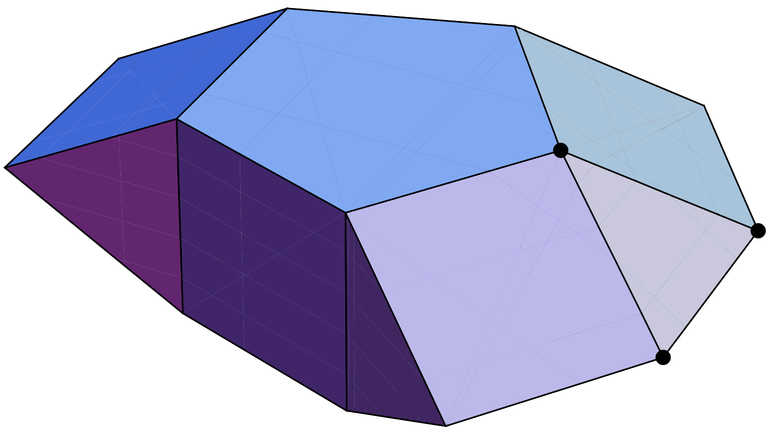
*

ISSN 1350-8539

M500 192



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

Glimpses of infinity

Jim James

Sebastian Hayes's article 'Why does calculus work?' published in M500 **185**, together with correspondence from John Hudson and Sheldon Attridge in M500 **188**, all encroach upon the age-old problem of the existence or non-existence of a mathematical infinity. I find that the more that one learns about mathematics, the more the infinity debate crops up in various guises, with arguments being expressed both for and against by academics, clerics, concerned researchers and gifted amateurs alike. Here are a few fragments aimed solely at providing a little food for thought on this interesting topic. . . .

> *It is true that all mathematics comes from the things of the real world.*
>
> Paul Halmos, *c.* 1970

> *Mathematical truth is immutable; it lies outside physical reality.*
>
> Joel Spencer, *c.* 1970

When I was in the sixth form at school, mathematics was taught as two different subjects, pure and applied. In the latter, under Mr Bullen, we studied practical mathematics, the stuff used by scientists and engineers. Rigour came a poor second to getting results. We novices were not concerned whether infinity existed or not, we treated it as just another mathematical object, quite happily summing series to infinity and integrating functions from minus infinity to plus infinity with reckless abandon. It was all so straightforward and our crude experimentation in physics-practical showed that it worked. It certainly served as a firm foundation for my later studies in engineering.

For Miss Hill, our pure mathematics teacher, rigour was everything. She introduced complications which we found much more difficult to follow. For some reason, which she never explained adequately, she talked about the sum of a series as n tends to infinity, never its value at infinity. Her approach to calculus was also quite different. In Mr Bullen's class we multiplied and divided by dx and dy, serenely cancelling infinitesimals wherever it was convenient and whenever it led to the solution of a practical problem. Miss Hill would not countenance such a procedure even though it always gave the correct results. She made uncomplicated mathematics complicated, easy mathematics hard. For us, at age 17, mathematics really was two quite different subjects. It was all very interesting, but at the same time, just a little confusing. . . .

> *Zeno's intention was to discredit the senses, which he sought to do through a brilliant series of arguments, or paradoxes, on time and space that have remained complex intellectual puzzles to this day.*
>
> *Encarta Encyclopaedia, 2000*

The concept of infinity has concerned mankind for well over 2000 years. Thus in the fifth century BC, the famous Greek philosopher Zeno of Elea expounded his famous paradox. He argued that motion was logically impossible, because no matter how short a distance specified, one had first to traverse half of it, then half of the remainder, and so on. Such bisections could go on indefinitely, and no matter at what speed one travelled, it was not possible to traverse an infinite number of spatial distances in finite time. This, and similar logical paradoxes, taxed the minds of philosophers, mathematicians and scientists over the following centuries and as more understanding was gained, so more theoretical difficulties appeared. The classical forms of reasoning seemed inadequate to achieve a complete resolution of such problems. The easy way out, adopted by the majority, was to avoid, or even deny, the concept of infinity and to concentrate on matters they understood. Infinity had to wait. ...

> *The good Christian should beware of mathematicians and all those who make empty prophesies. The danger already exists that the mathematicians have made a covenant with the devil to darken the spirit and to confine man in the bonds of hell.*
>
> St Augustine, *c.* 600 AD

Europe was somewhat slow in becoming involved in mathematics research. The early Church must bear its share of responsibility for this. As applied to real-world problems, classical mathematics was much valued by Church leaders and often successfully adapted to suit their purposes; witness the acceptance and wide application of geometry in the design and construction of monasteries, churches and cathedrals.

But the liberal application of rational argument, by intellectuals, was another matter. It purported to expose truths independent of divine revelation or papal edict and these only too often contradicted theological dogma. If the Church was to maintain and extend its position of power and authority over the lives of the masses, such freedom of expression was not to be encouraged; it had to be regulated, it might even need to be suppressed. Thus, in the Middle Ages, cultural advance was stifled by the Church on many fronts, and these included mathematics. Infinity had to wait a little longer. ...

> *Infinity and indivisibility are, in their very nature, incomprehensible to us.*
>
> Galileo Galilei, *c.* 1600

Galileo was a product of the Renaissance, as also were Descartes, Newton, Leibnitz, Fermat, and many other now famous men of mathematics. These were the new thinkers, born of the reawakening of intellectualism, inspired at first by the arts, that swept across Europe from the 14^{th} to the 17^{th} centuries. Trade, commerce, access to paid employment and simple everyday economics attracted people from the country into the towns and gradually gave rise to the formation there of a stable middle class. The long established feudal culture was becoming challenged and as the grip of State and Church on the masses was loosened, so intellectual freedom blossomed as never before. The golden age of European mathematics was about to come into being.

One of Galileo's many mathematical achievements concerned the natural numbers. He discovered what we would now refer to as a 1:1 correspondence between them and the even natural numbers, thus $\mathbb{N} \leftrightarrow 2\mathbb{N}$: $1 \leftrightarrow 2$, $2 \leftrightarrow 4$, $3 \leftrightarrow 6$, $4 \leftrightarrow 8$, and so on.

If we assume the concept of the infinitude of the natural numbers, claimed Galileo, then the 1:1 correspondence could go on for ever and this would suggest that there were as many even numbers as odd and even numbers combined. Since to him this was clearly impossible, he concluded that we must avoid considering infinite processes completely. Unfortunately Galileo failed to make the correct interpretation of this apparent contradiction; it had to await another 300 years for its resolution. In the meantime Galileo's deduction gave substantial support to the prevailing philosophical wisdom. Infinity had to wait a little longer still. ...

> *I protest above all the use of an infinite quantity as a completed one. The infinite is only a 'façon de parler', in which one properly speaks of limits.*
>
> Carl Gauss, 1831

The concept of a limit, which a sequence might approach, but never actually attain, was first mooted towards the end of the 18^{th} century. Its great attraction was the fact that infinity could be 'approached' as closely as one might desire using only finite quantities; reference to infinity itself, as a stand-alone mathematical object, therefore, was not required. In many respects this proved to be a great step forward.

Cauchy, using the idea of limits and convergence, made major advances in what came to be known as real and complex analysis. Amongst his many contributions to mathematics, he proved the validity of Newton's infinitesimal calculus, a problem which had been worrying mathematicians for nigh on 200 years. Such advances firmly established the technique and led many leading mathematicians to believe that this was the only acceptable way to handle problems which might otherwise require them having to admit to the existence of a mathematical infinity. But then came Georg Cantor. ...

