuition did not tell me that there will also be a local maximum.

Here is a particular example for which $A = (0, 0)$, $B = (1, 0)$ and the line is $y = x - 0.7$. I have plotted d against x. The local minimum at $x = 0.5$ is very sharp but there is also a much less sharp local maximum at about $x = 1.22$. For large $|x|$ values, d approaches a limiting value of about 0.7. What does this limit depend on? Can you find an expression for it in terms of the original configuration, and also predict the local maximum?



Titanic prime quintuplets

Tony Forbes

As any chemist will tell you, the word 'titanic' usually means 'pertaining to the tetravalent state of titanium, Ti, the 22nd element in the periodic table.' However, as applied to prime numbers the adjective has a specific and entirely different meaning, coined by Samuel Yates in 1985.

> A *titanic* prime is defined as a prime number which has at least 1000 decimal digits.

This same definition is used by Chris Caldwell in his database of large primes at www.utm.edu/research/primes/largest.

The first titanic prime was discovered by Alexander Hurwitz in 1961. He had programmed the computer to search for Mersenne primes and left it running overnight. When he looked at the results he found two, namely $2^{4253}-1$ and $2^{4423}-1$. The story goes that because of the way computer printers of that era worked the output was presented in a back-to-front manner. So, as he flipped over the pages, Hurwitz would have seen the larger prime first. Thus it generally agreed that first titanic prime discovered by a human was

$$2^{4423}-1 \qquad \text{(1332 digits).}$$

The *smallest* titanic prime, $10^{999}+7$, was discovered many years later. Although for quite a long time everybody 'knew' that $10^{999}+7$ was prime, it was not until 1998 that a proof (by Preda Mihailescu) appeared.

The first titanic twin primes were found in 1980 by Oliver Atkin and N. W. Rickert:

$$256200945 \cdot 2^{3426} \pm 1 \qquad \text{(1040 digits).}$$

During the 1990s I became interested in the subject and in December 1996, I was able to report the first proven titanic prime triplets [M500 **154**],

$$437850590(2^{3567}-2^{1189})-6\cdot 2^{1189}+d,\ d=-5,\ -1,\ +1 \qquad \text{(1083 digits).}$$

With more powerful equipment I carried out another computer search in September 1998, which resulted in the discovery of the 1003-digit titanic quadruplets,

$$76912895956636885(2^{3279}-2^{1093})-6\cdot 2^{1093}+d,\ d=-7,\ -5,\ -1,\ 1$$

[*Math. Gazette*, November 2000].

The challenge to find a set of titanic prime *quintuplets* (i.e. five 1000-digit primes packed together as closely as possible) was taken up by **Norman Luhn**. As you can imagine, this was a formidable task. Even as I write, there exist only a handful of known titanic prime quadruplets (the largest having 1284 digits); but quintuplets of similar magnitude are much rarer, and one would be forgiven for dismissing as unfeasible a search for such objects using the current generation of personal computers.

Therefore I was most surprised when, on 30 July 2002, Norman informed me that he had discovered the first ever set of *titanic prime quintuplets*,

$$31969211688 \cdot 2400\# + 16061 + d, \ d = 0, \ 2, \ 6, \ 8, \ 12 \qquad \text{(1034 digits)},$$

where $x\#$ denotes the product of all the primes not exceeding x. The first one (corresponding to $d = 0$) is written out in full on the front cover of this magazine. To save you counting, there are 20 lines of 50 digits and one line of 34; 1034 digits in all. The other four primes are the same except that they end in 0943, 0947, 0949 and 0953 respectively.

Some more prime number records, as at November 2002.

Largest prime

$2^{13466917} - 1$ (4053946 digits), Michael Cameron, George Woltman, Scott Kurowski, *et al.*

Largest prime twins

$33218925 \cdot 2^{169690} \pm 1$ (51090 digits), Daniel Papp and Yves Gallot.

