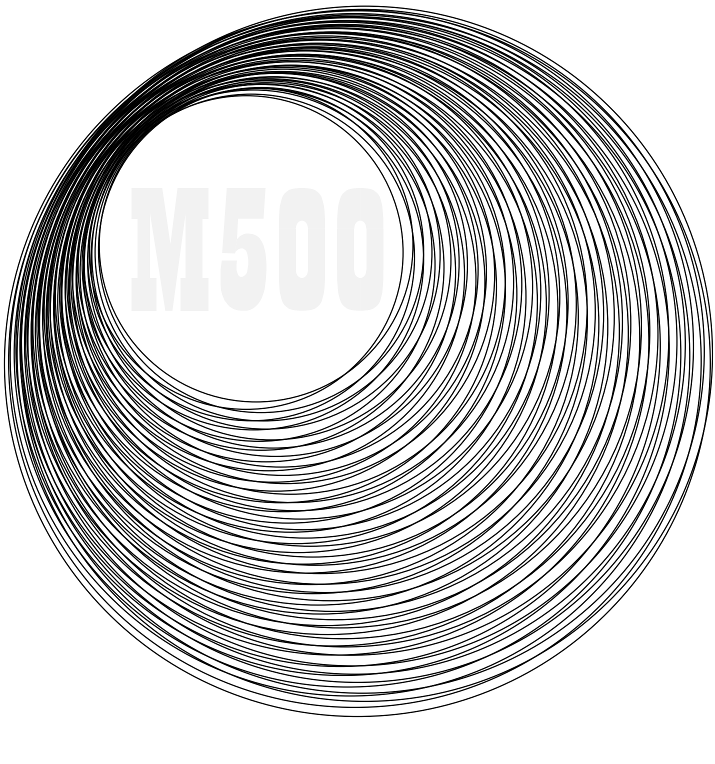
*

ISSN 1350-8539

M500 191



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

Easy ways with eigenvectors

Dilwyn Edwards

First, just in case it's needed, here is a quick reminder of what they are.

If you pre-multiply a column vector by a matrix you get another vector which is usually not obviously related to the vector you started with. Geometrically speaking it will be different in length and direction. For *eigenvectors* of the matrix, however, the resulting vector is a simple multiple of the original. It is just the same as if we had multiplied the vector by a scalar instead of by a matrix. This scalar is the corresponding eigenvalue. Geometrically, the vector has kept its direction but had its length multiplied by λ. In matrix algebra we write $Ax = \lambda x$; x is an eigenvector of the matrix A and λ is the corresponding eigenvalue. Incidentally, it is very common in textbooks to read the phrase 'eigenvalue and the corresponding eigenvector', which is really the wrong way round. To each eigenvector there can only be one eigenvalue (you soon find what it is by doing the matrix multiplication). But different eigenvectors can easily have the same eigenvalue. So the phrase should be 'eigenvector and the corresponding eigenvalue'. The reason they write it back to front is because that is the order in which we usually do things; we find the eigenvalues first.

For lots of reasons, eigenvectors and eigenvalues are very useful and important things, and there are various ways of finding them. Generally speaking, the choice is between numerical methods, which are clearly suited to computer work, and mathematical methods, which are clearly suited to making students sweat. I shall just discuss mathematical methods here, needing just pen, paper and (only a little) time. There are no quick or easy ways of finding the eigenvalues unless you are lucky enough to have a triangular matrix, in which case you don't have to do anything—the λ values are the numbers on the main diagonal. In most cases you can find the values by solving (much easier said than done!) the characteristic equation, which is the polynomial equation of degree n (when the matrix A is $n \times n$)

given by $|A - \lambda I| = 0$. I will assume all the eigenvalues have already been found and that we just want the eigenvectors.

There are various methods of proceeding, both easy and hard. OU courses tend to favour the harder methods (naturally, to give students value for money). The quicker and easier methods do not seem to be generally well known, which gives me my motivation for writing this article.

Quick method 1

For each λ write down the matrix $A - \lambda I$ and simply fill in the elements of the vector X which makes $(A - \lambda I)X = 0$ (not counting the trivial answer $X = 0$).

For 2×2 matrices this is very easy. Take the example $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ for which the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 4$. The matrix $A - \lambda_1 I$ is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and the vector we need to make $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} * \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is obviously $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. (Remember that $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is not really a different answer.)

For the other eigenvalue we have

$$(A - \lambda_2 I)X = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} * \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

so the eigenvector this time is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Did I take a particularly easy example? Not really; it can never be much harder—not when the eigenvalues are real numbers anyway.

Take another example, $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$ for which the eigenvalues are $\lambda_1 = 5$, $\lambda_2 = -2$. The relevant matrices $A - \lambda I$ are $\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}$ and $\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}$ so the corresponding vectors are clearly $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Three-by-three matrices require a bit more thought, but not a lot. Here's one with obvious eigenvalues.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -2 & 3 \end{bmatrix}, \qquad \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$$

Take

$$(A - \lambda_1 I)X = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} * \\ * \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

row by row. The first row works OK. For the second it is clear that the first two elements in the eigenvector need to be like 1, -1. Using these and moving on to the third row, we find the third value then has to be -2. So we have found our eigenvector $[1, -1, -2]^T$. Take

$$(A - \lambda_2 I)X = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} * \\ * \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The first element has to be 0 to blot out that -1. This also blots out the 1 on row 2. Moving to row 3 we see that we need 1 and 2 in the last two positions, giving the eigenvector $[0, 1, 2]^T$. Take

$$(A - \lambda_3 I)X = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} * \\ * \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We have to have 0 in the first position to knock out the -2 and from row 2 we need another 0 in the second position to knock out the -1. We then find in row 3 we have the luxury of putting anything for the third element so it may as well be 1 and the eigenvector is $[0, 0, 1]^T$. (No point putting a zero there because a vector of all zeros will be a trivial eigenvector for any matrix.)