> *By an 'aggregate' we are to understand any collection into a whole, M, of definite and separate objects, m, of our intuition or our thought. These objects are called the 'elements' of M.*
>
> Georg Cantor, 1895

Cantor had other ideas. While he accepted Cauchy's work when applied to the topics with which he was specifically dealing, his researches indicated that under certain circumstances infinity could, and should, be treated as an acceptable and worthwhile mathematical object in its own right. Infinity did not need always to be 'approached' as the analysts insisted; it could be met head-on.

Cantor's early work on trigonometric series led him to consider the fundamental properties of sets. He pioneered the development of formal set theory from first principles, starting with his very simple definitions of a set (as in the quotation above) and the notion of set equivalence, or similarity; two sets being similar if and only if their elements are in 1:1 correspondence with each other.

> *In introducing new numbers, mathematics is only obliged to give definitions of them, by which they can be definitely distinguished from one another. As soon as a number satisfies all these conditions it can and must be regarded as existent and real in mathematics.*
>
> Georg Cantor, *c.* 1880

Cantor showed that the simplest infinite set could be taken as $\mathbb{N}$, the set of natural counting numbers. He declared all sets similar to $\mathbb{N}$ to be 'countable', since by definition they exhibited a 1:1 correspondence with $\mathbb{N}$ and therefore had the same infinite element count, which he denoted by the distinguishing symbol $\aleph_0$ ($\aleph$ being 'aleph', the first letter of the Hebrew language). Cantor claimed that just as an irrational number was the limit to which a certain rational sequence converges, but which it never equals, so too was $\aleph_0$ the limit to which the natural numbers converge, but which they never equal. In this respect, he claimed, $\aleph_0$ to be just as valid a number, albeit a 'transfinite' one, as an irrational number.

Cantor showed, surprisingly, that the rationals were also countable, but that the irrationals and reals were not. The irrationals and reals were certainly infinite sets, so he deduced that there must be at least two different degrees of infinity; the infinity of the irrationals and reals being considerably greater than that of the natural and rational numbers.

This is but one easy to comprehend example of Cantor's involved and sometimes rather convoluted arguments, spread over 30 years during the late 19^{th} century. During this period, he also showed that for any set containing, say, k elements (where in an infinite set k is a transfinite number), the number of elements in the set of all its subsets must be greater than k. This means that given any non-empty set, finite or infinite, it is always possible to construct a larger set. Such a process can be continued indefinitely, thus proving that there must be an infinite number of successively larger infinities. This type of argument led Cantor to develop his celebrated theory of transfinite sets and transfinite numbers. It turned out to be a logically consistent arithmetic of the infinite. Infinity was waking from its long sleep at last. ...

> *The results of modern function theory and set theory are of no real significance.*
>
> Leopold Kronecker, 1884

But Cantor's revelations had to face the wrath of the ultra-conservative mathematicians, who preached that the existence of mathematical objects must be demonstrated clearly before they could be accepted as such. To them, arguments based upon symbols were meaningless unless the significance of the symbols could be intuitively understood; proofs had to be constructive; abstract objects and axiomatic methods were not acceptable. Kronecker, originally one of Cantor's tutors, was an extremist in this regard. He vociferously opposed the new style of abstract mathematics. To him the arguments proposed by Cantor, and all others who advocated or promoted the existence of a mathematical infinity, were fundamentally flawed. Infinity was now facing severe hostility. ...

> *Later generations will regard Cantor's work as a disease from which one has recovered.*
>
> Henri Poincaré, 1908

> *No one shall expel us from the paradise which Cantor has created for us.*
>
> David Hilbert, *c.* 1908

Although Cantor's work was rejected by many mathematicians, he nevertheless had strong support in high places. His friend Dedekind and the influential Hilbert, in particular, disagreed intensely with the narrow vision of Kronecker and his associates. Hilbert actively rebutted the opposition's unjust criticism of Cantor; he believed that mathematical theories of whatever type, abstract or otherwise, were perfectly acceptable so long as they did not lead to logical contradictions. Consistency was the important factor, not the meaning or significance of the symbols employed in the mathematical manipulations. Cantor's transfinite number theory had been shown to be consistent, so, for Hilbert, this, in itself, was sufficient proof of its validity.

Fortunately Hilbert's view prevailed and the era of 'mathematical correctness', which had been dictated by the establishment for so long, came to an end. Today we treat such abstractions as an essential part of mathematics; our OU courses abound with them.

> *My theory stands as firm as a rock; every arrow directed against it will return quickly to its archer. How do I know this? Because I have studied it from all sides, because I have examined all objections that have ever been made against infinite numbers.*
>
> Georg Cantor, *c.* 1900

But all was not as perfect as it seemed. Further work on Cantor's set theory by a number of researchers exposed certain logical paradoxes, which could not be resolved within the theory. In particular, Bertrand Russell produced his famous antinomy, or paradox: let $\boldsymbol{T}$ be a set which does not contain itself as an element and let $\boldsymbol{S}$ be the set of all such sets, that is $\boldsymbol{S} = \{\boldsymbol{T} : \boldsymbol{T} \notin \boldsymbol{T}\}$, then is $\boldsymbol{S} \in \boldsymbol{S}$? Clearly, if it is then it is not and vice-versa. Russell claimed that such contradictions arose because of the vagueness of Cantor's set definition; some accused Cantor of basing his theory on too many intuitive, naïve notions; others questioned the logic employed by the paradox creators.

True to the prevailing wisdom of the time, it was suggested that all would be made clear if set theory were to be properly axiomatized, displacing Cantor's preferred appeal to intuition and reasonableness. In 1908 Ernst Zermelo assisted by Abraham Fraenkel achieved just that and their axioms gave rise to a standard form of set theory, consistent with Cantor's. The validity of Cantor's infinity now seemed to be assured. ...

> *The fact that objects described by these axioms actually may exist in the real world is irrelevant to the process of formal deduction.*
>
> Paul Cohen, 1967

But was it so assured? Cantor's transfinite set theory relied heavily on what became known as the 'axiom of choice', namely that given any collection of non-empty sets, it was always possible to generate a new set by taking one element from each of the constituent sets. Although this sounds perfectly plausible to most of us, it did not satisfy all the experts and as a result the axiom of choice was submitted to intense scrutiny. Then in 1963, building on Gödel's work in the 1930s, Paul Cohen showed that the axiom of choice was independent of the other Zermelo-Fraenkel axioms and that it was possible to construct a 'non-Cantorian set theory' by rejecting it (recall how Lobatchevsky and Bolyai had constructed their non-Euclidean geometries by rejecting Euclid's parallel postulate).

But Cantor's transfinite numbers also play an important role in Cohen's non-Cantorian set theory. This means, surely, that whether one accepts the axiom of choice or not, whatever doubts one might have, or whatever objections others might raise, there is now sufficient positive evidence to uphold Cantor's heroic claim that a mathematical infinity does exist and that it is a perfectly legitimate mathematical object in its own right. ...

> *God created the natural numbers, and all the rest is the work of man.*
>
> Leopold Kronecker, *c.* 1880

Hmm, I wonder ...

Solution 189.2 – Brown eyes

> One day the elders of a village issued the following order: *If you discover that you have brown eyes, you must take the 12:00 train on the next day and leave this village permanently.* Nothing happened for a few years until one evening a passing tourist announced to everybody that he had seen a brown-eyed inhabitant of the village. As a consequence of the elders' order, ten days later all the brown-eyed people left on the noon train. Explain.

David Porter

It seems reasonable to assume that in the years since the elders made their decree every villager will have checked on the eye colour of each of the other villagers and therefore knows how many of them have brown eyes. Thus if there was only one brown-eyed villager she would say to herself on hearing the tourist's pronouncement 'No one else in the village has brown eyes, oh dear, it must be me' and will up and leave the next day—i. e. she will leave on the first day after tourist's visit.

Let's now proceed by induction with the inductive hypothesis being 'If and only if there are n brown-eyed villagers they will all leave on the nth day after the tourist's visit'.

