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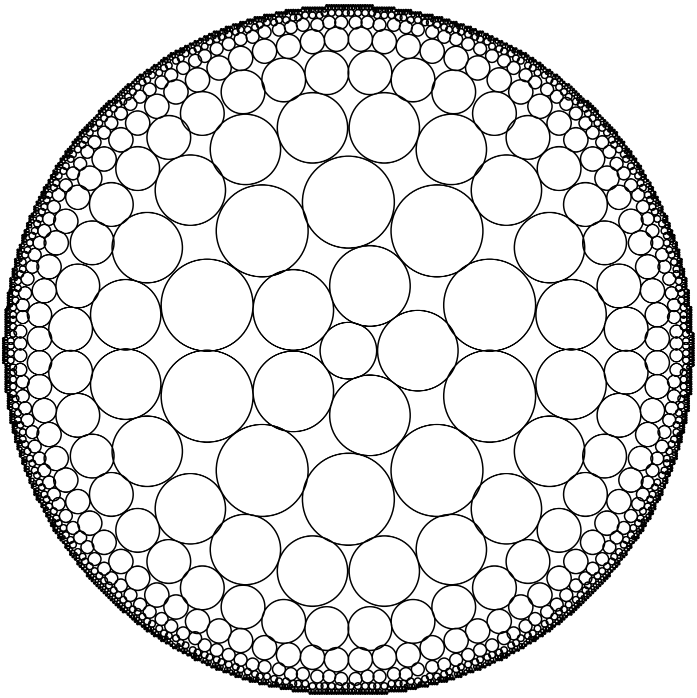

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

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A Golconda of golden numbers

Dennis Morris

The nugget The number $(1+\sqrt{5})/2 = 1.61803\ldots$ has from ancient times been called the golden ratio. It is an irrational number that is usually signified by ϕ and thus is called phi. It is a number with some remarkable properties. Thus

$$\phi \;=\; \frac{1}{\phi}+1$$

from which (multiplying throughout by ϕ) we have

$$\phi^2 = 1+\phi, \quad \phi^3 = \phi+\phi^2, \quad \phi^4 = \phi^2+\phi^3, \;\ldots.$$

But ϕ has a brother; it is the number $(1-\sqrt{5})/2 = -0.61803\ldots$, which is signified by ϕ_{brother}. Phi's brother, like ϕ, is an irrational number. Phi's brother is also a number with some remarkable properties. Thus

$$\phi_{\mathrm{brother}} \;=\; \frac{1}{\phi_{\mathrm{brother}}}+1, \quad \phi^2_{\mathrm{brother}} = 1+\phi_{\mathrm{brother}},$$

$$\phi^3_{\mathrm{brother}} = \phi_{\mathrm{brother}}+\phi^2_{\mathrm{brother}}, \quad \phi^4_{\mathrm{brother}} = \phi^2_{\mathrm{brother}}+\phi^3_{\mathrm{brother}}, \ldots.$$

The brothers often appear together:

$$\phi\cdot\phi_{\mathrm{brother}} \;=\; -1, \quad \frac{1+\sqrt{5}}{2}\cdot\frac{1-\sqrt{5}}{2} \;=\; -\frac{4}{4},$$

$$\phi+\phi_{\mathrm{brother}} \;=\; 1, \quad \frac{1+\sqrt{5}}{2}+\frac{1-\sqrt{5}}{2} \;=\; 1.$$

Phi is a number which turns up all over the place. As a continued fraction it is special:

$$\phi \;=\; [1;1,1,1,1,1,\ldots], \qquad \phi_{\mathrm{brother}} \;=\; [0;1,1,1,1,1,\ldots].$$

It is the ratio in which two diagonals of a regular pentagon cut each other. It is also the ratio of the side length of a regular decagon to the radius of the circumcircle of that regular decagon. It also occurs in the regular pentagram (the secret sign of the Pythagoreans) in the forms ϕ^3, ϕ^2, ϕ^1, ϕ^0, ϕ^{-1} and ϕ^{-2}.

However, the most surprising thing about ϕ is its association with Fibonacci type sequences. Take two numbers, any numbers will do, integers, fractions, negative numbers, irrational numbers, even complex numbers; call one of them the first number and the other the second number. Create a third number by adding the chosen two numbers together, then create a fourth number by adding together the second chosen number and the newly made third number, then create a fifth number by adding together the third number and the newly made fourth number, and carry on to infinity. For example: if we choose the first number to be 3 and the second number to be 17, our Fibonacci type sequence is then

$$S = [3, 17, 20, 37, 57, 94, 151, 245, 396, 641, \ldots].$$

Let us refer to the first of these numbers as S_1, to the second as S_2, etc. Now let us calculate the ratios of successive pairs of these numbers, S_{m+1}/S_m. Starting with $m = 1$, the first fifteen such ratios are

$$\begin{array}{lll}
5.66666666667, & 1.17647058824, & 1.85000000000, \\
1.54054054054, & 1.64912280702, & 1.60638297872, \\
1.62251655629, & 1.61632653061, & 1.61868686869, \\
1.61778471139, & 1.61812921890, & 1.61799761621, \\
1.61804788214, & 1.61802868199, & 1.61803601576.
\end{array}$$

The ratios converge towards the number ϕ. If we consider the ratio S_m/S_{m+1}, these ratios converge to $-\phi_{\text{brother}}$.

Quite amazingly, the ratios converge towards ϕ no matter what initial numbers are used. If complex numbers are initially chosen, then the ratios converge towards $\phi + 0i$. That this happens for complex numbers is at first quite shocking. However, when one considers a ratio of such a complex number sequence, though still remarkable, it becomes clearer. Choose the first number to be $a + ib$. Choose the second number to be $c + id$. The Fibonacci type sequence is:

$$\begin{aligned}
&a + ib,\ c + id,\ (a + c) + i(b + d),\ (a + 2c) + i(b + 2d),\\
&(2a + 3c) + i(2b + 3d),\ (3a + 5c) + i(3b + 5d),\\
&(5a + 8c) + i(5b + 8d),\ (8a + 13c) + i(8b + 13d),\ \ldots.
\end{aligned}$$

Now calculate the ratio of the last two of these terms:

$$\frac{(8a + 13c) + i(8b + 13d)}{(5a + 8c) + i(5b + 8d)}.$$

The imaginary part of this is

$$\frac{-bc + da}{25a^2 + 80ac + 64c^2 + 25b^2 + 80bd + 64d^2},$$

where a, b, c and d in this expression are, of course, the chosen initial numbers. The numbers in the denominator of this expression will increase as successively later terms are used to calculate the ratio. Hence the denominator will increase without bound whereas the numerator is fixed by the values of the initially chosen numbers. Thus the imaginary part will tend to zero.