This all takes more time to explain than to actually do. If there are repeated eigenvalues, to find different eigenvectors try a zero in a different place. Here's an example to explain what I mean.

The matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ has eigenvalues $\lambda = 1, 1, 5$. The matrix

$A - I$ is $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ and putting a 0 in the first position we see that $[0, 1, -2]^T$ works very nicely. Putting a zero in the second position we find that another eigenvector with eigenvalue equal to 1 for this matrix is $[1, 0, -1]^T$. For the third, we take the matrix $A - 5I$ and (perhaps after an initial slight hesitation) we quickly see that the eigenvector which goes with eigenvalue $\lambda = 5$ is $[1, 1, 1]^T$.

Quick method 2

Having the eigenvalues, write down the matrix $A - \lambda I$ for each λ. For 2×2 matrices the columns of each $A - \lambda I$ matrix are just repetitions of the eigenvector which goes with the *other* eigenvalue. For example, the matrix $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$ has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 7$. The matrix $A - \lambda_1 I$ is $A = \begin{bmatrix} 4 & 5 \\ 4 & 5 \end{bmatrix}$ whose columns are multiples of $[1, 1]^T$ which is the eigenvector that goes with $\lambda_2 = 7$. The matrix $A - \lambda_2 I$ is $A = \begin{bmatrix} -5 & 5 \\ 4 & -4 \end{bmatrix}$, which tells us that the eigenvector $[-5, 4]^T$ goes with $\lambda_1 = -2$. Check:

$$\begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -8 \end{bmatrix} = 2 \begin{bmatrix} 5 \\ -4 \end{bmatrix}.$$

This will work for complex eigenvalues too. For example, the matrix $A = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$ has eigenvalues $\lambda = 3 \pm 2i$. The two matrices $A - \lambda I$ are $\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}$ and $\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}$; so the eigenvectors are $[i, 1]^T$ and $[1, i]^T$. Check:

$$\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = (3 + 2i) \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = (3 - 2i) \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

The method fails only when the eigenvalues are equal, but for 2×2 matrices that's a trivial situation anyway. The explanation for the method is found in the fact that any matrix satisfies its own characteristic equation. For 2×2 matrices we can write this as $(A - \lambda_1 I)(A - \lambda_2 I) = 0$ and if you compare with $(A - \lambda_1 I)X_1 = 0$, you can see that X_1 can be identified with $A - \lambda_2 I$, and similarly for X_2. The method extends to 3×3 but is not so convenient because the characteristic equation is now $(A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I) = 0$, so to find X_1 we need to multiply the two matrices $(A - \lambda_2 I)(A - \lambda_3 I)$ together. Let's try this method on our previous 3×3 example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -2 & 3 \end{bmatrix}, \qquad \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$$

I write down each of the $(A - \lambda I)$ matrices and underneath each I put the product of the other two:

$$\begin{array}{ccc} \lambda_1 = 1 & \lambda_2 = 2 & \lambda_3 = 3 \\ \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix} & \begin{bmatrix} -2 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 0 & 0 \\ -2 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ -2 & -2 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & 2 \end{bmatrix}. \end{array}$$

The columns of these product matrices either are all zeros or give us the eigenvectors $[1, -1, -2]^T$, $[0, 1, 2]^T$, $[0, 0, 1]^T$. What determines how many of the columns are zero vectors? (I don't know.) Again, we need the eigenvalues to be distinct for the method to work. Although I have described both these methods as mathematical rather than numerical, *Quick method 2* could be obviously be programmed too. Finally, if you fancy a challenge, see if you can find the eigenvectors of

$$\begin{bmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{bmatrix},$$

whose eigenvalues are 1, 2, 5 and 10.

Can a number be equal to the sum of the sum and the product of its digits?

David Singmaster

Thomas Koshy [*Fibonacci and Lucas Numbers with Applications*, Wiley, 2001, p. 10] observes that $8 + 9 + 8 \cdot 9 = 89$ and asks if other numbers have the same property. Letting S be the sum of the digits of an integer N and P the product of the digits, we are asking when $N = S + P$. In 1998, I studied the question of when $N = SP$ ['On the ratio of a number to the sum of its digits', unpublished], so Koshy's question immediately intrigued me. Here I show that $N = S + P$ occurs if and only if $N = 0$ (an exceptional case) or $N = 19, 29, 39, \ldots, 99$. Then I examine a few related topics.

If N has only one digit, $N = a > 0$, then $S = a$, $P = a$ and $S + P = 2N$. The case $N = 0$ is a bit exceptional as it really has no digits at all! Then $S = 0$, $P = 1$ as these are an empty sum and an empty product, so $N = 0$ while $S + P = 1$.

If N has two digits, $N = (ab)_{10}$, then $S = a + b$, $P = ab$ and $N = S + P$ gives us $10a + b = a + b + ab$ or $9a = ab$ or $b = 9$ (since $a \neq 0$).

If N has three digits, $N = (abc)_{10}$, then $N = S + P$ gives $100a + 10b + c = a + b + c + abc$, or $99a + 9b = abc$. But $abc \leq a \cdot 9 \cdot 9 = 81a$, which shows that $N = S + P$ cannot hold. A similar argument shows that $N = S + P$ cannot hold for any larger number of digits.

In fact, the above shows that $N \geq S + P$ for two or more digits and hence that $N < S + P$ only holds for $N = 0, 1, 2, \ldots, 9$.