Let this be true for n brown-eyed villagers and consider what happens if there are $n+1$ brown-eyed villagers. Each of these will reason as follows. If I have not got brown eyes then the n people I know to have brown eyes will all leave on the nth day; if they don't leave then, then I too must have brown eyes. Thus when no one leaves on the nth day all $n+1$ brown-eyed villagers realize that they have brown eyes and thus leave on the $n+1$th day.

Thus since the hypothesis is true for $n = 1$ it is true for all n and hence there were 10 brown eyed villagers. Fortunately these were all of the village elders, each of whom had hoped to become sole elder and hence village dictator. Now the village has a new, younger set of elders who welcome eye colour diversity and the villagers have taken to openly admiring each other's eyes and the threat of closure of the village school has evaporated.

Simon Gardiner

The explanation to this event assumes that each villager has the same logical mind as that required to solve the problem. One needs to imagine the same scene through the eyes of the brown-eyed inhabitants.

If, for the intervening years between the proclamation and the tourist's visit, you have gone about your business and never seen anyone with brown eyes, then the tourist's comment can only apply to yourself. Enough to send you scurrying for your suitcase and on the noon train the following day.

One person with brown eyes ($n = 1$) leaves the village one day after the tourist ($t = 1$). Imagine the relief felt by all the remaining villagers as they watched you board the 12.00 to nowhere. Because if you had not caught that train, something would be amiss, they would see someone with brown eyes but unaware of that fact, implying that a second person had brown eyes.

In fact, if you know of just one person with brown eyes you would watch them and if they did not catch the train then you would know that you also had brown eyes; having missed the noon train on day 1, the pair of you ($n = 1$) would catch the train on the following day ($t = 2$).

This scenario extrapolated indefinitely as $t = n$ until all the brown-eyed people leave on the tenth day; so there must have been ten of them.

ADF

In case the above is not clear, we thought you might like to see the argument written out in detail. It goes something like this.

On the 9th afternoon following the announcement each brown-eyed person thinks thus: 'Suppose my eyes are not brown. I see 9 brown-eyed people. On the 8th afternoon following the announcement each brown-eyed person would have thought: "Suppose my eyes are not brown. I see 8 brown-eyed people. On the 7th afternoon following the announcement each brown-eyed person would have thought: 'Suppose my eyes are not brown. I see 7 brown-eyed people. On the 6th afternoon following the announcement each brown-eyed person would have thought: "Suppose my eyes are not brown. I see 6 brown-eyed people. On the 5th afternoon following the announcement each brown-eyed person would have thought: 'Suppose my eyes are not brown. I see 5 brown-eyed people. On the 4th afternoon following the announcement each brown-eyed person would have thought: "Suppose my eyes are not brown. I see 4 brown-eyed people. On the 3th afternoon following the announcement each brown-eyed person would have thought: 'Suppose my eyes are not brown. I see 3 brown-eyed people. On the 2nd afternoon following the announcement each brown-eyed person would have thought: "Suppose my eyes are not brown. I see 2 brown-eyed people. On the 1st afternoon following the announcement each brown-eyed person would have thought: 'Suppose my eyes are not brown. I see 1 brown-eyed person. On the evening of the announcement the brown-eyed person would have thought: "Suppose my eyes are not brown. I see no brown-eyed people. The tourist said there exists a brown-eyed person. It is I! I will leave at noon tomorrow." But he/she didn't. Therefore my eyes are brown. I will leave at noon tomorrow.' But they didn't. Therefore my eyes are brown. I will leave at noon tomorrow." But they didn't. Therefore my eyes are brown. I will leave at noon tomorrow.' But they didn't. Therefore my eyes are brown. I will leave at noon tomorrow." But they didn't. Therefore my eyes are brown. I will leave at noon tomorrow.' But they didn't. Therefore my eyes are brown. I will leave at noon tomorrow." But they didn't. Therefore my eyes are brown. I will leave at noon tomorrow.' But they didn't. Therefore my eyes are brown. I will leave at noon tomorrow." But they didn't. Therefore my eyes are brown. I will leave at noon tomorrow.' The ten brown-eyed people leave at noon ten days after the announcement.

'Only one in two cameras are actually in operation, but this could increase to as many as one in three.'—*Watford Observer* [sent by Peter Fletcher]

Solution 189.3 – Amazing object

A convex solid looks like this from three orthogonal views:

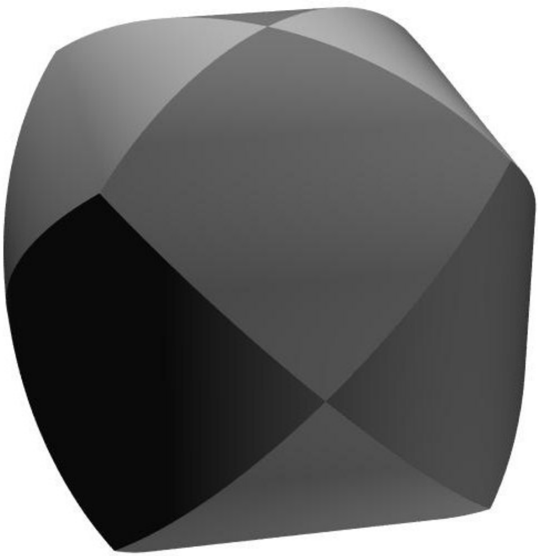
$$\bigotimes \quad \bigotimes \quad \bigotimes .$$

What is it? What is its volume?

Dick Boardman

Call the mutually orthogonal directions X, Y and Z. It is evident that the object, whatever it is, is contained within a cylinder of unit radius whose axis is the X-axis. Similarly, the object, whatever it is, is contained within a cylinder of unit radius whose axis is the Y-axis. Finally, the object, whatever it is, is contained within a cylinder of unit radius whose axis is the Z-axis.

This suggests that the solid whose points are within all three cylinders is well worth looking at. Closer investigation shows that this is the solution.



For those of you with a computer, do you have a copy of POV-Ray? If not, why not? It is a high quality program, whose documentation shames most professional programs. It is easy to use, great fun and *free*. It allows you to create camera-quality pictures in full colour, using ray tracing, the one above being a very simple example.

Now for the volume! The solid may be thought of as a cube with a cap on each face. The length of the side of each edge of the cube is the square root of 2 so the volume of the cube is $2^{3/2}$. A slice through the cap parallel to the face of the cube is a square. So its volume is $\int(\text{side of square})^2 dz$. But

$$\frac{(\text{side of square})^2}{4} + z^2 \;=\; 1.$$

Hence the volume of a cap is

$$\int_{1/\sqrt{2}}^{1} (4 - 4z^2)dz = \frac{4}{3}\left(2 - \frac{5}{2\sqrt{2}}\right).$$

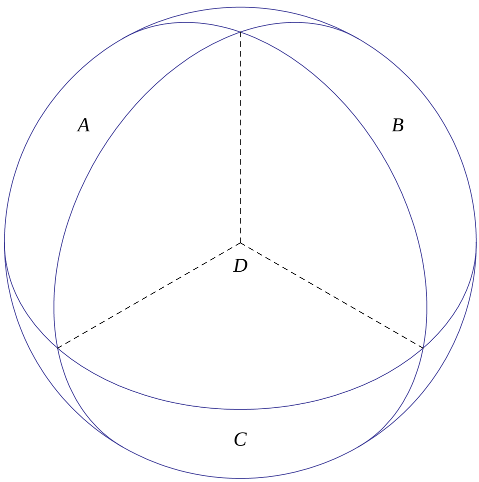
The volume of the object is therefore

$$\text{volume of cube} + 6(\text{volume of cap}) = 8(2 - \sqrt{2}).$$

Barbara Lee

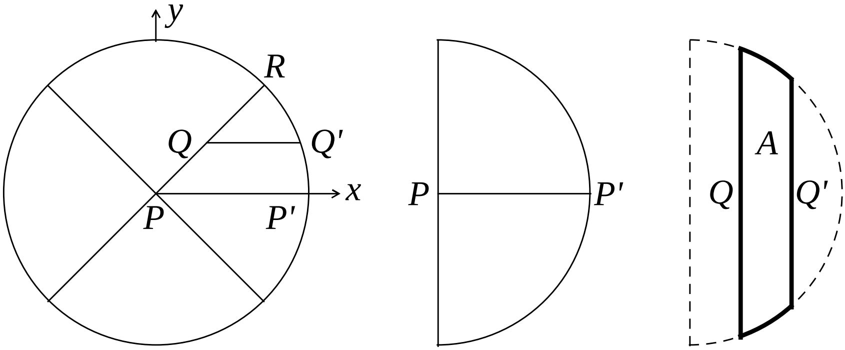
I cannot find anything similar in my engineering drawings. Ignoring convexity, it is clearly a sphere with three eighths of the volume cut away. (So the volume is $5\pi/6$.)