If we had taken the ratio of the 6th and 7th terms rather than the ratio of the 7th and 8th terms, then we would have found the imaginary part to be

$$-\frac{-bc+da}{9a^2+30ac+25c^2+9b^2+30bd+25d^2}.$$

Quite remarkable.

One of the properties of ϕ listed above is $\phi^2 = 1+\phi$; so ϕ is a root of $x^2-x-1=0$. Solving this equation gives ϕ as one root and ϕ_{brother} as the other root.

The mother lode It might surprise readers to discover that ϕ has some cousins; in fact, ϕ has an infinite number of cousins. Phi has a trivial cousin, which we signify as ϕ_1, defined as the solution of the equation $x-1=0$. So $\phi_1 = 1$. Phi itself is more properly signified as $\phi_2 = (1+\sqrt{5})/2$ and is the largest positive root of the equation $x^2-x-1=0$. Similarly, ϕ_3 is the largest positive root of the equation $x^3-x^2-x-1=0$, ϕ_4 is the largest positive root of the equation $x^4-x^3-x^2-x-1=0$, and so on up to ϕ_∞.

$\boldsymbol{\phi_3}$ The roots of $x^3-x^2-x-1=0$ are

$$\frac{1}{3} - \frac{\Delta}{6} - \frac{2}{3\Delta} - i\sqrt{3}\left(\frac{\Delta}{6} - \frac{2}{3\Delta}\right),$$
$$\frac{1}{3} - \frac{\Delta}{6} - \frac{2}{3\Delta} + i\sqrt{3}\left(\frac{\Delta}{6} - \frac{2}{3\Delta}\right)$$

and

$$\frac{1}{3} + \frac{\Delta}{3} + \frac{4}{3\Delta},$$

where $\Delta = (19+\sqrt{33})^{1/3}$. The last of these can be rewritten as

$$\frac{1}{3}\left((19+3\sqrt{33})^{1/3} + (19-3\sqrt{33})^{1/3} + 1\right).$$

Looking at these roots we see that there is one real root and two imaginary roots. The imaginary roots are conjugates, as we would expect. The real root is ϕ_3, and the two imaginary roots are ϕ_3's brothers. Evaluating the roots gives

$$\phi_3 \approx 1.83928, \qquad \phi_{3\,\text{brother}_1} \approx -0.41964 + 0.60629i,$$
$$\phi_{3\,\text{brother}_2} \approx -0.41964 - 0.60629i.$$

Now, in the case of ϕ_2 we found that $\phi \cdot \phi_{\text{brother}} = -1$ and $\phi + \phi_{\text{brother}} = 1$. In the case of ϕ_3 we have

$$\phi_3 \cdot \phi_{3\,\text{brother}_1} \cdot \phi_{3\,\text{brother}_2} \;=\; 1, \qquad \phi_3 + \phi_{3\,\text{brother}_1} + \phi_{3\,\text{brother}_2} \;=\; 1.$$

Similarly, from the basic equation, we get $\phi_3^3 = \phi_3^2 + \phi_3 + 1$ and all its variations.

Remarkably, ϕ_3 has a similar role to ϕ_2 as the limit to which ratios of successive numbers in Fibonacci type sequences converge. There is a difference. Instead of initially choosing two numbers and forming each successive number by adding the previous two numbers, one forms a ϕ_3-Fibonacci type of sequence by initially choosing three numbers and forming each successive number by adding the previous three numbers. For example: if we choose our first number to be 2, our second number to be 2 and our third number to be 1, our sequence is

$$S = [2, 2, 1, 5, 8, 14, 27, 49, 90, 166, 305, 561, 1032, \ldots].$$

With these numbers the first fifteen ratios S_{m+1}/S_m for successive m are

$$\begin{array}{lll}
0.50000000000, & 5.00000000000, & 1.60000000000,\\
1.75000000000, & 1.92857142857, & 1.81481481481,\\
1.83673469388, & 1.84444444444, & 1.83734939759,\\
1.83934426230, & 1.83957219251, & 1.83914728682,\\
1.83930453109, & 1.83930105987, & 1.83927737113,
\end{array}$$

converging towards ϕ_3. The ratios S_m/S_{m+1} converge towards the product of the two 'brother' numbers: $\phi_{3\,\text{brother}_1} \cdot \phi_{3\,\text{brother}_2}$.

The number ϕ_4 The roots of the equation $x^4 - x^3 - x^2 - x - 1 = 0$ are

$$\phi_4 \;\approx\; 1.92756, \qquad \phi_{4\,\text{brother}_1} \;\approx\; -0.07637 + 0.81470i,$$
$$\phi_{4\,\text{brother}_2} \;\approx\; -0.07637 - 0.81470i, \qquad \phi_{4\,\text{brother}_3} \;\approx\; -0.77480.$$

Again, we have

$$\phi_4 \cdot \phi_{4\,\text{brother}_1} \cdot \phi_{4\,\text{brother}_2} \cdot \phi_{4\,\text{brother}_3} = -1,$$
$$\phi_4 + \phi_{4\,\text{brother}_1} + \phi_{4\,\text{brother}_2} + \phi_{4\,\text{brother}_3} = -1.$$

The number ϕ_4 has a role similar to ϕ_2 and ϕ_3 as the limit to which ratios of successive numbers in Fibonacci type sequences converge. In this case, we form our ϕ_4-Fibonacci sequence by initially choosing four numbers and form each successive number by adding the previous four numbers. For example, if we choose -2, 12, 1000 and 1.2, our sequence is

$$S \;=\; [-2, 12, 1000, 1.2, 1011.2, 2024.4, 4036.8, 7073.6, 14146, 27280.8, \ldots].$$

With these numbers the first fifteen ratios S_{m+1}/S_m for successive m are

$$\begin{array}{lll} 0.00120000000, & 842.666666667, & 2.00197784810, \\ 1.99407231772, & 1.75227903290, & 1.99983035512, \\ 1.92851689524, & 1.92579396499, & 1.92316301592, \\ 1.92999041941, & 1.92745700548, & 1.92741711480, \\ 1.92747823955, & 1.92764060072, & 1.92755239368, \end{array}$$

converging towards ϕ_4. The ratios S_m/S_{m+1} converge towards the product of the three 'brother' numbers: $\phi_{4\,\mathrm{brother}_1} \cdot \phi_{4\,\mathrm{brother}_2} \cdot \phi_{4\,\mathrm{brother}_3}$.