I computed examples where $S + P$ divides N. Results are summarized below; d is the number of digits, 'Number' is the number of d-digit numbers with $S + P \mid N$, 'Min R' is the minimum value of $R = N/(S + P)$, '#' is the number of occurrences of the minimum value of R, and in the last column are the values of N giving the minimum R.

d	Number	Min R	#	
1	0			
2	19	1	9	$19, 29, \ldots, 99$
3	78	3	2	$285, 594$
4	672	7	1	2793
5	5482	7	1	36498
6	48417	4	1	979968
7	439263	19	2	$3989943, 3989962$

The maximum value of R for d-digit numbers is easily determined.

$$R = \frac{N}{S+P} \le \frac{N}{S} = \frac{\sum_{i=0}^{d-1} a_i 10^i}{\sum_{i=0}^{d-1} a_i} \le \frac{\sum_{i=0}^{d-1} a_i 10^{d-1}}{\sum_{i=0}^{d-1} a_i} = 10^{d-1}.$$

Equality can hold if and only if $a_i = 0$ for $i < d-1$, i.e. for $N = a \cdot 10^{d-1}$, except when $d = 1$, for which R is always 2, and when $d = R = 0$.

If one doesn't require $S + P \mid N$, the minimum value of the ratio $R = N/(S+P)$ for d-digit numbers seems to occur when N is all 9s, i.e. $N = 10^d - 1$, assuming $d > 2$. (As already seen: for $d = 0$, $R = 0 = 10^0 - 1$; for $d = 1$, R is always 2; and for $d = 2$, we have $R = 1$ for $N = 19, 29, \ldots, 99$.) This is borne out by my limited calculations. However, I have not been able to prove this.

One obvious approach is to show that R decreases when one increases its digits in some way. But when $N = (10a999999999999)_{10}$, we have that R increases as a increases. The same occurs when one considers the 13th or 14th places (starting from the 0th place at the right end), so that increasing the number of 9s in N_0 by one fails to decrease the value of R.

A second approach is to directly compare $N/(S+P)$ with the value for $N_d = 10^d - 1$. However, the ratio R_d for N_d is $(10^d - 1)/(9d + 9^d)$, which is awkward to deal with, and one finds that all the digits of N need to be considered, especially as d gets larger and when P gets small. It may even be that the minimum does not occur for $10^d - 1$.

I'd be delighted to hear from anyone who can make progress on determining when the minimum of R occurs.

Solution 189.9 – Magic square

Fill in the missing numbers to make a magic square.

15			14
		12	
	5		

Béla Bodó

As this is a variation of Dürer's *Melancholia*, the solution can easily be found by 'Magic square arithmetic', described in M500 **120**, **121**, **134** and **135**, where the elements of the square were denoted as shown on the diagram, below.

Being Dürer's square, its magic constant ($\square$) is given by:

$$\square = 2(g + j) = 2(12 + 5) = 34.$$

From M500 **134**, we have $q = g + j - a = 17 - 5 = 2$. Also, $m + g + j + d = \square$; therefore $m = \square - g - j - d = 34 - 17 - 14 = 3$. Similarly, $a + b + c + d = \square$; therefore $b + c = 34 - 15 - 14 = 5$.

Hence b or c can only take the value 1 or 4. If we choose $c = 1$, then the other elements can be found from the table below, constructed similarly to Table 1 in M500 **135**, and the magic square is completed as shown.

a	b	c	d
e	f	g	h
i	j	k	l
m	n	p	q

15	4	1	14
6	9	12	7
10	5	8	11
3	16	13	2

$g + j - 2$	$4c$	c	$g + j - 3c$
$j + c$	$g - 3c$	g	$j + 2c$
$g - 2c$	j	$j + 3c$	$g - c$
$3c$	$g + j - c$	$g + j - 4c$	$2c$

Tony Forbes

I cannot resist the brute force approach to solving this kind of problem. It might even be instructive as an exercise in solving linear equations.

Just about every set of four symmetrically placed numbers sums to 34. Let us work with these 32 equations in 16 variables:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \\ a & e & i & m \\ b & f & j & n \\ c & g & k & p \\ d & h & l & q \\ a & f & k & q \\ d & g & j & m \\ a & b & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c & d & g & h \\ i & j & m & n \\ k & l & p & q \\ f & g & j & k \\ a & d & m & q \\ b & c & n & p \\ e & h & i & l \\ b & e & l & p \\ c & h & i & n \\ a & b & p & q \\ c & d & m & n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \end{bmatrix}, \quad \begin{bmatrix} a & e & l & q \\ d & h & i & m \\ a & c & n & q \\ b & d & m & p \\ a & i & h & q \\ e & m & d & l \\ e & g & j & l \\ f & h & i & k \\ b & j & g & p \\ c & k & f & n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \end{bmatrix}.$$

Obviously there is a certain amount of duplication and in fact there are really only 12 independent equations. Hence any general solution will have four parameters. Here is one possibility, with unknowns l, m, p and q:

$$\begin{gathered} a = 17 - q,\ b = 17 - p,\ c = m + p + q - 17,\ d = 17 - m,\ e = 17 - l, \\ f = l + p + q - 17,\ g = 17 + l - m - p,\ h = 17 - l + m - q, \\ i = l - m + q,\ j = m + p - l,\ k = 34 - l - p - q,\ n = 34 - m - p - q. \end{gathered}$$

Plugging in the known values, $a = 15$, $d = 14$, $g = 12$ and $j = 5$, doesn't quite yield a unique answer:

$$\begin{gathered} a = 15,\ b = 17 - p,\ c = p - 12,\ d = 14,\ e = 19 - p, \\ f = 2p - 17,\ g = 12,\ h = 20 - p,\ i = p - 3,\ j = 5, \\ k = 34 - 2p,\ l = p - 2,\ m = 3,\ n = 29 - p,\ q = 2, \end{gathered}$$

but things look promising because those values which are independent of p are different integers in $[1, 16]$. From $p - 12 = c \geq 1$ and $k = 34 - 2p \neq q = 2$, we obtain $13 \leq p \leq 15$. Therefore, since 14 and 15 are already taken, it must be that $p = 13$.

Problem 191.1 – Bee

Tony Huntington

My daughter e-mailed me a simple problem to keep me occupied this Ramadan. Or at least she said it was supposed to be simple, and I thought so too when I read it, but neither of us can make much headway with it. This is what she sent me.