Largest prime triplets

$$(108748629354 \cdot 4436 \cdot 3251\#(4436 \cdot 3251\# + 1) + 210)\frac{4436 \cdot 3251\# - 1}{35}$$
$+ d, \ d = 7, \ 11, \ 13$ (4135 digits), David Broadhurst.

Largest prime quadruplets

$10271674954 \cdot 2999\# + 3461 + d, \ d = 0, \ 2, \ 6, \ 8$ (1284 digits), Michael Bell, Michael Davison, Matt Jack, Ronald Lau, Graeme Leese and Ben Lowing.

Largest prime sextuplets

$110282080125 \cdot 700\# + 6005887 + d, \ d = 0, \ 4, \ 6, \ 10, \ 12, \ 16$ (301 digits), Norman Luhn.

Largest prime septuplets

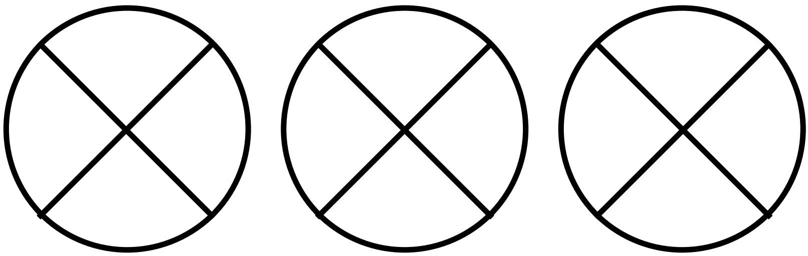
$497423806097 \cdot 400\# + 380284918609481 + d, \ d = 0, \ 2, \ 6, \ 8, \ 12, \ 18, \ 20$ (173 digits), Norman Luhn.

Problem 189.3 – Amazing object

Andrew Pettit

Some time ago my brother-in-law set me this puzzle, which he'd come across in *Model Engineer* **4170** (31 May 2002), in an article by Jaques Maurel. I attempted a solution but, like my brother-in-law, I failed to produce the correct answer.

Behold, three views of a convex solid object, taken from three mutually orthogonal directions.



Reconstruct the solid from the pictures. And when you have done that, calculate its volume.

Problem 189.4 – 100 members

David Kerr

I have a list of the names of 100 members of a society. They are all different and in random order. I will read them out one by one. You can stop me at any time, and your objective is to stop me immediately after I have read out the longest name.

Two questions: (i) what tactic should you use, and (ii) what is the probability of success?

Problem 189.5 – 40 years

John Reade

True or false? Everybody's 40th wedding anniversary falls on a Sunday.

The binomial coefficient ${}^{2m}C_m$

Sebastian Hayes

The central entry of an even-numbered row of Pascal's triangle, ${}^{2m}C_m$, is given by $(2m)!/(m!)^2$. The first entries are 1, 2, 6, 20, 70, …; they are the Catalan numbers multiplied by $m + 1$.

The coefficient ${}^{2m}C_m$ has certain curious properties. It is what we obtain when we multiply matrix-wise any row m of Pascal's triangle by itself; for example,

$$[1\ 4\ 6\ 4\ 1] \cdot [1\ 4\ 6\ 4\ 1]^{\mathrm{T}} \ = \ 70 \ = \ {}^{8}C_4.$$

Indeed, as mentioned in an earlier article, if we arrange so-many rows of Pascal's triangle to form a diamond, the matrix multiplication of any two rows symmetrically placed about the 'diameter' is constant—in this example, 20. This is a consequence of the the rule of addition for Pascal's triangle:

$$\begin{aligned} 1 \cdot 20 \ &= \ 1 \cdot 10 + 1 \cdot 10 \ = \ (4+6) + (4+6) \\ &= \ 1 \cdot 4 + 2 \cdot 6 + 1 \cdot 4 \end{aligned}$$

and so on. The indices of the pairs being multiplied must add to $2m$ and the whole procedure comes to an end when we have as first entry

$${}^{m}C_0 \, {}^{2m-m}C_{m-m} + \dots \ = \ {}^{m}C_0 \, {}^{m}C_0 + \dots ;$$

i.e. we get row m matrix-multiplied by itself.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ 1 & & 3 & & 3 & & 1 \\ & 4 & & 6 & & 4 & \\ & & 10 & & 10 & & \\ & & & 20 & & & \end{array}$$