Higher ϕs This type of behaviour continues as we consider ϕ_n for increasing n. Our basic ϕ_n-equation is

$$x^n - x^{n-1} - \dots - x^2 - x - 1 \;=\; 0.$$

Each equation has one positive real root, ϕ_n.

ϕ_1	1	ϕ_2	1.61803	ϕ_3	1.83928
ϕ_4	1.92756	ϕ_5	1.96594	ϕ_6	1.98358
ϕ_7	1.99196	ϕ_8	1.99603	ϕ_9	1.99802
ϕ_{10}	1.99901				

For odd n, the other roots come in conjugate pairs. For even n, the other roots come in conjugate pairs plus one other real root. This other real root is always less than zero. Its values are as follows.

$\phi_{2\,\mathrm{brother}}$	-0.61803	$\phi_{4\,\mathrm{brother}}$	-0.77480	$\phi_{6\,\mathrm{brother}}$	-0.84030
$\phi_{8\,\mathrm{brother}}$	-0.87628	$\phi_{10\,\mathrm{brother}}$	-0.89903	$\phi_{12\,\mathrm{brother}}$	-0.91471
$\phi_{14\,\mathrm{brother}}$	-0.92617	$\phi_{12\,\mathrm{brother}}$	-0.93492	$\phi_{14\,\mathrm{brother}}$	-0.94181
$\phi_{20\,\mathrm{brother}}$	-0.94738				

In all cases, the sum of all the roots is $+1$. In half the cases, the product of the roots is $+1$. In the other half of the cases, the product of the roots is -1. The $+1$ occurs for ϕ_n-equations with odd n; the -1 occurs for ϕ_n-equations with even n. All the imaginary roots of all the equations have both real and imaginary parts between -1 and $+1$.

The connection with ϕ_n-Fibonacci type sequences continues with ϕ_n being the limit to which the ratios S_{m+1}/S_m converge and the modulus of the product of all the other roots being the limit towards which the ratios S_m/S_{m+1} converge. We immediately notice that ϕ_n converges towards 2 with increasing n. ($\phi_{20} \approx 1.99999$). We are inclined to think that $\lim_{n\to\infty} \phi_n = 2$. This is certainly true for ϕ_n considered as a root of $x^n - \dots - 1 = 0$ because $2^n - 2^{n-1} - \dots - 2^2 - 2 - 1 = 1$. It is also true for ϕ_n considered as the limiting ratio of successive terms of a ϕ_n-Fibonacci sequence. For ϕ_2-Fibonacci sequences we have $u_{n+1} = u_n + u_{n-1}$. The ratio is

$$\frac{u_{n+1}}{u_n} = \frac{u_n + u_{n-1}}{u_n} = 1 + \frac{u_{n-1}}{u_n} = 1 + \frac{u_n - u_{n-2}}{u_n} = 2 - \frac{u_{n-2}}{u_n}.$$

So $u_{n+1}/u_n = 2 - u_{n-2}/u_n$. But for large n, u_{n-2}/u_n is always less than 1, and hence the ratio is always less than 2.

For ϕ_3-Fibonacci sequences we have $u_{n+1} = u_n + u_{n-1} + u_{n-2}$. In a similar manner it can be shown that $u_{n+1}/u_n = 2 - u_{n-3}/u_n$. But for large n, u_{n-3}/u_n is always less than 1, and so the ratio is always less than 2.

In general, for ϕ_k-Fibonacci sequences, $u_{n+1}/u_n = 2 - u_{n-k}/u_n$. Also, regarding ϕ_n as a root of $x^n - \cdots - 1 = 0$, we have

$$(-1)^n - 1 = (-1)^{n-1} + (-1)^{n-2} + \cdots + (-1)^2 + (-1) + 1.$$

Rearranging,

$$(-1)^n - (-1)^{n-1} - (-1)^{n-2} - \cdots - (-1)^2 - (-1) - 1 = 1.$$

For even n, our equation is

$$x^n - x^{n-1} - x^{n-2} - \cdots - x^2 - x - 1 = 0.$$

Obscurely, $\phi_{n\,\mathrm{brother}}$ is bounded below by -1.

The connection with ϕ_n-Fibonacci type sequences Consider the ϕ_2-Fibonacci sequence associated with ϕ_2. We have the definition of the ϕ_2-Fibonacci sequence $u_{n+1} + u_n = u_{n+2}$. We consider the case as u_n approaches infinity because, strictly speaking, it is only at infinity that the ratios of ϕ_2-Fibonacci sequences are associated with ϕ_2. As u_n approaches infinity, $u_n/u_{n-1} \sim u_{n+2}/u_{n+1}$. So

$$\begin{aligned}
u_{n+1} + u_n &\sim u_{n+1}\frac{u_n}{u_{n-1}}, \\
u_{n-1}(u_{n+1} + u_n) &\sim u_{n+1}u_n, \\
(u_{n+1} - u_n)(u_{n+1} + u_n) &\sim u_{n+1}u_n, \\
\frac{u_{n+1}^2 - u_n^2}{u_n u_{n+1}} &\sim 1.
\end{aligned}$$

We can therefore write our polynomial as

$$x^2 - x\left(\frac{u_{n+1}^2 - u_n^2}{u_n u_{n+1}}\right) - 1 \sim 0,$$

which can be rearranged as

$$\left(x - \frac{u_{n+1}}{u_n}\right)\left(x + \frac{u_n}{u_{n+1}}\right) \sim 0.$$

As u_n approaches infinity, this last equation approaches

$$(x - \phi_2)(x + 1/\phi_2) = 0, \quad (x - \phi_2)(x - \phi_{2\,\mathrm{brother}}) = 0, \quad x^2 - x - 1 = 0.$$