Parts A, B and C are the cut away sections. Part D is now only attached by a thin rod at the centre of the sphere. It is not easy to draw [Editor agrees] but you can see what I mean if you cut it out of an apple.



David Kerr

The object is the intersection of three identical orthogonal cylinders whose axes meet at the object's centre.



If we let P be the origin and P' the point $(1,0)$, the co-ordinates of Q, Q' and R are (x,x), $\sqrt{1-x^2}, x)$ and $(1/\sqrt{2}, 1/\sqrt{2})$. The volume of the part of the object between PP' and R is got by integrating the shaded area, A, as QQ' goes from PP' to R. Hence the total volume is given by

$$V = \int_0^{1/\sqrt{2}} A dx.$$

The first step is to find A. The area of a segment of a unit circle is given by $\theta - (\sin\theta)(\cos\theta)$, where the segment subtends an angle of 2θ at the centre. Hence

$$\begin{aligned} A &= \left(\cos^{-1} x - x\sqrt{1-x^2}\right) - \left(\cos^{-1}\sqrt{1-x^2} - x\sqrt{1-x^2}\right) \\ &= \cos^{-1} x - \sin^{-1} x. \end{aligned}$$

Hence

$$V = 8\int_0^{1/\sqrt{2}} (\cos^{-1} x - \sin^{-1} x) dx = 8(2-\sqrt{2}) \approx 4.6863.$$

As expected, this volume is slightly greater than that of the unit sphere, $4\pi/3 \approx 4.1888$.

A stitch in time saves nine. Too many cooks spoil the broth. You should not make mountains out of molehills. One swallow does not make a summer. Yes, we have no bananas, we have no bananas today. A spoonful of sugar helps the medicine go down.

Solution 189.7 – All the sevens

> If N is any non-negative integer, prove that the last digit of its 77th power is the same as the last digit of N.

Patrick Lee

Let L be the last digit of N. If P is a positive integer, then the last digit of L^P is the same as the last digit of N^P. For let $N = K + L$; then $N^P =$ (terms in K and L) $+ L^P$. Tabulating all possible values of L and the last digit of its powers P, we get the following table.

L	0	1	2	3	4	5	6	7	8	9
$P = 1$	0	1	2	3	4	5	6	7	8	9
$P = 2$	0	1	4	9	6	5	6	9	4	1
$P = 3$	0	1	8	7	4	5	6	3	2	9
$P = 4$	0	1	6	1	6	5	6	1	6	1
$P = 5$	0	1	2	3	4	5	6	7	8	9

From this we see that, for any value of L, the last digit of its 5th power is L. Hence, for any positive integer r the last digit of its $(4r + 1)$th power is L. Put $r = 19$, then $4r + 1 = 77$ so the last digit of N^{77} is the same as the last digit of N.

David Porter

This is trivial for integers ending in 0, 1, 5 and 6 since every power of integers with these endings ends in the same digit as the number itself.

Now the 4th powers and hence the 76th powers ($76 = 4 \cdot 19$) of integers ending in 3, 7 and 9 all end in 1 and hence the 77th powers will end in the same digit as the integer itself.

Also the 4th powers and hence the 76th powers of integers ending in 2, 4 and 8 all end in 6 and hence the 77th powers will end in the same digit as the integer itself.

Note to experts: Those of you who are number-theoretically well educated know that it is possible to deliver the answer in one or two lines. In fact, all you need to do is utter the words 'Fermat's Little Theorem' and maybe 'Euler's totient function'. Nevertheless, the above proofs are of interest because they avoid both of these dizzying concepts.—**ADF**

Solution 189.6 – Three friends

> Alan, Bert and Curt each have a different positive integer written on their foreheads. Also they know that one of the numbers is the sum of the other two. They take it in turns in alphabetical order to attempt to deduce their own number. The conversation goes as follows. Alan: 'I cannot deduce my number.' Bert: 'I cannot deduce my number.' Curt: 'I cannot deduce my number.' Alan: 'My number is 50.' What are Bert's and Curt's numbers?

David Porter

Bert and Curt's numbers are 10 and 40.

Since one of the three numbers is the sum of the other two, each of the friends know that the number on his forehead is either the sum or the difference of the two numbers that he can see on the other two foreheads.

Assume that Bert's number is 10 and Curt's is 40 then the friends will reason as follows:

Alan: 'I can see 10 and 40 thus my number is either 30 or 50 but at the moment I cannot tell which.'

Bert: 'I can see 40 and 50 thus my number is either 10 or 90. If my number is 10 then Alan will know that his number is either 30 or 50 and if mine is 90 he will know that his number is either 50 or 130. In either case he would not be able to deduce his own number. None of these facts enables me to deduce my own number.'

Curt: 'I can see 10 and 50 thus my number is either 40 or 60.'

He would then work back through what Alan and Bert would have thought if his number were 40 and then again for if it were 60. However, none of this turns out to be any help in deducing his own number. (An exercise for the reader!)

Alan: 'If my number is 30 then Curt will have deduced that his number is 20 or 40 but he would then have reasoned that if his number is 20 I would have been seeing 10 and 20 and, since our three numbers are all different, I would have been able to say with certainty that my number was 30. But I did not, therefore he would have deduced that I must be seeing 40 on his forehead. However, he did not deduce that his number is 40 so my number cannot be 30 and so it must be 50.'

Solution 189.1 – Neighbours

> Some people are sitting at round tables in a restaurant, at least three to a table. Partition the diners into two sets, M and W, of m and w persons, respectively. Show that the number of M–M neighbours minus the number of W–W neighbours is equal to $m - w$.

David Porter

Consider a single table with a random arrangement of m Ms and w Ws, where $m + w > 2$. Let the number of M–M neighbours minus the number of W–W neighbours be the D value of the arrangement.

First let's consider a couple of special cases where one of the classes (W say) is either empty or only has one member. In both cases there are no W–W neighbours whilst in the first case there are m M–M neighbours and in the second there are $m - 1$. These will be the corresponding D values and in both cases this is $m - w$.

For all the remaining cases, $m, w \geq 2$, if we remove an M from the table either the M–M count will decrease by one or the W–W count will increase by one. On randomly replacing the M at the table either the M–M count will increase by one or the W–W count will decrease by one. In all four combinations of these pairs of possibilities both counts change by the same amount (1, 0 or -1) and hence leave the D value unchanged. Thus, since any arrangement of the m Ms and w Ws can be transformed to any other by a sequence of such moves, $D(m, w)$ is independent of the seating plan.

But if we consider the seating plan that places all the Ms on one arc of the table and all the Ws on the remaining arc we find that we have $m - 1$ M–M neighbours and $w - 1$ W–W neighbours and so $D(m, w) = (m - 1) - (w - 1) = m - w$.