Problem 189.6 – Three friends

David Kerr

I have three friends, all excellent logicians. Let's call them Alan, Bert and Curt. I write a different positive integer on the forehead of each of them and I tell them that one of the numbers is the sum of the other two. They take it in turns in alphabetical order to attempt to deduce their own number. The conversation goes as follows.

Alan: 'I cannot deduce my number.'

Bert: 'I cannot deduce my number.'

Curt: 'I cannot deduce my number.'

Alan: 'My number is 50.'

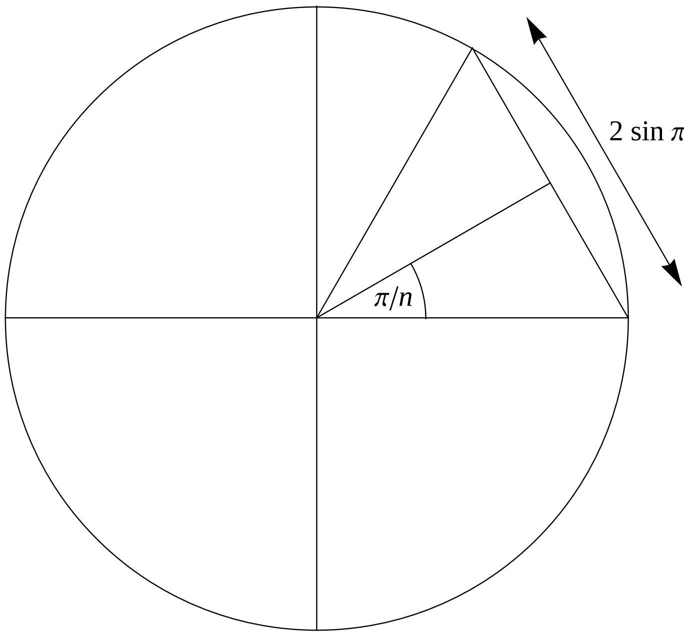
What are Bert's and Curt's numbers?

Sums of powers of chords (again)

Sebastian Hayes

The binomial coefficient ${}^{2m}C_m$ makes an unexpected appearance in the chord^2 sum problem for regular polygons.

To recapitulate: If we consider the side of a regular polygon inscribed in a circle of unit radius, its length will be $2\sin\pi/n$ and the chord to the next vertex will be $2\sin 2\pi/n$, and so on.



The sum of the squares of the chords (not the chords themselves) is thus

$$\sum_{r=1}^{n-1}\left(2\sin\frac{\pi r}{n}\right)^2 \;=\; \sum_{r=0}^{n-1}\left(2\sin\frac{\pi r}{n}\right)^2 \;=\; 2\sum_{r=0}^{n-1}\left(1-\cos\frac{2\pi r}{n}\right).$$

Incidentally, it is more convenient to sum from 0 to $n-1$, and this will not affect the result since $\sin 0 = 0$.

For a regular n-gon inscribed in a circle of unit radius—or indeed *any* inscribed polygon where the cosine sum is zero—the result is just $2n$, where n is the number of sides of the polygon (see M500 **175**).

What came as a surprise was discovering that continuing to sum the *even* powers of the chords gave numbers that seemed familiar: $2n$, $6n$, $20n$, $70n$, …; i.e. ${}^{2m}C_m\, n$, or so I conjectured.