A bee visiting a group of flowers behaves as follows. On arriving at a flower, it drinks the nectar and moves on, unless it drank the nectar on a previous visit, in which case it moves on at once. On leaving a flower, it goes to a flower chosen at random from the whole group of flowers (it cannot remember which it has visited before). Suppose it takes 3 seconds to drink the nectar from a flower, and one second to move from one flower to the next. Show that the expected value of the time taken to visit all the flowers is

$$n\left(4+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n-1}\right)$$

and find its variance.

Problem 191.2 – LCM

ADF

Denote by $[a, b, c]$ the lowest common multiple of a, b and c. Show that $[a, b, c] \le (n/3)^3$ for all sufficiently large n, where $a+b+c=n$, $1 \le a < b < c \le n$.

Investigate the difference between the maximum value of $[a, b, c]$ and $(n/3)^3$.

What about splitting n into four distinct parts, $n=a+b+c+d$? Five? ...

Why does

eleven plus two = twelve plus one?

Are there any more?—**Colin Davies**

Problem 191.3 – Tetrahedron

ADF

How many ways are there of colouring the elements of a regular tetrahedron such that two vertices are red, two vertices are green, two faces are blue, two faces are yellow, three edges are pink and three edges are black.

You can reasonably interpret the problem in three entirely different ways, depending on which symmetry group you choose to adopt:

(i) The trivial group. This is like having the tetrahedron fixed in 3-space.

(ii) You are allowed to rotate the tetrahedron. The relevant group is A_4, the group of even permutations of four objects. If we number the faces 1, 2, 3, 4, the twelve face permutations are generated by the cycles $(1, 2, 3)$ and $(1, 2, 4)$.

(iii) You are allowed to rotate tetrahedron or look at it in a mirror. The group is S_4, the group of all permutations of four elements. The 24 face permutations are generated by the cycles $(1, 2, 3)$ and $(1, 4)$.

Problem 191.4 – What's next?

Here are the first few terms of an infinite sequence, S_n, if we haven't made a mistake:

$$10,\ 10,\ 11,\ 12,\ 15,\ 16,\ 21,\ 21,\ 23,\ 27,\ 27,\ \ldots.$$

(i) What's the next term?

(ii) What's the rule?

When you've answered (i) and (ii), have a go at the following.

(iii) Prove that the terms always have two digits.

(iv) What can you say about the behaviour of the first digit of S_n as $n \to \infty$?

Hint: $S_{12} = 2\mathrm{B}$.

Add one w to stall to get a word with a similar meaning.—**JRH**

Solution 189.8 – 30 degrees

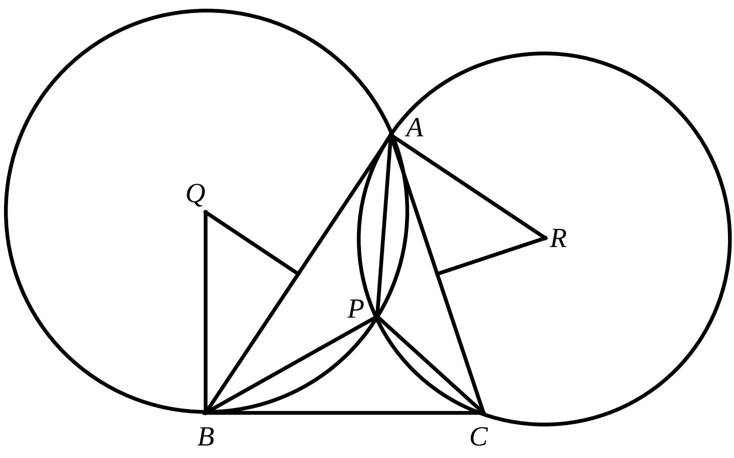
> If ABC is any triangle and P is any point inside ABC, show that not all of the angles PAB, PBC and PCA can exceed 30 degrees.

Dick Boardman

This solution is in two parts. First, the case where the three angles are equal is examined. In this case, P is one of the two *Brocard points*, named after Henri Brocard, a French army officer, who described them in 1875. It is shown that the Brocard angle can never exceed 30 degrees and only reaches 30 degrees in the case of an equilateral triangle. Secondly, it is shown that moving P away from the Brocard point must always decrease at least one of the angles. Thus all three angles can never simultaneously exceed the Brocard angle and hence never all exceed 30 degrees.

Part 1

Given: a triangle ABC. Draw the perpendicular bisector to the side AB. Draw a perpendicular to BC at B. Let them meet at Q. Draw a circle, centre Q radius QB. Since QB is perpendicular to BC, BC is a tangent to the circle at B. Furthermore, since Q is on the perpendicular bisector to AB, the circle passes through A.



There is a well known theorem in elementary geometry which states that the angle between a tangent to a circle and a chord equals the angle subtended by the chord in the alternate segment. Thus for any point P on the circle, $\angle CBP = \angle PAB$

Similarly, draw the perpendicular bisector to the side AC. Draw a perpendicular to AB at A. Let them meet at R. Draw a circle, centre R, radius RA. These circles will meet at A and at a second point, P. By the same theorem $\angle PCA = \angle PAB$. Hence for this point P, the three angles are equal, and P is a Brocard point for triangle ABC.

Let $\angle BAC = a$, $\angle CBA = b$, $\angle ACB = c$ and $\angle PAB = w$. Then $\angle PBA = b - w$, $\angle BPA = 180° - (b - w) - w = 180° - b$. Similarly, $\angle PAC = a - w$ and $\angle APC = 180° - a$. Applying the sine rule to triangle APB gives $AP = AB(\sin(b - w))/(\sin b)$. Applying the sine rule to triangle APC gives $AP = AC(\sin w)/(\sin a)$. Eliminating AP gives

$$AB\,\frac{\sin(b-w)}{\sin b} \quad = \quad AC\,\frac{\sin w}{\sin a}.$$

Considerable simplification using the formula $AB/AC = (\sin(a + b))/(\sin b)$ shows that this reduces to $\cot w = \cot a + \cot b - \cot(a + b)$, which matches the published equation for the Brocard angle, $\cot w = \cot a + \cot b + \cot c$. But $\cot w = \cot a + \cot b - \cot(a + b)$ is symmetrical in a and b, and will have a maximum for w when $a = b$. Call this w_1. Then $w_1 = \operatorname{arccot}(2 \cot a - \cot 2a)$.