Since this is true for any of the permitted table sizes it is also true over any set of such tables.

Problem 192.1 – Root 33

If $\theta = 2\pi/33$, show that

$$\cos\theta + \cos 2\theta + \cos 4\theta + \cos 8\theta + \cos 16\theta \;=\; \frac{1 + \sqrt{33}}{4}.$$

Mr Schmidt proposed as the only way to save Europe's remaining fish stocks that all EU fleets should be cut back by 40 per cent, hitting Spain proportionally more than anyone else.—*Sunday Telegraph* [sent by Peter Fletcher]

Fermat numbers

Tony Forbes

A number of the form $2^N + 1$ cannot be prime unless N is a power of two. The proof is not difficult. If q is odd,

$$x^q + 1 = (x + 1)(x^{q-1} - x^{q-2} + x^{q-3} - \ldots + 1).$$

Therefore if N has an odd factor, q, then $2^N + 1$ is divisible by $2^{N/q} + 1$.

On the other hand, for numbers of the form $2^{2^n} + 1$, usually called *Fermat numbers* and denoted by F_n, there is no corresponding algebraic factorization; so it is quite possible for them to be prime. Perhaps it was this observation, together with the evidence of the first four cases, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$ and $F_4 = 65537$, which led Fermat to conjecture that *all* F_n *are prime.*

However, unlike that other famous conjecture of Fermat, this one seems to have been amazingly wide of the mark. According to *Fermat factoring status*, the Web site http://www.prothsearch.net/fermat.html managed by Wilfrid Keller, as at 21 February 2003 a total of 212 composite F_n are known. And if that wasn't enough to invalidate Fermat's conjecture, *not one single further example of a prime Fermat number has been found.*

Fermat numbers come in a variety of flavours.

(i) The known primes: F_1, F_2, F_3 and F_4. It is likely that in the forseeable future there will be no additions to this list—unless there is a major mathematical breakthrough, or a sudden enormous increase in computer speeds. The next possible candidate is F_{33}, which at 2,585,827,973 digits is far too large for any of the current primality proving programs. And in any case, billion-digit primes are not all that common.

(ii) Fermat numbers which are composite and completely factorized: F_5 (Euler, 1732), F_6 (Landry & Le Lasseur, 1880), F_7 (Morrison & Brillhart, 1970), F_8 (Brent & Pollard, 1980), F_9 (Lenstra, Manasse *et al.*, 1990), F_{10} (Selfridge, 1953; Brillhart 1962; Brent, 1995) and F_{11} (Cunningham, 1899; Brent & Morain, 1988).

Richard Guy has a personal interest in this section. He says, 'Keep at it! If no other Fermat number has been completely factored by my 100th birthday, then I have to pay John Conway \$20.00, but if you manage completely to factor one, then he pays me—the sooner the better!'

(iii) Proved composite but with no known factor: F_{14} (Selfridge & Hurwitz, 1963), F_{20} (Buell & Young, 1987), F_{22} (Crandall, Doenias, Norrie & Young, 1993), F_{24} (Mayer, Papadopoulos & Crandall, 1999).

(iv) Proved composite, a prime factor found but known to be incompletely factorized: F_{12} (a prime factor was found by Lucas & Pervushin, 1877; the cofactor was proved composite by Baillie, 1986), F_{13} (Hallyburton & Brillhart, 1974; Brent, 1995), F_{15} (Kraitchik, 1925; Brent, 1997), F_{16} (Selfridge, 1953; Brent, 1996), F_{17} (Gostin, 1978; Baillie, 1987), F_{18} (Western, 1903; Crandall, 1999), F_{19} (Riesel, 1962; Crandall, Doenias, Norrie & Young, 1993), F_{21} (Wrathall, 1963; Crandall, Doenias, Norrie & Young, 1993), F_{23} (Pervushin, 1878; Mayer, Papadopoulos & Crandall, 2000).

(v) Don't know. The only infinite class. A new Fermat prime must come from here. The smallest is F_{33}, followed by F_{34}, F_{35}, F_{40}, ...

(vi) Proved composite but not included in any of the above. There is at least one known prime factor but—unlike (iv)—the status of the unfactorized part has not been determined: F_{25} (Wrathall, 1963), F_{26} (Wrathall, 1963), F_{27} (Wrathall, 1963), F_{28} (Taura, 1997), F_{29} (Gostin & McLaughlin, 1980), F_{30} (Wrathall, 1963), F_{31} (Kruppa & Forbes, 2001), F_{32} (Wrathall, 1963), F_{36} (Seelhoff, 1886), F_{37} (Gostin, 1991), F_{38} (Cullen, Cunningham & Western, 1903), F_{39} (Robinson, 1956), ... many omitted ..., F_{303088} (Young, 1998), F_{382447} (Cosgrave & Gallot, 1999), $F_{2145351}$ (Cosgrave, Jobling, Woltman & Gallot, 2003).

The last one is a new world record, discovered on 16 February 2003 by John Cosgrave, using Paul Jobling's program NEWPGEN, George Woltman's PRP, and Yves Gallot's PROTH. What John actually discovered was that $F_{2145351}$ is divisible by the 645817-digit prime $3 \cdot 2^{2145353} + 1$. The prime factor itself is a significant achievement in its own right—at time of writing, $3 \cdot 2^{2145353} + 1$ ranks 5th in the list of the largest known primes and first amongst the non-Mersenne primes.

The number $F_{2145351}$ is not small. I was having difficulty trying to explain this concept to someone. Here is a part of the conversation: 'So what you are saying is that the 2145351st Fermat number has approximately 645815 digits.' 'No,' I replied, 'What I said was that *the number of digits* in the 2145351st Fermat number has approximately 645815 digits.'

According to some measurements I took, 500 sheets of standard A4 paper occupy a volume of about $11.75 \times 8.25 \times 2.0$ cubic inches, a page of M500 has about 4.5×7.0 square inches of print area, and a square inch of typical text holds 84 digits. Armed with this information, you might like to work out how big a building (in cubic light-years, say) would be required to store one copy of the resulting issue of M500 if the Editor were foolish enough to publish the decimal representation of $F_{2145351}$ in full.

Solution 190.2 – Nested roots

Given $\sqrt{4+\sqrt{4^2+\sqrt{4^3+...}}} = 3$, find $\sqrt{4-\sqrt{4^2-\sqrt{4^3-...}}}$.

Jim James

The clue is to explore the properties of 2^n+1, when we find that

$$2^n+1=\sqrt{4^n+(2^{n+1}+1)},$$

so the given equation follows with $n=1$.

The solution for $\sqrt{4-\sqrt{4^2-\sqrt{4^3-...}}}$ is now clear, for

$$2^n-1=\sqrt{4^n-(2^{n+1}-1)}.$$

Dick Boardman

There are many problems in mathematics which become very simple if you can guess the answer—and this is one of them. First, replace the sequence of nested roots by a function with a sequence of values.

Consider a function $f(n)$, where

$$f(1)=\sqrt{4+f(2)},\quad f(2)=\sqrt{4^2+f(3)},\quad f(3)=\sqrt{4^3+f(4)},$$

and so on. Then $f(1)$ is the number we require.

The problem says that the sequence of nested roots continues indefinitely but in order to calculate approximate values for the function we must specify a limit, say $f(n)=0$ for $n>20$. Using this definition, my computer can calculate values for $f(n)$, and to within its working precision they look like this:

1	2	3	4	5	6	7	8	9
3	5	9	17	33	65	129	257	513.

Now where have I seen this sequence of numbers before? Clearly, this is the moment for an intelligent guess. Suppose $f(n)=2^n+1$. Then

$$f(n-1)=\sqrt{4^{n-1}+2^n+1}=\sqrt{2^{2n-2}+2\cdot 2^{n-1}+1}=2^{n-1}+1.$$

Thus the function $f(n)$ converges to 2^n+1 and $f(1)=3$.