It is quite easy to prove this for small values of m simply by repeated application of the double angle formulae. But the only way I have found to prove the general case is to use the expansion formula for $(\sin\phi)^{2m}$ given below, which was first brought to my attention by Barry Lewis (who arrived independently at the same result):

$$(\sin\phi)^{2m} \;=\; \frac{(-1)^m}{2^{2m}}\sum_{r=0}^{2m}(-1)^r\,{}^{2m}C_r\cos(2m-2r)\phi.$$

Note that the central term, corresponding to $r = m$, is just ${}^{2m}C_m/2^{2m}$.

Now

$$\sum_{r=0}^{n-1}\cos r\phi \;=\; 0 \;\text{ if }\; \phi = \frac{2\pi}{n}$$

and the same goes for $4\pi/n$, $8\pi/n$, …, provided we have $n > m$. Thus all the terms $\cos 2m\phi = \cos 2\pi m/n$, $\cos(2m-2)\phi$, &c. go to zero and we are left with just the central term, which has to be multiplied by 2^{2m}. Hence the result.

But why do we have this requirement that $n > m$? Because otherwise the formula might not work, as I found to my chagrin when trying out a few easy values. If $n \le m$, the sum of the cosines is not necessarily zero for multiples of $2\pi/n$. For example, if we replace $\cos 2\pi r/n$ by $\cos(6\cdot 2\pi r/n)$ the summation is no longer zero for $n = 3$. We are now summing $\cos 4\pi r$ with $r = 0$, l, 2 and the result is $1 + \mathrm{l} + 1 = 3$.

This point is discussed in detail in the more extensive article on the same topic written by Barry Lewis and due to appear in the *Mathematical Gazette.*

Problem 189.7 — All the sevens

Patrick Lee

If N is any non-negative integer, prove that the last digit of its 77th power is the same as the last digit of N.

Solution 182.6 – n balls

> There are n coloured balls, no two having the same colour. Remove two balls at random, paint the second of the pair to match the first and replace both balls. Repeat until all balls have the same colour. What is the expected number of turns?

John Smith

Think instead of a bag of n balls, r of them red and $n - r$ of them black. Initially we have one red ball and the others black.

Let $E(r)$ be the expected number of steps before the bag contains just red balls. Step sequences which result in the bag becoming full of black balls will be considered to have length 0.

Then we can write a recurrence relation of the form

$$\begin{aligned} E(r) \;=\; & (1 \text{ step}) \text{ times the probability that ultimately the bag} \\ & \qquad \text{fills with red} \\ & + E(r-1) \text{ times the probability that we colour a red ball black} \\ & + E(r+1) \text{ times the probability that we colour a black ball red} \\ & + E(r) \text{ times the probability that we draw two reds or two blacks.} \end{aligned}$$

Putting some expressions to these things, we have

$$\begin{aligned} \Pr(\text{bag ultimately fills with red}) &= \frac{r}{n}, \\ \Pr(\text{colour a red ball black}) &= \frac{r(n-r)}{n(n-1)}, \\ \Pr(\text{colour a black ball red}) &= \frac{(n-r)r}{n(n-1)}, \\ \Pr(\text{draw two reds, or two blacks}) &= 1 - 2\,\frac{r(n-r)}{n(n-1)}, \end{aligned}$$

which gives

$$E(r) \;=\; \frac{r}{n} + \frac{r(n-r)}{n(n-1)}\big(E(r-1) - 2E(r) + E(r+1)\big) + E(r),$$

which is nicer written as $n - 1$ equations of the form

$$\frac{n-1}{n-r} \;=\; -E(r-1) + 2E(r) - E(r+1) \qquad \text{for } 1 \le r < n,$$

with the extreme cases of $E(0) = E(n) = 0$.

More magic: multiply the rth equation by $n - r$ and add them all up. Conveniently, the only $E(\cdot)$ which survives is $E(1)$, in the form

$$\big(2(n-1) - (n-2)\big)E(1) \;=\; (n-1)^2.$$

Hence $E(1) \ = \ \dfrac{(n-1)^2}{n}$.