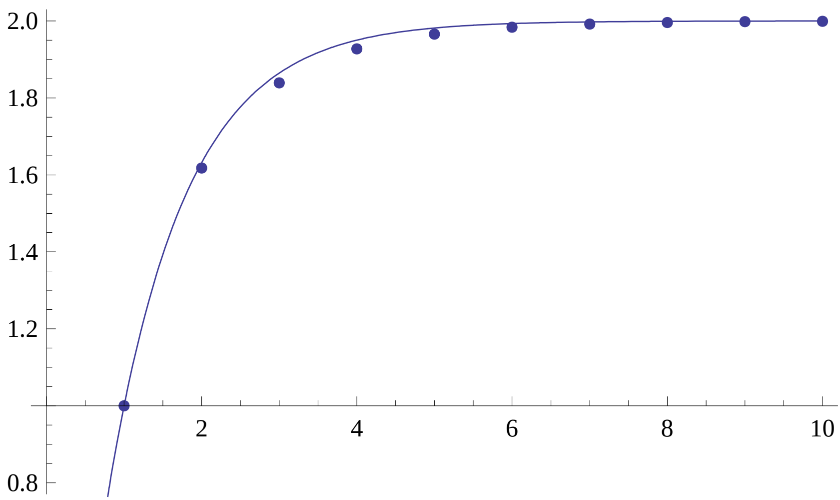
Similar algebra deals with the other ϕ_n-Fibonacci sequences.

Finally, we compute the series expansion about $x = 1$ of $\dfrac{1}{x^2 - x - 1}$:

$$\begin{aligned}
&-1 - (x-1) - 2\,(x-1)^2 - 3\,(x-1)^3 - 5\,(x-1)^4 - 8\,(x-1)^5 \\
&- 13\,(x-1)^6 - 21\,(x-1)^7 - 34\,(x-1)^8 - 55\,(x-1)^9 - 89\,(x-1)^{10} \\
&- 144\,(x-1)^{11} - 233\,(x-1)^{12} - 377\,(x-1)^{13} - 610\,(x-1)^{14} \\
&- 987\,(x-1)^{15} - 1597\,(x-1)^{16} - 2584\,(x-1)^{17} - 4181\,(x-1)^{18} \\
&- 6765\,(x-1)^{19} + O(x-1)^{20}.
\end{aligned}$$

Notice the coefficients.

The sequence of ϕ_ns closely fits the curve $y = 2 - \exp(1 - x)$. The limit as x tends to infinity of this expression is 2. Its value at $x = 1$ is 1.



References *The Divine Proportion* by H. E. Huntley; *The Golden Ratio and Fibonacci Numbers* by Richard A. Dunlap.

Problem 198.1 – Two knights

David Hughes

What is the probability that two knights attack each other on an $n \times n$ board?

Solution 195.1 – Two queens

> Two queens are placed on different squares of an $n \times n$ chessboard. What is the probability they 'attack' each other?

David Hughes

It is sufficient to consider the number of cells attacked by *one* queen.

A single *rook* placed anywhere attacks $2(n-1)$ cells out of $n^2 - 1$ empty cells. The probability that an empty cell is attacked is therefore

$$R = \frac{2(n-1)}{n^2-1} = \frac{2}{n+1}.$$

A single *bishop* attacks the number of cells illustrated on this 4×4 board. For example, a bishop in the top left cell attacks $4 + 1 - 2 = 3$ cells, along the diagonal. The pattern is evident. The numbers in all cells sum to

$4+1-2$	$3+2-2$	$2+3-2$	$1+4-2$
$3+2-2$	$4+3-2$	$3+4-2$	$2+3-2$
$2+3-2$	$3+4-2$	$4+3-2$	$3+2-2$
$1+4-2$	$2+3-2$	$3+2-2$	$4+1-2$

$$2\left(n^2 + 2\sum_{i=1}^{n-1} i^2\right) - 2n^2 = \frac{2}{3}(n-1)n(2n-1)$$

for n^2 possible locations of the bishop. The probability that an empty cell is attacked by a randomly placed bishop is therefore

$$B = \frac{2}{3}\,\frac{(n-1)n(2n-1)}{n^2(n^2-1)} = \frac{2}{3}\,\frac{2n-1}{n(n+1)}.$$

Note that $B < R$.

For a *queen*, the required probability is

$$Q = R + B = \frac{2(5n-1)}{3n(n+1)}.$$

When $n = 8$, $R = 2/9$, $B = 5/36$, $Q = 13/36$.

David Kerr

The method is to take each square in turn and calculate how many other squares a queen attacks. Call this function Q_x, where x defines the square. Given that the first queen is on square x, the probability that the second queen attacks the first (and of course vice versa) is just $Q_x/(n^2-1)$. We simply need to 'sum' these probabilities as x ranges over the whole board. Hence the required probability is given by $P = \sum Q_x/(n^2(n^2-1))$.

What we need, therefore, is to express $\sum Q_x$ as a function of n. It is easy to see that $Q_x = R_x + B_x$, where R_x is the number of the squares attacked by a rook, and B_x is the number of squares attacked by a bishop.

The first is easy; $\sum R_x = 2(n-1)n^2$. The B_x term is a bit more difficult. One way is to break the board into a series of concentric square rings. Clearly B_x is the same for all squares in a given ring. Number the rings starting with 1 at the outside ring. If n is even, the central ring will contain 4 squares and will be numbered $n/2$. If n is odd, the central ring will contain one square and will be numbered $(n+1)/2$.

Let S_r be the number of squares in ring r and let B_r be number of possible bishop moves that can be made from a square in ring r. Then $\sum B_x = \sum_{r=1}^{n/2} S_r B_r$ for even n and $\sum B_x = \sum_{r=1}^{(n+1)/2} S_r B_r$ for odd n.