Differentiating with respect to a and equating to zero shows that the maximum value of w_1 is 30 degrees and occurs when $a = b = 60$ degrees. Thus the Brocard angle is always less than or equal to 30 degrees.

Part 2

The problem states that P is inside the triangle. Moving P away from the Brocard point must involve moving it closer to one of the sides of triangle ABC and this must reduce one of the angles PAB, PBC or PCA and hence not all of the angles can exceed the Brocard angle; hence not all of the angles can exceed 30 degrees.

Solution 187.7 – Task

> You have a task, T, to perform on your computer. Normally the task takes t seconds to complete. However, in any interval of duration one second while it is running T will fail with probability p. When T fails it has to be started again from the beginning.

David Kerr

We start with some definitions and assumptions. Let t and p be as in the statement of the problem; let $q = 1 - p$. Each new attempt at the task is known as a *run*. We assume that a failure can only occur at an integral number of seconds after the start of a run; i.e. at 1 or 2, etc. up to t seconds.

Random variables are defined as follows. Let X be the time required to complete the task. The range of X is $\{t, t+1, \dots\}$. Let N be the number of failed runs before the successful run. The range of N is $\{0, 1, 2, \dots\}$. Let Y denote the length of a failed run. The range of Y is $\{1, 2, \dots, t\}$.

Lower case letters x, n and y represent specific values of X, N and Y. The probability that $X = x$ is given by $\mathbb{P}(x)$, and the mean, or expectation, of X is given by $\mathbb{E}(X)$. Similarly for N and Y.

The problem is to find $\mathbb{E}(X)$.

It is easy to see that

$$\mathbb{E}(X) = \mathbb{E}(N) \cdot \mathbb{E}(Y) + t. \tag{1}$$

Prima facie, this might seem obvious, but if you think that it does need to be proved, the details are provided on p. 16.

The problem therefore reduces to finding $\mathbb{E}(N)$ and $\mathbb{E}(Y)$. We start with $\mathbb{E}(N)$. The probability of a given run being successful is q^t. The probability of n failed runs followed by a successful run, i.e. $\mathbb{P}(n)$, is $(1 - q^t)^n q^t$. Hence N has a geometric distribution and the mean is given by $\mathbb{E}(N) = (1 - q^t)/q^t$. This is a standard result but it follows, after some calculation, from $\mathbb{E}(N) = \sum_{n=0}^{\infty} n(1 - q^t)^n q^t$.

The other mean, $\mathbb{E}(Y)$, is a bit more complicated. The method is to find $\mathbb{P}(y)$ and use $\mathbb{E}(Y) = \sum_{y=1}^{t} y\mathbb{P}(y)$.

Let A be the event that a run fails at the yth second and let B be the event that a run fails (anywhere). Then $\mathbb{P}(y) = \mathbb{P}(A \mid B)$. This says that the probability that a failed run fails at the yth second is the same as the probability that a run fails at the yth second given that it is known to fail somewhere. By the theorem of conditional probability,

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)},$$

as $A \cap B$ is clearly just A.

Now $\mathbb{P}(A) = pq^{y-1}$ and $\mathbb{P}(B) = 1 - q^t$. Hence $\mathbb{P}(y) = pq^{y-1}/(1 - q^t)$ for y in the range $[1, t]$ and

$$\mathbb{E}(Y) = \sum_{y=1}^{t} y \frac{pq^{y-1}}{1 - q^t} = \frac{1 - q^t(1 + tp)}{p(1 - q^t)}$$

after some more work. Finally,

$$\mathbb{E}(X) = \frac{1 - q^t}{pq^t} \cdot \frac{1 - q^t(1 + tp)}{p(1 - q^t)} = \frac{1 - q^t}{pq^t}.$$

Notice that the limit of $\mathbb{E}(X)$ as $p \to 0$ is t. Also, if $tp = 1$, the limit of $\mathbb{E}(X)$ as $p \to 0$ is $t(e - 1)$. For example, if $t = 1000$ seconds and $p = 0.001$, the mean of X is 1720 to the nearest second.

The above fully solves the problem as posed but it is interesting to find the probability generating function of X. The usual convention is to denote this by $\Pi_X(s)$, where s is a dummy variable. The probability generating function is such that the coefficient of s^x gives the probability that $X = s$. We already know that X is given by the sum of a number of failed runs given by the random variable N each of length Y, plus t seconds for the final successful run. Using a standard result (that students of M343 will know), the probability generating function of X can therefore be written as

$$\Pi_X(s) = \Pi_N(\Pi_X(s))s^t. \tag{2}$$

It is therefore sufficient to find Π_N and Π_Y.

The first, Π_N, is straightforward; from $\mathbb{P}(n) = (1-q^t)^n q^t$ we can see that $\Pi_N(s) = q^t/(1-(1-q^t)s)$.

The other one, Π_Y, is a bit more difficult. We know that $\mathbb{P}(y) = pq^{y-1}/(1-q^t)$ for $y = 1, 2, \ldots, t$. From this and some fairly horrendous algebra we get

$$\Pi_Y(s) = \frac{ps(1-qs)^t}{(1-q^t)(1-qs)}.$$

Hence, after lots more calculation,

$$\Pi_X(s) = \frac{(qs)^t(1-qs)}{1-s+pq^ts^{t+1}}.$$

The mean of X is given by $\Pi_X'(1)$. I won't bother with the gory details, but after differentiating and letting $s = 1$ this comes out as $(1-q^t)/pq^t$, confirming the result above.