Going back to the original question with balls of many colours: my $E(1)$ is the product of the expected number of steps until all the balls are the same colour, and the probability that all the balls become the colour of one arbitrary ball from the initial bag. Since it is clear that ultimately all the balls become one colour, and all colours are equally likely, then this probability is $1/n$. Thus the expected number of steps must be $(n-1)^2$.

As a distraction from MSc revision this has been most successful!

Rates

Dilwyn Edwards

Some time ago I read some comments by Jeremy Humphries in the magazine ['What do they mean', M500 **180**, 17], which struck a chord with me in relation to the quoting of rates. It seems part of the standard sloppiness of thought amongst journalists and the like is to forget that a rate 'x per y' is meaningless unless both x and y are specified. Around Budget time we get streams of statements such as 'the average family will be £10 better off.' Often this means £10 a week but sometimes it means £10 a month, or it could be £10 a year, or even just a one-off £10.

The legal limit for drink-driving in this country is 80 mg of alcohol per 100 ml of blood, but this is rarely quoted correctly. I once heard a government minister refer to the limit as '80 mg of alcohol *in your blood*.' The average person has about 8 litres of blood and so can have about 6400 mg of alcohol in their blood and still be inside the limit. When I wrote to point out that the minister was wrong by a factor of about 80 I got no response.

If reporters have difficulty with a simple rate, we have to expect that a rate of change, or even a change in the rate of change, is a concept completely beyond them. I once saw a report that a high proportion (I forget the figure, but it was over 70 per cent) of people believe that if the inflation rate falls, that means prices have come down.

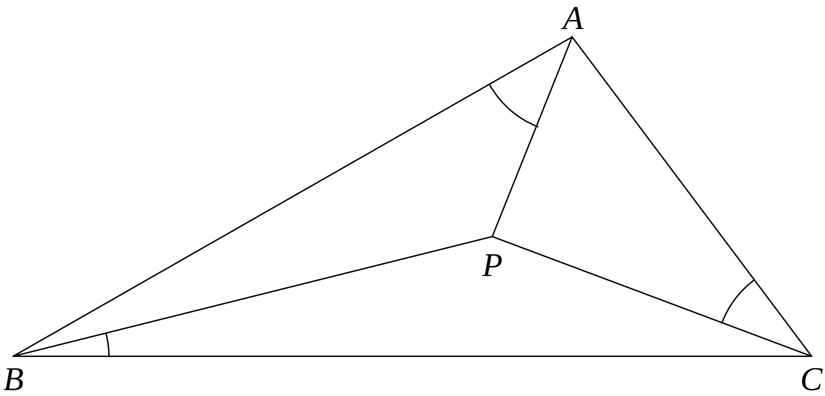
'Winning £1000 has never been easier—or more rewarding.'

From an investment newsletter. I don't know about the easier bit, but the rewarding bit is surely what mathematicians call *false*. Is it not the case that £1000 is always worth more yesterday than today, because inflation is always positive? Or as near always as makes no difference.—**JRH**

Problem 189.8 – 30 degrees

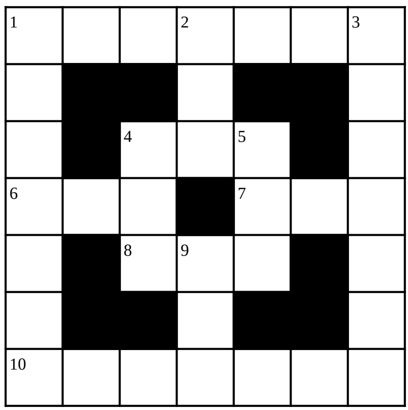
David Kerr

If ABC is any triangle and P is any point inside ABC, show that not all of the angles PAB, PBC and PCA can exceed 30 degrees.