First consider even n. It can be seen that $S_r = 4n+4-8r$ and $B_r = n-3+2r$ for r in the range from 1 to $n/2$. Hence

$$\sum_{r=1}^{n/2} B_x = \sum_{r=1}^{n/2} (4n+4-8r)(n-3+2r) = \frac{2n(2n-1)(n-1)}{3}.$$

Now suppose n is odd. If r is in the range from 1 to $(n-1)/2$, $S_r = 4n+4-8r$ and $B_r = n-3+2r$, as before. However, when $r = (n+1)/2$, $S_r = 1$ and $B_r = 2n-2$. Therefore

$$\sum_{r=1}^{(n+1)/2} B_x = 2n-2+\sum_{r=1}^{(n-1)/2} (4n+4-8r)(n-3+2r) = \frac{2n(2n-1)(n-1)}{3}.$$

The final values are the same. Hence for all $n \geq 2$,

$$\sum Q_x = 2(n-1)n^2 + \frac{2n(2n-1)(n-1)}{3} = \frac{2n(5n-1)(n-1)}{3},$$

$$P = \frac{2n(5n-1)(n-1)}{3n^2(n^2-1)} = \frac{2(5n-1)}{3n(n+1)}.$$

Problem 198.2 – Two students

David Hughes

Take a random group of 23 or more people, and the odds are better than evens that two of them share a birthday. A few years ago, in a tutorial group of about this size, two students found that they shared the same first name and family name. About how likely is this?

Problem 198.3 – Primes

Bryan Orman

One of the questions I posed in last year's Staff–Student Xmas Quiz was:

> What's special about the three consecutive primes 953, 967, 971?

One answer, 'all start with a 9', was of no significance; 'delete the 9s and you're left with primes' was more interesting. I was actually looking for the fact that reversing the digits, giving 359, 769, 179, produces primes.

I have two posers; the first is as follows.

> *Is there a longest sequence of consecutive primes with this reversal property?*

In my list of primes up to 12919 there is one decuplet, starting with 1193.

Observing that 7, 47, 347, 2347, 12347, 812347 are all prime, the second poser is:

> *Is there a longest sequence with this construction?*

[This is a kind of reversal of something we did involving the sequence of primes 3, 37, 373, 3733, 37337, 373379, 3733799, 37337999 (Eddie Kent, 'Prime primes', M500 **194**). — **ADF**]

Problem 198.4 – Determinant

ADF

Determine the determinant of the matrix

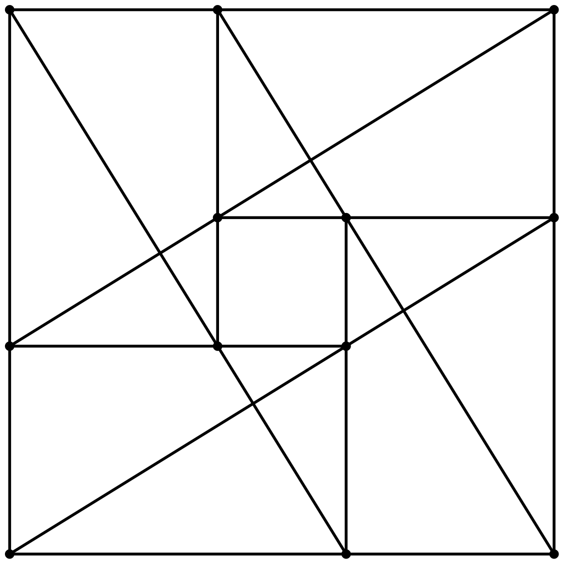
$$\begin{bmatrix} 1 & x & x^2 & \ldots & x^{n-2} & 0 \\ 0 & 1 & x & x^2 & \ldots & x^{n-2} \\ x^{n-2} & 0 & 1 & x & x^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x^2 & \ldots & x^{n-2} & 0 & 1 & x \\ x & x^2 & \ldots & x^{n-2} & 0 & 1 \end{bmatrix}.$$

Problem 198.5 – Divided polygon

ADF

Look at the diagram. Twelve special points are marked by black blobs. There are 12 lines, each containing three special points. The lines are divided in the same ratio by the interior point.

What is this ratio? Devise a ruler-and-compasses construction for the diagram. Generalize to an n-sided regular polygon.



Problem 198.6 – Snap

What is the probability that a game of *snap* will be played without a snap occurring.

To play the game, you inspect two randomly sequenced decks of cards. In each deck there are $n = rs$ cards, designated by ordered pairs (X, Y), $X = 1, 2, \ldots, r$, $Y = 1, 2, \ldots, s$. A snap occurs if for some i, $1 \le i \le n$, the ith card in one deck has the same value of X as the ith card in the other deck. Traditionally $r = 13$ and $s = 4$. But note that if $s = 1$ the answer is

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!},$$

which for large n is approximately $1/e$; see Nick Pollock, 'Hats', M500 **178**.

Relativity

Sebastian Hayes

Galileo first proposed the principle that there was no way of distinguishing (from inside) whether an 'inertial system' was at rest or in constant straight line motion. In a more careful formulation this becomes

The laws of physics take the same form in all inertial frames.

Newton assumed the principle, which is why he speaks of a particle being 'at rest or in constant straight line motion' in his first law of motion.

A striking feature of the behaviour of bodies within an inertial frame is that they automatically partake of the motion of the frame. Thus, if I roll a marble on the floor in a train, it will have the motion of the train as well as its own motion and thus the resultant speed of the marble will be (speed of train + speed of marble), which will be *greater* than the speed of the train and/or speed of marble when at rest relative to the train. However, this will not show up if I carry out measurements inside the train. Supposing I have a toy gun which fires off rubber balls and I stand at the back of the last compartment. I fire off the rubber ball and we suppose it rebounds from a wall at the other end and comes back to me. The time it takes to go down the compartment will be the same as the time it takes to come back—allowing for some small deduction for air resistance.

Now, however, suppose there is a window at the back of the train and that as the train goes through the station I (from the platform) manage to fire off a rubber bullet into the back window which we can assume to be made of very thin glass or plastic. The rubber bullet has the speed with which it left the gun (muzzle velocity) and according to Newton's laws will retain this velocity unchanged unless some outside force intervenes. In such a case the time taken for the rubber bullet to go down the length of the compartment will not be the same as the time it takes when it rebounds—the first time will be greater than the time taken when firing from the inside, the rebound time will be less. If the muzzle velocity is exactly the same as that of the train the rubber bullet will not penetrate the glass but remain apparently glued to it for a time, while if the muzzle velocity is less than that of the train it has no chance of breaking the glass.