As promised, we can now give a simple proof of (1). Differentiating (2), we have

$$\Pi_X'(s) = \Pi_N'(\Pi_Y(s))\Pi_Y'(s)s^t + \Pi_N(\Pi_Y(s))ts^{t-1}$$

and therefore

$$\begin{aligned}\mathbb{E}(X) &= \Pi_X'(1) = \Pi_N'(\Pi_Y(1))\Pi_Y'(1) + \Pi_N(\Pi_Y(1))t \\ &= \Pi_N'(1)\Pi_Y'(1) + t \\ &= \mathbb{E}(N)\mathbb{E}(Y) + t,\end{aligned}$$

since $\mathbb{E} = \Pi'(1)$ and $\Pi_N(1) = \Pi_Y(1) = 1$.

The variance of X, σ_X^2, is given by $\Pi_X''(1) + \mu_X - \mu_X^2$, where μ_X is the mean and σ_X is the standard deviation of X. After differentiation and another load of algebra we get

$$\sigma_X^2 = \frac{1-pq^t(1+2t)-q^{2t+1}}{(pq^t)^2}.$$

It can be shown that the limit of σ_X as $p \to 0$ is also 0, which offers some reassurance that the algebra is correct.

Using the above example, i.e. $t = 1000$ seconds and $p = 0.001$, we get $\sigma_X^2 = 977$. I'm not really sure exactly what this means but I think we can say that the distribution has a very long tail. Although the mean is 1720 seconds, there is a not insignificant probability that it could take, say, 4000 seconds to complete the task.

Solution 188.2 – Cylinder

> A cylindrical container is gradually filled a liquid. When is the centre of gravity at its lowest point?

David Kerr

Without loss of generality we can let the height and weight of the cylinder be 1 unit. Let the full weight of the liquid be w units. We assume that the weight of the base of the cylinder is negligible. Let x be the height of the liquid and y the height of the centre of gravity.

Taking moments about the base,

$$\begin{aligned} y(1+wx) &= \frac{1}{2}\cdot 1 + \frac{x}{2}\cdot wx \\ \Leftrightarrow y &= \frac{1+wx^2}{2(1+wx)} \qquad (1) \\ \Leftrightarrow \frac{dy}{dx} &= \frac{2(1+wx)(2+wx)-2w(1+wx^2)}{4(1+wx^2)}. \end{aligned}$$

Thus y will be a minimum when

$$\begin{aligned} 2(1+wx)(2+wx)-2w(1+wx)^2 &= 0 \\ \Leftrightarrow wx^2+2x-1 &= 0 \\ \Leftrightarrow x &= \frac{\sqrt{1+w}-1}{w}. \qquad (2) \end{aligned}$$

By substituting (2) into (1) we get, rather neatly, that the minimum y also equals $(\sqrt{1+w}-1)/w$. For example, if $w=3$, we get the minimum $y=x=1/3$; $w=8 \Rightarrow y=x=1/4$; $w=15 \Rightarrow y=x=1/5$.

Solution 187.6 – Iteration

Dilwyn Edwards

The answer must be $a_n = \dfrac{1}{2}\left(a_{n-1}+\dfrac{A}{a_{n-1}^{m-1}}\right)$ which, if it converges to anything, will converge to L, where $L=\dfrac{1}{2}\left(L+\dfrac{A}{L^{m-1}}\right)$. So $L=A^{1/m}$.

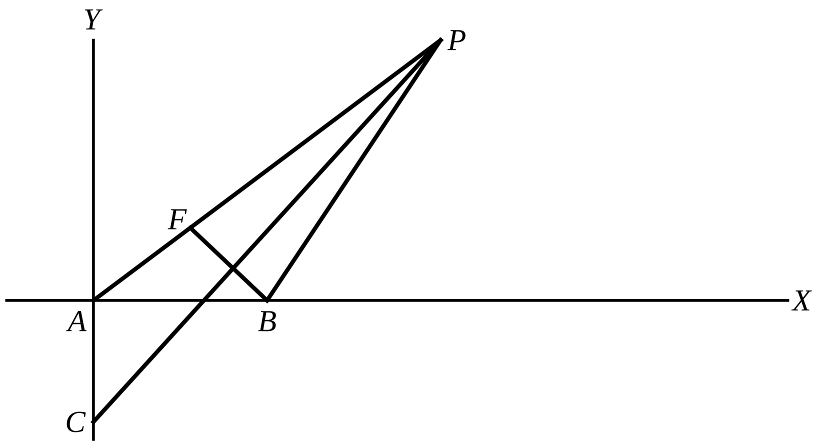
More moving points

Martyn Lawrence

Following on from 'Moving point' by Dilwyn Edwards [M500 **189** 11], in the diagram, below, BC is given by $y = mx + c$, where in this case $m = 1$ and $c = -0.7$. Thus $y = x - 0.7$ and P has co-ordinates $(x, x - 0.7)$.

For large values of positive x, the slopes of AP and BP approach m. Also for large positive x, angle APB tends to zero. If we let $PB = PF$, then $AF = AP - PB$. For large positive x, angle AFB tends to $\pi/2$. Thus $AB^2 = AF^2 + FB^2$.

With the slope of AP approaching m, by simple trigonometry it can be shown that AF tends to $1/\sqrt{m^2+1}$. For this particular example, where $m = 1$, AF tends to $1/\sqrt{2}$, which supports Dilwyn's figure of 0.7. A similar argument applies to large values of negative x. Hence for large $|x|$, $|AP - PB|$ tends to $1/\sqrt{m^2+1}$.