Crossnumber

Tony Forbes



Across

1. $(8 \text{ ac.})^{5/2}$
4. $(2 \text{ dn.}) - 4(6 \text{ ac.})$
6. $(1 \text{ dn.})^{1/3}$
7. $(2 \text{ dn.}) - (9 \ \text{ dn.})$
8. $(1 \text{ ac.})^{2/5}$
10. $9(4 \text{ dn.})^2$

Down

1. $(6 \text{ ac.})^3$
2. $\sqrt{(3 \text{ dn.})}/3$
3. $9(2 \text{ dn.})^2$
4. $\sqrt{(10 \text{ ac.})}/3$
5. $(9 \text{ dn.}) + 9\sqrt{(6 \text{ ac.})}$
9. $(2 \text{ dn.}) - (7 \text{ ac.})$

Problem 189.9 – Magic square

Claudia Gioia

A slight variation of the gnomon magic square that appears in the engraving *Melancholia I*, by Dürer (1514). Fill in the missing numbers such that the rows, columns and various other symmetrically placed groups of four add up to the same 'magic' total.

15			14
		12	
	5		

Twenty-five years ago . . .

. . . M500 published *Gaberbocchus*, a Latin poem by Hassard Dodgson, a relative of Lewis Carroll. Here it is again, except that for the benefit of readers whose Latin is a bit rusty we have translated it into English. The original begins: *Hora aderat briligi. Nunc et Slythæia Tova / Plurima gyrabant gymbolitare vabo; . . . .*

The hour was brilig. And now the slythean toves
Did much gyrate and gymbol in the vabe;
And the borogoves were mimzy formed on all sides,
All the momiferious raths exgrabued.

'Beware of the gaberbocch I warn you, son, beware.
(Talons that tear. Teeth that kill.)
And fear the jubbjubb, which is not that dangerous bird,
And of the bandersnatch which snorts all the time, beware.'

But that vorpal sword he took and the
Manxonian enemy he long earnestly sought.
Next, he rested in the shade under a tumtumm tree.
Steadfast and calm in a certain frame of mind he had many thoughts.

While resting with uffish thoughts, a monster,
Behold! present, a wild beast whose eyes flashed with fire.
In person the gaberbocch, siffling through the rough thicket
At that very moment, with fearful burbling he came.

Three times! Four times! And so again, quickly the most vorpal sword,
Snic-snac, dissected deep internal organs.
The hero left the lifeless body, took away the head
And whithersoever much galumph, he returned home.

'Have you been able to kill the gaberbocch, son?
Beamish boy! Come to my arms.
Oh frabious day! repeatedly caloque calaque,
I am happy again,' he chortled, chortling in the manner of a grand old man.

The hour was brilig. . . .

Solution 181.4 – Four points
Robin Marks ... 1
Problem 189.1 – Neighbours
Tony Forbes ... 3
Solution 184.7 – Deux nombres
Tony Forbes ... 4
Problem 189.2 – Brown eyes ... 7
Solution 186.5 – Horse
Simon Geard ... 8
Do men and women design gardens differently?
Nick Pollock ... 10
Moving point
Dilwyn Edwards ... 11
Titanic prime quintuplets
Tony Forbes ... 12
Problem 189.3 – Amazing object
Andrew Pettit ... 14
Problem 189.4 – 100 members
David Kerr ... 14
Problem 189.5 – 40 years
John Reade ... 14
The binomial coefficient $^{2m}C_m$
Sebastian Hayes ... 15
Problem 189.6 – Three friends
David Kerr ... 15
Sums of powers of chords (again)
Sebastian Hayes ... 16
Problem 189.7 – All the sevens
Patrick Lee ... 17
Solution 182.6 – n balls
John Smith ... 18
Rates
Dilwyn Edwards ... 19
Problem 189.8 – 30 degrees
David Kerr ... 20
Crossnumber
Tony Forbes ... 20
Problem 189.9 – Magic square
Claudia Gioia ... 21
Twenty-five years ago 21