When speaking of an object inside the train, if our viewpoint is that of the train, we say it is at rest and we *feel* at rest if the train does not suddenly accelerate. However, if pushed we would concede that this 'state of rest' is not absolute but relative—relative to the Earth, which we never observe or feel to be other than motionless. The Earth thus functions as an 'absolute' frame of reference for earthbound persons. However, since Galileo we believe the Earth is not at rest in an absolute sense. On the analogy with vehicles and the Earth, we might wonder whether there is not some absolute reference frame with regard to which planets and heavenly bodies are in motion. Newton seemed to think that there was such a thing as a state of 'absolute rest', the appropriate frame of reference being the 'fixed stars' — at his time practically everyone assumed the stars did not move. So, according to Newton, states of rest and constant straight line motion were not completely identical since in principle at least some experiments could distinguish between the two.

This appeared academic at the time. However, by the end of the 19th century it looked as if there *was* a way of distinguishing between absolute rest and straight line constant motion. Most physicists at this time still envisaged light as a disturbance propagated within a medium and they suggested that there was an invisible 'substance' permeating the whole of space, the so-called luminiferous ether. If this were so, it should be possible to distinguish between rest and motion by an optical experiment even though it was agreed that the distinction could not be made by a mechanical experiment. Although a bullet fired from inside a moving train already has the velocity of the train as it were, it was assumed that this would not happen with respect to a light beam. Light would propagate at a constant speed *relative to the ether* whether or not the beam was produced from inside the train or from the outside. The ether itself was viewed as 'absolutely at rest' — there was nothing more basic with which to compare its motion.

The argument was that the Earth, though following an elliptical orbit, could be considered to be in constant straight line motion over (astronomically) small distances. We had thus a moving inertial reference frame ready-made. On the analogy with the train, if a beam were directed *down* the train exactly in the direction of motion and then reflected back up again, there should be a discrepancy between the times taken, there and back. But given the colossal speed of light this would be undetectable. However, what could be done was to compare the time taken when sending a light beam down a square room and back again with the time taken when beaming a torch across the same square room. If the room—inertial frame—were in motion, the effective distances traversed by the ray of light would not be the same.

For the purposes of illustration, suppose the speed of light to be very much less than it actually is, e.g. 10 metres per second. If the square room is just 10 metres long, the light beam will take 1 second to cross the room and so take 2 seconds to go down and back, or across and back. However, now suppose the room to be moving at the rate of 5 metres per second. Assuming the light ray does not participate in the motion of the room, the distance down is now rather more than 10 metres. The ray will thus take 2 seconds to reach the mirror, but in compensation it will only take 2/3 second on the way back. The total time taken will be 2.667 seconds approximately.

The time T, say, taken to cross the room is rather different, however, and requires Pythagoras' theorem. By the time the light ray has crossed the room, the point on the opposite wall has moved on $5T$ m. So the distance is somewhat greater, $\sqrt{10^2 + (5T)^2}$ m. But this distance is also equal to $10T$ m. Hence $T = 2/\sqrt{3}$ s, or 1.1547 s approximately. Since the light ray is travelling at 10 m/s, the distance will be 11.547 m. This will be the same as on the return journey since there is no 'compensation' if you are going across the current. The total time taken going across there and back will thus be 2.309 seconds—somewhat less than the 2.667 seconds taken there and back in the other direction.

The details of the Michelson–Morley experiment need not concern us. Basically what they did was to 'divide light from an extended source into two beams by partial reflection. These beams are sent in different directions (at right angles to each other), reflected against mirrors and brought back again ...'. Once they have returned the beams form an 'interference fringe' — analogous to the interference of two sets of water waves coming from different directions. If the distances traversed are exactly equal, the times taken will be the same and a known fringe pattern will result. However, if this is not the case there will be a noticeable shift in the pattern.

The experiment was done in order to *prove* the ether theory, since if the experiment showed a noted discrepancy this would be due to the motion of the Earth through the ether. Of course, one could not be sure that the way the laboratory mirrors were placed would really be exactly 'down' and 'across' with respect to the Earth's motion, but there would in any case be *some* discrepancy, or so it was believed. In fact no difference was found and the same null result was obtained six months later.

This surprised everyone and various explanations were offered, one being that the Earth 'carried its own ether' along with it—though there were serious difficulties about this interpretation. In 1905 Einstein proposed to simply drop the ether altogether—or more specifically he suggested that the 'ether had no physical properties, only geometric ones'. He propounded two basic principles which are, in essence, the Special Theory of Relativity.

1. **The laws of physics take the same form in all inertial frames** (and so there is no way of distinguishing between a state of rest and constant straight line motion).

2. **The speed of light in a vacuum is strictly constant.**

The latter may appear obvious or banal but it is not. It means for example that the measured speed of light would be the same if a spaceship travelling with a headlamp were travelling right towards us, were motionless relative to us, or receding (tail-lamp in this case). This is not speculation today—though it was when Einstein wrote his paper—since the speed of radiation from particles travelling at very high speeds (relative to the observing apparatus) has been measured and compared with radiation from stationary particles, with no appreciable difference recorded.

Various consequences follow from Einstein's two principles. It follows for example that the description of motion of two or more 'inertial' frames is optional; it depends on the one you are situated in. If I am in a spaceship, I consider the one going past me to be in motion, but then so does the person in the other one. Normally, we would assume that we are not both right—either we are 'really' both of us in motion, or one of the two is in motion and the other is not. Special Relativity says that, provided we are talking about inertial frames and straight line motion, *either* description is correct—we are at liberty to consider ourselves at rest or in *any* state of motion provided it is unaccelerated. As someone said, 'there is no truth of the matter' — if you accept Einstein. On the other hand it must be stressed that whatever observations are made by a single observer in his own frame, the observations will be perfectly consistent to him. Trouble only comes when we confront one set of observations with another set from a different reference frame.

Einstein seems to have viewed the speed of light as a sort of all-time absolute—nothing can exceed it. At any rate, there is no way of sending a signal or otherwise influencing another person or object. (This is in fact why he didn't like Quantum Mechanics.) The speed of light thus placed a boundary on the operation of cause and effect, not only actual cause and effect but *possible* cause and effect, because to influence someone or something else you have to do or send something to them and you can't go faster than light. Thus certain galaxies will remain forever out of reach: they can't interact with us or us with them.