Next, we have

$$AP = \sqrt{(mx+c)^2 + x^2} = \sqrt{x^2(m^2+1) + 2mcx + c^2}$$

and hence

$$\frac{d(AP)}{dx} = \frac{2xm^2 + 2x + 2mc}{2\sqrt{x^2(m^2+1) + 2mxc + c^2}}. \tag{1}$$

Also

$$\begin{aligned} BP &= \sqrt{(mx+c)^2 + (x-1)^2} \\ &= \sqrt{x^2(m^2+1) + 2x(mc-1) + c^2 + 1} \end{aligned}$$

and

$$\frac{d(BP)}{dx} = \frac{2xm^2 + 2x + 2mc - 2}{2\sqrt{x^2(m^2+1) + 2x(mc-1) + c^2 + 1}}. \tag{2}$$

To look for any minimum or maximum points on the graph of $AP - PB$ we need to subtract (2) from (1) and equate to zero, obtaining

$$x = \frac{-c}{m} \quad \text{and} \quad x = \frac{-c(2mc + m^2 - 1)}{2c + 2m^2c + m^3 + m}.$$

Observe that $x = -c$ or $-c^2/(2c+1)$ for $m = 1$, and when $c = 0.7$ these become 0.7 and 1.225, as found by Dilwyn. Also $-c/m$ is merely the point on the x-axis where the line given by $y = mx + c$ crosses it.

Further investigation on the computer (by varying values of m and c) suggests that a maximum value of $|AP - BP|$ occurs at

$$x = \frac{-c(2mc + m^2 - 1)}{2c + 2m^2c + m^3 + m},$$

when the line $y = mx + c$ passes between A and B.

If $y = mx + c$ passes outside A or B, then the maximum value of $|AP - BP|$ occurs at $x = -c/m$, the value where the line crosses the x-axis. For example, if $m = 0.6$ and $c = -0.7$, then the maximum of $|AP - BP|$ occurs at 1.167 and has the value 0.857.

Letters to the Editors

Horses

Dear Tony,

'Problem 186.5 – Horse' is quite old. My earliest reference is the following, which deals with the outside of a circle!

Upnorensis, proposer; Mr Heath, solver; *Ladies' Diary*, 1748–49. Also Leybourn, II: 6-7, question 302 [Leybourn, Thomas (1770–1840), *The Mathematical Questions, Proposed in the Ladies' Diary, And their original answers, Together with some new solutions, From its commencement in the year 1704 to 1816. In Four Volumes*, J. Mawman, London 1817]. (I have a reference to p. 41 of the *Ladies Diary.*) A circular pond is enclosed by a circular railing of circumference 160 yards. The horse is tethered to a post of the railing by a rope 160 yards long. How much area can he graze?

The fortuitous constants of Problem 186.5 simplify the solution enormously to the point where it is not obvious how to generalize the problem. But if we go back to basics, we can get a solution of sorts.

Let O be the circumference of the circle. Let $OA = OB = OC = r$ be the radius of the circle and $AB = AC = L$ be the length of the rope. It turns out that $\gamma = \angle AOC$ is the easiest parameter to use. Then $\angle CAB = \pi - \gamma$, $L = 2r \sin \frac{1}{2}\gamma$ and $L^2 = 2r^2(1 - \cos\gamma)$. The area of sector CAB is $(\pi - \gamma)L^2/2 = (\pi - \gamma)(1 - \cos\gamma)r^2$. Sectors COA and BOA each have area $\gamma r^2/2$. Triangles COA and BOA each have area $\frac{1}{2}r^2 \sin\gamma$. Thus the area that the horse can reach is

$$\begin{aligned}(\pi - \gamma)(1 - \cos\gamma)r^2 + \gamma r^2 - r^2 \sin\gamma \\ = \bigl(\pi - (\pi - \gamma)\cos\gamma - \sin\gamma\bigr)r^2.\end{aligned}$$

Normally the reachable area is specified and we want to find L. A classic version of the problem gives the area as half of the circular pasture, i.e. $\pi r^2/2$. So we get $\pi/2 = (\pi - \gamma)\cos\gamma - \sin\gamma$. This kind of equation cannot be explicitly solved, but we can always use numerical methods. The simplest approach is to try and convert the equation into the form $\gamma = f(\gamma)$ such that the iterating function f will produce a convergent sequence. After several trials I found that

$$\gamma = \cos^{-1}\left(\frac{\pi/2 - \sin\gamma}{\pi - \gamma}\right)$$

worked well. Starting with the estimate 1.25, three iterations produced $\gamma = 1.235896924 = 70.81167767^\circ$ and $L/r = 2\sin\frac{1}{2}\gamma = 1.158728473$. For Problem 186.5, again I started with 1.25 and four steps gave $\gamma = 1.5707963267 = \pi/2 = 90^\circ$ and $L/r = \sqrt{2}$.

I gave the classic form of the problem in my department newsletter in 1987 and Victor Nikola misinterpreted it to produce an interesting problem and answers that may amuse you.

David Singmaster

[It and they did, so we have included it in this issue as 'Problem 191.6 – Porthole.'—**ADF**]

Rates again

Further to Dilwyn Edwards's article in M500 **189**, it is common for journalists to 'calculate' the annual rate of inflation by subtracting last January's retail price index from that for the current January. For example, the RPI for January 2002 was 173.3 and, for January 2001, it was 171.1 giving, so they said, an inflation rate of 2.3 percent. This ignores the fact that current RPIs are based on January 1987 being 100. (The RPI is 'rebased' every few years at irregular intervals) so that the true annual rate of increase for January 2002 was 1.29 percent. This was brought home to me when I compared the annual increase in my 'index-linked' pension with the increase in RPI quoted in the news media.

Another thing: what exactly do journalists think they are talking about when they use the term 'slide rule' as in 'running a slide rule over' something or, as I heard Andrew Marr say recently in *Start the Week*, 'putting a slide rule against' something?

Patrick Lee

There are 10 types of mathematician—those who understand binary numbers and those who don't.

M500 189

Problem 189.5. Not only is it not true that everyone's 40th wedding anniversary is on a Sunday but, in a sense, it fails fairly spectacularly. It's a nice example where one should not be seduced into seeing a pattern by looking at a lot of handy examples.