A very similar argument shows that the second set of nested roots converge to 1.

Tony Forbes

This is not a solution. All I want to do is investigate the function $g(n,x)$, defined by

$$g(1,x) = x,$$
$$g(n+1,x) = g(n,x)^2 - 4^n, \quad n = 1, 2, \ldots.$$

From the definition, $g(n,x)$ is a polynomial of degree 2^{n-1} in x,

$$\begin{aligned} g(2,x) &= x^2 - 4 = (x-2)(x+2), \\ g(3,x) &= x^2(x^2 - 8), \\ g(4,x) &= (x^4 - 8x^2 - 8)(x^4 - 8x^2 + 8), \\ g(5,x) &= (x^8 - 16x^6 + 64x^4 - 80)(x^8 - 16x^6 + 64x^4 - 48), \end{aligned}$$

and for $n > 5$ the pattern of two factors continues, the polynomials approximately doubling in length at each step.

When $x = 3$, it turns out that $g(n,x)$ is particularly easy to evaluate; in fact, one can prove by induction that $g(n,3) = 2^n + 1$. This is interesting. If you try to compute $g(n,x)$ for values of x other than 3, you will see that it blows up to enormous levels even for n of quite modest size. For example, assuming my computations were sufficiently precise, I found that

$$g(16, 2.9999999999) \approx 4 \cdot 10^{454}$$

and

$$g(16, 3.0000000001) \approx 2 \cdot 10^{470},$$

which, you will agree, are a long way away from $g(16,3) = 2^{16} + 1 = 65537$.

Now let us define

$$h(n) = \sqrt{4^n + \sqrt{4^{n+1} + \sqrt{4^{n+2} + \sqrt{\ldots}}}}.$$

Squaring and rearranging, we obtain

$$h(n+1) = h(n)^2 - 4^n,$$

which is reminiscent of the equation that we used to define $g(n,x)$. If we set $x = h(1)$, the correspondence is exact: $h(n) = g(n,x)$ and, as we have seen, $g(n,x)$ apparently diverges to infinity unless $x = 3$.

Problem 192.2 – 10 degrees

Let $x = 1 + 4\sin 10^\circ$. Show that

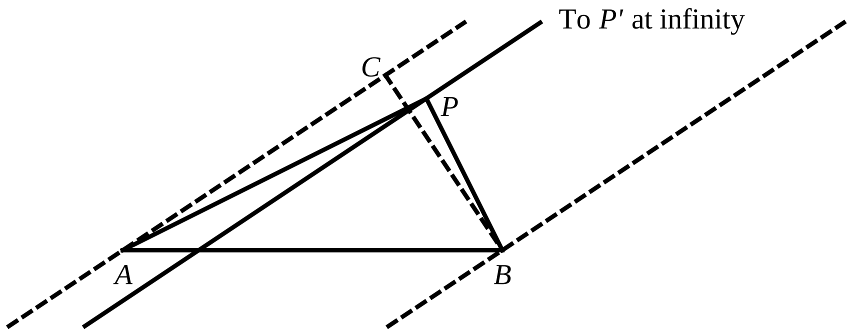
$$x = \sqrt{11 - 2\sqrt{11 + 2\sqrt{11 - 2x}}}.$$

Moving point

Patrick Lee

In M500 **189**, Dilwyn Edwards posed some questions about mathematical intuition applied to a problem concerning a point P moving on a straight line passing between two fixed points A and B (see below).

His first question was about the limiting value of the difference between the distances PA and PB. In the diagram I have tried to show the situation when P is at a large distance from A and B (P 'at infinity'). In that case the lines PA and PB are effectively parallel so that the difference between their length approaches AC and, because the line $y = x - 0.7$ has slope 1, the limiting value of the angles that PA and PB make with AB is $q = 45°$. Hence $AC = AB \cos 45° \approx 0.7071$.



His second question, about the local maximum in the value of d, the difference between PA and PB, is not quite so easy. To answer it we have to recognize that the locus of a point, the difference of whose distances from two fixed points is constant, is a hyperbola whose foci are the fixed points. The illustration opposite shows a family of hyperbolae with A and B as their foci. The centre of the hyperbolae is labelled O and the vertical line through O is the locus of points equidistant from A and B ($d = 0$). As $|d|$ increases, the curvature of a hyperbola increases until a limiting value is reached when $|d| = 1$ when the hyperbola becomes a straight line starting at B and coincident with the positive x-axis for positive d and starting at A, coincident with the negative x-axis for negative d. The line $y = x - 0.7$ is also shown and we can see that it intersects some of the hyperbolae and will be tangential to one particular hyperbola on the right side of the diagram (d positive). The hyperbola to which the line is tangential will be the one corresponding to the largest value of d that can be attained and the point of tangency is therefore that of the local maximum.

To calculate the local maximum, it is convenient to work with the origin at the centre of the family of hyperbolae, rather than at point A as in the original problem. The equation of the line then becomes $y = x - 0.2$. The standard formula for a hyperbola is $x^2/a^2 - y^2/b^2 = 1$, i.e.

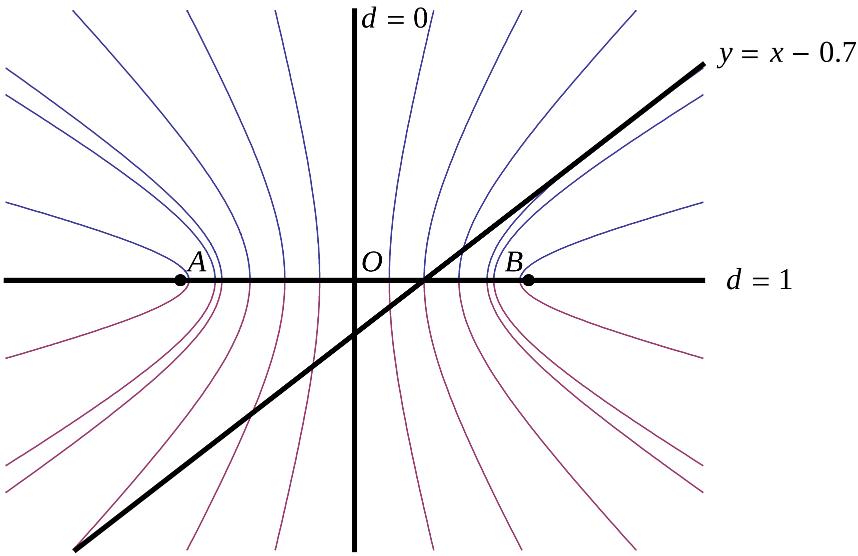
$$b^2x^2 - a^2y^2 = a^2b^2. \tag{1}$$

The tangent point must also satisfy

$$y = x - 0.2. \tag{2}$$

Differentiating (1) gives $2b^2x - 2a^2y \cdot dy/dx = 0$. And, since, at the tangent point, the slope of the hyperbola must be the same as the slope of the line $y = x - 0.2$, we have $dx/dy = 1$. Thus

$$b^2x = a^2y. \tag{3}$$



Substituting (3) in (1), we have $b^2x^2 - b^2xy = a^2b^2$, i.e. $x^2 - xy = a^2$. Substituting from (2), we have $x^2 - x^2 + 0.2x = a^2$. Hence

$$x = 5a^2. \tag{4}$$

Substituting (2) in (3) gives $b^2x = a^2(x - 0.2)$; thus

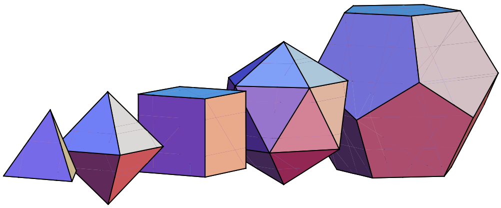
$$(a^2 - b^2)x = 0.2a^2. \tag{5}$$

It is known that for a hyperbola, $c^2 = a^2 + b^2$, where $2c$ is the inter-focal distance (= 1 in this case). Hence $c^2 = 0.25$ and $b^2 = 0.25 - a^2$. Substituting for b^2 in (5) gives $(2a^2 - 0.25)x = 0.2a^2$. Substituting for x from (4) gives $5a^2(2a^2 - 0.25) = 0.2a^2$, which reduces to $10a^2 = 1.45$. So $a \approx 0.3808$. But $d = 2a$, so the local maximum in the value of d is approximately 0.7616 and occurs when $x \approx 0.7250$. In the original co-ordinates where point A is at $(0, 0)$ this is $x \approx 1.2250$.