What happens if something travels faster and faster? Mass, viewed by Newton as the quantity of matter possessed by an object, is today viewed more abstractly: it is merely a 'property' by means of which a body resists any attempt to change its state of rest or straight line motion. In principle, Newtonian mass won't change with increasing speed: if you pump more and more energy into an object with a small mass you will make it take on fantastic speeds, exceeding the speed of light. But this can't happen in Special Relativity because the so-called 'relativistic mass' increases rapidly after a certain point and acts as a sort of in-built brake. It would require an 'infinite' amount of energy to accelerate a body to the speed of light. All experiments have confirmed this particular prediction of Einstein's.

Of course, this elementary account of Special Relativity is intended to appeal to persons who are meeting this sort of material more-or-less for the first time. To take the next step in understanding this fascinating subject, we suggest Albert Einstein, *Relativity: The Special & the General Theory*, and perhaps Joseph Schwartz, *Einstein for Beginners*.

Problem 198.7 – Sums of powers

ADF

Let

$$S_k(n) = \sum_{i=1}^{n} i^k,$$

the sum of the kth powers of the positive integers up to and including n. For the first few values of k we have

$$\begin{aligned}
S_1(n) &= \frac{1}{2}\, n(n+1),\\
S_2(n) &= \frac{1}{6}\, n(n+1)(2n+1),\\
S_3(n) &= \frac{1}{4}\, n^2(n+1)^2,\\
S_4(n) &= \frac{1}{30}\, n(n+1)(2n+1)(3n^2+3n-1),\\
S_5(n) &= \frac{1}{12}\, n^2(n+1)^2(2n^2+2n-1),\\
S_6(n) &= \frac{1}{42}\, n(n+1)(2n+1)(3n^4+6n^3-3n+1),\\
S_7(n) &= \frac{1}{24}\, n^2(n+1)^2(3n^4+6n^3-n^2-4n+2),\\
S_8(n) &= \frac{1}{90}\, n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3),\\
S_9(n) &= \frac{1}{20}\, n^2(n+1)^2(n^2+n-1)(2n^4+4n^3-n^2-3n+3).
\end{aligned}$$

It is clear from the above that $S_3(n) = S_1(n)^2$, and with a modicum of computation we find the interesting formula

$$2S_5(n) \;=\; 3S_2(n)^2 - S_3(n).$$

Motivated by this observation, our first question is: (i) Are there any other similar equalities involving powers of $S_k(n)$?

Also we see that $S_k(n)$ contains the product $n(n+1)(2n+1)$ for even k and $n^2(n+1)^2$ when k is odd and greater than 1. No doubt this follows immediately from the general formula for $S_k(n)$ (involving Bernoulli numbers). However, it is not a trivial exercise to obtain this formula. So instead we ask: (ii) Is there an easy proof that $S_k(n)$ is a rational multiple of $S_2(n)$ for even $n \geq 2$ and $S_k(n)$ is a rational multiple of $S_1(n)^2$ for odd $n \geq 3$.

Solution 195.2 – Six tans

If $\theta = \pi/13$, prove that

$$\tan\theta\ \tan 2\theta\ \tan 3\theta\ \tan 4\theta\ \tan 5\theta\ \tan 6\theta\ =\ \sqrt{13}.$$

John Reade

The solutions of the equation $\sin 13x = 0$ are $x = n\pi/13$, where n is an integer. If we write $S = \sin x$, $C = \cos x$, then we can get an expansion of $\sin 13x$ in terms of S, C as follows:

$$\begin{aligned}\sin 13x\ &=\ \Im\, e^{13ix}\ =\ \Im\,(C+iS)^{13}\\ &=\ 13\,C^{12}S - 286\,C^{10}S^3 + 1287\,C^8S^5 - 1716\,C^6S^7\\ &\qquad\qquad + 715\,C^4S^9 - 78\,C^2S^{11} + S^{13}.\end{aligned}$$

Therefore $\sin 13x = 0$ gives

$$S^{13} - 78\,C^2S^{11} + \cdots + 13\,C^{12}S\ =\ 0,$$

which on dividing by C^{13} and putting $T = \tan x = S/C$ gives

$$T^{13} - 78\,T^{11} + \cdots + 13\,T\ =\ 0.$$

The 13 roots of this equation are

$$T\ =\ 0, \tan\frac{\pi}{13}, \tan\frac{2\pi}{13}, \ldots, \tan\frac{12\pi}{13}\ =\ 0, \pm\tan\frac{\pi}{13}, \pm\tan\frac{2\pi}{13}, \ldots, \pm\tan\frac{6\pi}{13}.$$

Therefore the 12 roots of the equation $T^{12} - 78T^{10} + \cdots + 13\ =\ 0$ are

$$T = \pm\tan\frac{\pi}{13}, \pm\tan\frac{2\pi}{13}, \ldots, \pm\tan\frac{6\pi}{13}.$$

It follows that the six roots of the equation

$$U^6 - 78U^5 + \cdots + 13\ =\ 0$$

must be $\tan^2\pi/13, \tan^2 2\pi/13, \ldots, \tan^2 6\pi/13$.

Hence, using the formula for the product of the roots of a polynomial equation, we get

$$\tan^2\frac{\pi}{13}\ \tan^2\frac{2\pi}{13}\ \ldots\ \tan^2\frac{6\pi}{13}\ =\ 13$$

as required.

Platonic solids

Chris Pile

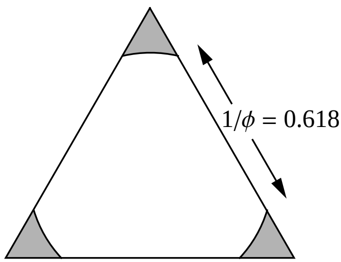
Further to my letter and the Editor's query in M500 **195**, I have spent many evenings trying unsuccessfully to fit the cube inside the icosahedron!