1. Not everyone gets married on a Saturday. For anyone married during the 20th century on a non-Saturday, their 40th anniversary is not on a Sunday. (Proof as exercise for reader)

2. For anyone whose 40 years of marriage spans a non-leap year century and who did get married on a Saturday, the 40th anniversary is not a Sunday either. (ditto)

It reminds me of the fact that $n^2 + n + 41$ produces primes for $n = 1, 2, \ldots, 39$ but not for 40 or (even more obviously) 41.

Problem 189.2. Owners of the now very rare *Chez Angelique* collected problems booklet know a more entertaining and bloodthirsty version of this called the Faithless Spouses (OK, OK it was faithless wives in those less egalitarian days). The editors may care to resurrect the original for everyone's amusement.

Problem 189.3. As a regular reader of the said *Model Engineer* magazine, I saw this. Having had the privilege of being a colleague of Richard Ahrens, I have seen his beautiful model of the gadget drawn here. To see the object roll across a horizontal plane is a truly weird experience. The drawings show that it *must* roll smoothly. One's eyes refuse to totally believe what they are seeing when it does; the object simply looks too angular for it to be possible. (And that's the only clue I'm giving as to the solution, except to say that anyone who ever attended M101 summer school has the solution in the official literature. The shape is there, and a hint as to how to calculate the volume.)

Bob Margolis

'Queen's horse scores first win since 1949'—Headline in *The Times*, 12 February 2003. [Spotted by **JRH**]

Twelve tarts

Tony Forbes

Colin Davies kindly sent me a cutting from the *Telegraph* concerning one of the classic conundrums of tart weighing.

> There are 12 jam tarts. All weigh the same, with one exception. In three weighings, determine the odd tart and whether it is lighter or heavier than the others.

I have to admit that I was dismayed by the hideous complexity of what I saw. The solution offered by Ian Stewart & Martin Golubitsky [*Telegraph*, 8 February 2003] involved a tedious case-by-case analysis, where the instructions for the third weighing depend upon the outcome of the second, the details of which, in turn, depend on the result of the first.

Then I remembered that I did some work on the very same problem while I was analyzing Dick Boardman's 'Nine tarts' [M500 **182** and **184**]. I had also managed to find a solution to 'Twelve tarts' in which the three weighings are fixed in advance. Here it is:

1st weighing:	A B C D	against	E F G H
2nd weighing:	A B C E	against	D I J K
3rd weighing:	A D F I	against	B G J L

With the results of the three tests it is easy to identify the tart with the odd weight. Although it is not really needed, the following table shows how. The possible results of the weighings are shown as movements of the left-hand pan, U: up, D: down, B: balanced.

UUU ⇒ A light	UUB ⇒ C light	UUD ⇒ B light
UBU ⇒ G heavy	UBB ⇒ H heavy	UBD ⇒ F heavy
UDU ⇒ D light	UDB ⇒ E heavy	UDD ⇒ –
BUU ⇒ J heavy	BUB ⇒ K heavy	BUD ⇒ I heavy
BBU ⇒ L heavy	BBB ⇒ –	BBD ⇒ L light
BDU ⇒ I light	BDB ⇒ K light	BDD ⇒ J light
DUU ⇒ –	DUB ⇒ E light	DUD ⇒ D heavy
DBU ⇒ F light	DBB ⇒ H light	DBD ⇒ G light
DDU ⇒ B heavy	DDB ⇒ C heavy	DDD ⇒ A heavy

Problem 191.5 – Another magic square

Claudia Gioia

On the Web there are lots of hints on how to create magic squares. Here is a 9×9 one. Fill in the missing numbers.

47	58	69	80		12	23	34	45
	68	79	9					
		8	10					
			20					
				41				
					62			
					72	74		
					73	3	14	
37	48	59	70		2	13	24	35

Problem 191.6 – Porthole

David Singmaster

Consider a circular porthole of radius r. One end of a windscreen wiper of length L is attached to a point on the circumference, and the wiper arm turns on this point, thereby sweeping out a circular sector on the porthole. This sector is half the area of the porthole. How long is the wiper?

There is something unusual about the solution—do you see it? Can you generalize it? What length of wiper clears the maximum area?

Problem 191.7 – Sum and reciprocal

There are n positive numbers, $a_1, a_2, \ldots, a_n$. Show that

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2.$$

Problem 191.8 – Infinite exponentiation

Dilwyn Edwards

I want to know the value of $y = x^{x^{x^{x^{\cdots}}}}$ (where the powers go on for ever) when $x = 1.1$. Writing it as $y = x^y$ I get $\log y = y \log x$; so $\log x = (\log y)/y$. Putting $x = 1.1$, I find this has two solutions: $y = 1.111782011...$ and $y = 38.22873285....$ They can't both be right. What is the explanation?

Problem 191.9 – Switch

There is a game show involving a host, H, a switch, S, and n contestants. The switch has two settings, ON and OFF, but it is not connected to anything. At the start, H sets the switch one way or the other. The game then proceeds in stages.

At each stage, H chooses a contestant, C, who can either (i) change the setting of S, or (ii) leave S unchanged, or (iii) announce that all n contestants have been chosen by H. The game continues until case (iii) occurs, when the game ends. The contestants win a Valuable Prize if and only if C's assertion is true.

The rules are: (a) the contestants may consult with each other before the game begins; (b) no communication is allowed after the game has started; (c) if the game goes on for ever, each contestant will be chosen infinitely often.

Formulate a strategy which will guarantee a win for the contestants. If that is too difficult, try relaxing rule (b).

Yes, it's different. M500 is now set in LaTeX. This issue uses 11-point type instead of the usual 10 because I didn't notice until it was half done. I must admit that if it wasn't for a 21 percent increase in paper costs I would be inclined to keep it that way for future issues. I still have a lot to learn about the new system. I don't know how to draw diagrams, nor have I figured out how to start an article with the traditional big letter. (Any advice from experts?) But I'm slowly getting there, and sifting through the huge amount of LaTeX related material available via the Internet.—**ADF**

Contents — M500 191 – April 2003