It is interesting to note that the value of d when P is 'at infinity' can be derived by approximating the relevant hyperbola by one of its asymptotes which, in order for it to approximate to the line of slope 1, must itself have slope 1. This means that $a = b$ and, since $a^2 + b^2 = 0.25$, $a = b = \sqrt{2}/2$. So $d = 2a = \cos 45^\circ \approx 0.7071$.

Problem 192.3 – Platonic solids

Arranged in order of volume, the five regular polyhedra are the tetrahedron, with volume $\sqrt{2}/12$, the octahedron, $\sqrt{2}/3$, the cube, 1, the icosahedron, $(5(3 + \sqrt{5}))/12$, and the dodecahedron, $(15 + 7\sqrt{5})/4$.



Can they be fitted one inside the next, Russian doll style?

'Our plan is not to inconvenience passengers more than is humanly possible.'—Ryanair spokesman talking on R4 about the takeover of Buzz.

JRH— I put that on an English usage message board which I frequent, and nobody seemed to grasp that there was anything odd about it. Clearly I should look for a board with brighter subscribers. Compare these equivalent reassurances:

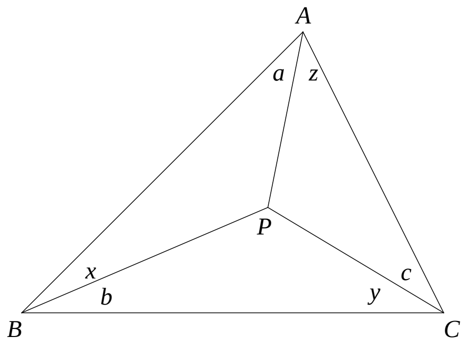
Our plan is not to rob you of more money than we can humanly carry.

Our plan is not to sleep with your spouse more often than we can humanly manage.

Our plan is not to make more noise outside your house than we can humanly achieve by state-of-the-art amplification.

Solution 189.8 – 30 degrees

If ABC is any triangle and P is any point inside ABC, show that not all of the angles PAB, PBC and PCA can exceed 30 degrees.



Ted Gore

In the triangle above it is easy to show (using the sine rule) that

$$\sin a \sin b \sin c \;=\; \sin x \sin y \sin z. \tag{1}$$

Now let $x + y + z = 3m$, $x = m + \theta$ and $y = m + \phi$. Then

$$\begin{aligned} V &= \sin x \sin y \sin z \\ &= \sin(m+\theta)\sin(m+\phi)\sin(m-\theta-\phi), \\ \frac{dV}{d\theta} &= \sin(m+\phi)\big[\cos(m+\theta)\sin(m-\theta-\phi) \\ &\qquad - \cos(m-\theta-\phi)\sin(m+\theta)\big] \\ &= \sin(m+\phi)\sin(-2\theta-\phi) \end{aligned}$$

and

$$\begin{aligned} \frac{dV}{d\phi} &= \sin(m+\theta)\big[\cos(m+\phi)\sin(m-\theta-\phi) \\ &\qquad - \cos(m-\theta-\phi)\sin(m+\phi)\big] \\ &= \sin(m+\theta)\sin(-\theta-2\phi). \end{aligned}$$

These two derivatives are zero when $\theta = \phi = 0$ and this point is a local maximum. Hence $\sin x \sin y \sin z \le \sin^3 m$.

Now assume that a, b and c are all greater than 30°. Then $m < 30^\circ$, since $3m = 180^\circ - a - b - c$, so that

$$\sin x \sin y \sin z \;\le\; \sin^3 m \;<\; \sin^3 30^\circ.$$

Therefore $\sin x \sin y \sin z < \sin a \sin b \sin c$, which contradicts (1). Hence it is not possible for all of a, b and c to be greater than 30°.

Solution 184.1 – Twelve boxes

> There are twelve closed boxes numbered $1, 2, \ldots, 12$. On each turn you throw a pair of dice and you must open closed boxes whose numbers add up to the sum of the numbers shown by the dice. If this is impossible, the game stops and you lose. If you manage to open all the boxes, the game stops and you win. If neither, the game continues. What's the probability of winning?

Dick Boardman

I was fascinated by John Smith's solution of 'Twelve boxes' [M500 **190** 18] and I have coded my version of his algorithm. Unfortunately, there is a small discrepancy in the final answer. His best probability of winning is 0.003622181; mine is 0.003730.

My program interprets the pattern of open and closed boxes as a binary number; closed boxes corresponding to noughts and open boxes to ones, so that all boxes closed corresponds to 0 and all boxes open to 4095. I call this number the *state number*. There are eleven possible dice sums at any throw $(2, 3, \ldots, 12)$. For each state number I store

(i) the total probability,
(ii) the eleven next states, one for each dice sum, and
(iii) the eleven next probabilities, again one for each dice sum.

The total probability is the sum of the eleven 'next probabilities' and is calculated last. The final answer is the total probability of state 0.

States must be processed in order of increasing 'sum of closed boxes', starting with all boxes open (4095), whose probability of winning is 1, and all boxes open except box 1 (4094), whose probability of winning is zero since no throw can total less than 2. Any operation opening boxes reduces the 'sum of closed boxes' so that processing states in this order means that any node processed will only refer to a node already processed. The sum of the numbers $1, 2, \ldots, 12$ is 78 so that (including zero) there are 79 possible 'sums of closed boxes'. To allow processing in the required order, I create an index. This index has 79 lists of state numbers, one for each 'sum of closed boxes'.

To see how this index is created we must first learn a little about binary arithmetic. Consider the decimal number 1037. This is the sum $7+3\cdot 10+0\cdot 100+1\cdot 1000$. It is also equal to $1+0\cdot 2+1\cdot 4+1\cdot 8+0\cdot 16+0\cdot 32+0\cdot 64+0\cdot 128+0\cdot 256+0\cdot 512+1\cdot 1024$. In binary arithmetic this is written as 10000001101. To convert a number from decimal to binary involves nothing more difficult than subtracting 1 if the result is odd and then dividing by two. First note that 1037 is odd so that the least significant binary digit is 1. Subtract it and divide by 2 to give 518. This is even; so the next binary digit is 0. Repeat this and show the result as a table.

1037	518	259	129	64	32	16	8	4	2	1
1	0	1	1	0	0	0	0	0	0	1
1	2	3	4	5	6	7	8	9	10	11

The binary version of the number is found by reading the second row from right to left. Now, remembering that a nought is a closed box and a one is an open box, we see that there closed boxes in positions 2, 5, 6, 7, 8, 9, 10 so that the 'sum of closed boxes' is 47. So state number 1037 is attached to list 47. The first few entries in the index are as follows.

sum	state			
0	4095			
1	4094			
2	4093			
3	4091	4092		
4	4087	4090		
5	4079	4086	4098	
6	4063	4078	4085	4088

Thus my program processes states in the order 4093, 4091, 4092, 4087, 4090, 4079 and so on. Obviously there might be many states to each sum. The largest number is 124 states to a sum of 39. For each state and for each possible throw there is a next state. The probability of each link to a next state is the 'probability of next state' multiplied by the 'probability of the throw'.

When there is more than one possible set of boxes that could be closed, the next state with the highest probability is chosen (this is different from John Smith's example). When all the possible links from a state have been calculated, the link probabilities are added together to give the total state probability.

Here are a few examples.

State 4093 (box 2 closed) has one link to state 4095. This results from a dice sum of 2. The probability of state 4095 is 1 and the probability of a dice sum of 2 is 1/36; so the next probability for a dice sum of 2 is 1/36. There are no other possible dice sums so the total probability is also 1/36. State 4091 (box 3 closed) has a link to 4095 from a dice sum of 3. This could be throws $(1, 2)$ or $(2, 1)$ so the probability is 2/36. The probability of state 4095 is 1 so that the probability for a dice sum of 3 and also the total probability is 2/36.

State 4092 (boxes 1 and 2 closed) also has a link to 4095 with the same probability, but it also has a link to state 4094 from a dice sum of 2. The dice sum probability is 1/36 but the probability of state 4094 is 0 so the link probability is 0. The total probability is $2/36 + 0 = 2/36$.

Skipping a few states, consider state 4088 (boxes 1, 2 and 3 closed). A throw of 3 could take it to either 4091 or 4092. As it happens, these two states have the same probability so it makes no difference but had they been different, the one with the higher probability would have been chosen.

And so on for all the rest of the states.

Finally, the state with a 'sum of closed boxes' = 78 (all boxes closed, state 0) is processed and its probability is the final answer.

Letters to the Editors

40 years

Dear Tony,

Re: John Reade's question: 'Everybody's wedding anniversary falls on a Sunday—true or false?' [M500 **189** 14].

This is both false and possibly true.

I know for a fact that my 40th wedding anniversary falls on a Thursday in July this year so the statement is false.

However, many people of my father's generation would say of me 'Porter's 40th wedding anniversary falls on a Thursday'. Thus if Mr Everybody (who has yet to publish his diary) was married three days after I was then these same people would say of him 'Everybody's 40th wedding anniversary falls on a Sunday'.

David Porter

The single sentence 'Everybody's wedding anniversary falls on a Sunday' may possibly be true if the word 'Everybody's' refers to a single object. For example there was once a magazine called 'Everybody's'.

Dick Boardman

Amazing object

Dear Eddie

I have been beating my brains out on Problem 189.3 – Amazing object. At first I tried a structure like a four-bladed fan with extra blades at the top and bottom, but it proved impossible to get the blades to overlap correctly and still fit inside a sphere. Also, it was not a solid but a collection of planes. Two senior geologists at Imperial College, to whom I described the problem, are equally stuck.

I am wondering whether 'three mutually orthogonal directions' is a weasel phrase, as it does not categorically state that the x and y axes have to go straight across and up the page. If the axes are rotated 45 degrees, the problem becomes trivial because an ordinary sphere will do the job and any fule can calculate its volume.

I note that no measurement is supplied, which makes it look as if the volume is expected to be zero. This could be achieved with axes tilted at 45 degrees by having a figure made of three intersecting circles, but that is not a convex solid.

The problem could be solved with conventional axes if you take a heretical attitude about what constitutes a view. Imagine a sphere the size of the world, and that you are standing on it, and that it is inscribed with three Xs large enough to stretch to the horizon when viewed from your eye level. They are placed, say, on the equator at 0 degrees longitude, on the equator at 90 degrees west, and at the north pole. If you photograph any of these Xs from head height with a fisheye lens pointed straight at the centre of the sphere, you will get an X in a circle, plus or minus your own ankles.

Ralph Hancock

Harmonic ratio

Dear Tony

Re: Problem 190.7. Extract from *Collins Dictionary of Mathematics* by E. J. Borowski and J. M. Borwein (1989 edition):

> *Harmonic Ratio*: (projective geometry) A cross-ratio of four points (harmonic points) that is equal to −1: that is, such that the directed ratio ...

$$(A, B; D, E) \quad = \quad (AC \cdot BD)/(AD \cdot BC) = -1.$$

The first bracket is as copied—that seems to be five points!

Regards,

Ken Greatrix

Problem 192.4 – Two boxes

ADF

You have an object of dimensions $a\times b\times c$ and a hole of dimensions $A\times B\times C$. The object fits in the hole. Make two boxes of internal dimensions a_1, b_1, c_1 and a_2, b_2, c_2 out of materials with thickness t_1 and t_2, respectively. Choose $a_1, b_1, c_1, t_1, a_2, b_2, c_2$ and t_2 at random, subject to the constraint that the object fits in each box and each box fits in the hole.

What's the probability that one box fits in the other box?

I have more than an academic interest in the answer. I wanted to store some CDs in a safe place and I was unsuccessful in finding a pair of suitable nested boxes out of my considerable collection of old biscuit tins, plastic Tupperware, etc.

Problem 192.5 – 16 polygons

ADF

There are two regular pentagons, eight squares and six equilateral triangles. All have the same side length.

Is it possible to make a polyhedron out of these 16 polygons?

Problem 192.6 – 500 factors

Dick Boardman

The number 48 has ten factors, including 48 and 1. What is the smallest number that has exactly 500 factors?

Problem 192.7 – 4-cycle-free graphs

ADF

Draw a (finite) graph in which (i) every vertex has degree at least 3 and (ii) there are no 4-cycles.

When you have done that, try drawing a graph which also has no 8-cycles. Then you can continue by drawing graphs that also avoid 16-cycles, 32-cycles, and so on. In view of a hitherto unsolved problem of Erdős and Gyarfas, it would be interesting to see how far you can go.

The problem asks whether there exists a graph which has no vertex of degree less than three and no cycle of length a power of two.

Proverbs and well-known sayings

1. An isolated, species-specific ornithological observation is not necessarily indicative of the seasonal transition at vernal termination.

2. In cases of difficulty with medicinal administration, the situation can usually be alleviated by the addition to the dose of approximately 5ml glucose.

3. It is inadvisable to engineer a situation where it is possible to initiate an orogeny by means of the excavations of a particular species of small mammal.

4. Excessive employee levels in the catering department is consistently cited as a primary causation of potagian failure.

5. Prompt appraisal of haberdasherial requirements can achieve substantial savings, often in the region of 900 per cent.

6. With reference to the question concerning the total depletion of banana stocks, we wish to respond in the affirmative. Furthermore, we confirm that it is the current date to which this situation applies.

The above is exhaustive of the limitations to our deliberations. Continuance of this feature is contingent upon future readership contribution supported, as usual, by editorial encouragement.

M500 Mathematics Revision Weekend 2003

The **29th M500 Mathematics Revision Weekend** will be held at **Aston University**, **Birmingham** over **12–14 September 2003**.

The Weekend is designed to help with revision and exam preparation, and is open to all OU students. We plan to present most OU maths courses. Sessions start at 19.30 on the Friday and finish at 17.00 on the Sunday.

On the Saturday evening we have a mathematical guest lecture. After the lecture **Charles Alder** will be running a disco. For the less energetic we plan to organize a ceilidh, to which you are invited to contribute—especially if you play a musical instrument.

See http://freespace.virgin.net/jeremy.humphries/sept.htm for full details and an application form, or send a stamped, addressed envelope to

Jeremy Humphries, M500 Weekend 2003.

(It's a Gaelic word; 'Ceilidh ($k\bar{a}'l\breve{i}$) ... Informal gathering for music, dancing, song, and story'—*COD*. Usually the songs are of the folky type, often concerning the trials and tribulations of whingeing lovesick fishermen.)

Contents M500 192 – June 2003