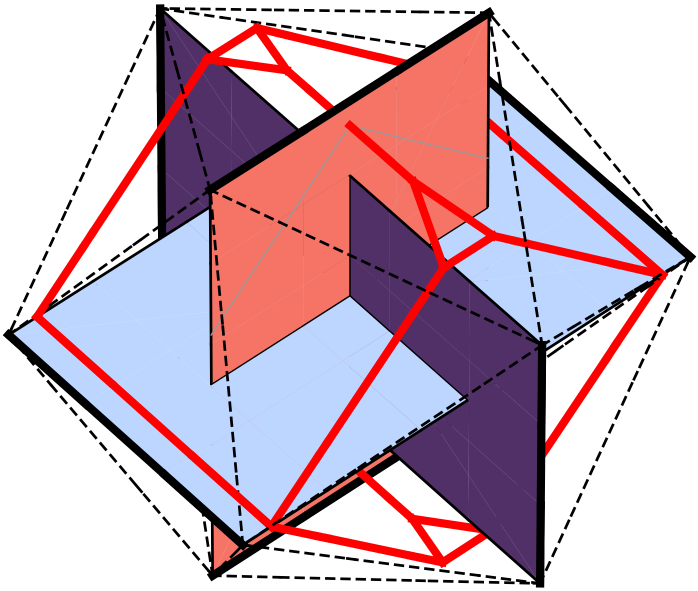
I do not have MATHEMATICA but I do have a cardboard model and a calculator. Let $\phi = (\sqrt{5}+1)/2$. For polyhedra with a unit edge length, the cube has a circumradius of $\sqrt{3}/2$ (≈ 0.8660) and the icosahedron has a circumradius of $(2 - 2/\sqrt{5})^{-1/2}$ (≈ 0.9511) and an inradius (radius of the sphere that touches the faces) of $\phi^2/(2\sqrt{3})$ (≈ 0.7557); so the vertices of the cube cannot be fitted between the faces of the icosahedron. The vertices of the icosahedron are at the corners of three mutually orthogonal golden section rectangles; so the diagonal plane of the cube can be aligned along one of these with four vertices inside (as $\sqrt{2} < \phi$). Unfortunately the other four vertices of the cube protrude through the faces of the icosahedron.

I have tried other symmetrical arrangements but the cube refuses to fit! It is frustratingly close and it seems that a small shift or twist might bring the vertices of the cube closer to the vertices of the icosahedron. I find it difficult to believe that God or Plato would have designed these polyhedra to be incompatible in this way! I have just about convinced myself that it is not possible to find even a non-symmetric solution but I would be very pleased if someone could fit the cube inside. (A slightly squashed cube of unit volume with two opposite rhombic faces having diagonals $2/\phi$ and ϕ could be accommodated).

In the diagram on the next page, the heavy black lines are the six icosahedron edges which coincide with the short sides of the three golden rectangles. The dashed lines are the other 24 edges of the icosahedron. The thick grey lines show the cube as it appears after truncation by the icosahedron.

A typical icosahedron face is illustrated on the right. The vertices of a centrally placed cube will protrude through the icosahedral face in the unshaded region. The distance along the side between permitted regions is $1/\phi$. The radius of the forbidden region is $\sqrt{3/4 - \phi^4/12} \approx 0.4229$.





Solution 189.4 – 100 members

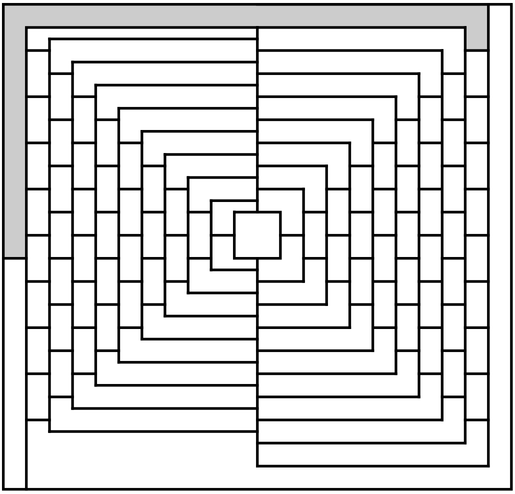
> I have a list of 100 names, all different and in random order. I read them out one by one. You can stop me at any time, and your objective is to stop me immediately after I have read out the longest name. What is the probability of success, assuming best strategy?

The best strategy (I think) is to sample the first $100/e$ names and then stop as soon as you hear a name that is longer than the longest name in the sample. But if there is anyone out there who knows why, please tell us! All I did was notice a similar problem in *Chez Angelique* by John Jaworski *et al.* At a time when political correctness was irrelevant they were concerned with a kind of beauty contest involving 10 'fair ladies'. The idea was that you had just one chance to select the fairest lady as the 10 fair ladies were paraded past you one by one. The recommended sample size was $10/e$ but *Chez Angelique* did not provide any explanation. — **ADF**

Problem 198.8 – Four colours

Roger Winstanley

The map shown below is the one that Martin Gardner reported in the April 1975 issue of *Scientific American* as a counter-example to the four colour conjecture. That April Fool hoax and some of its amusing consequences are also printed in Gardner's *Time Travel and Other Mathematical Recreations* (New York 1987). Of course, the conjecture became a theorem, proved by Appell & Haken in 1977 (see, for example, *Four Colours Suffice* by Robin Wilson); so the map must be four-colourable.



(i) Four-colour the map. The usual rule applies: two areas that share a common boundary must receive different colours. As you can see, we have given you a start by colouring one of the areas grey.

(ii) Is it possible to colour the centre square something other than grey?

'The astonishing truth is that the average person is at greater risk of being killed by an asteroid than dying in a plane crash.' — Sir Martin Rees, the Astronomer Royal.

(You find it is true, if you calculate with reasonable figures for event probability and associated deaths. — JRH)

(I can see why, but I don't have access to any reasonable figures. Perhaps someone can oblige? — ADF)

Letters to the Editor

Highest common factor

Dick Boardman writes on page 8 of M500 **195**, 'We were never told an algorithm to calculate LCM'. I am not entirely clear what he means by an algorithm in this case. I suppose you could call the system below an algorithm, but I was taught it as a method for calculating both LCM and HCF. It's easiest to show the method using an example.

Find the HCF and LCM of 2376, 1980 and 1728. Find all the prime factors of each number. Put each prime in a column:

$$\begin{array}{ccccccccc} 2376 & = & 2^3 & \cdot & 3^3 & \cdot & & & 11 \\ 1980 & = & 2^2 & \cdot & 3^2 & \cdot & 5 & \cdot & 11 \\ 1728 & = & 2^6 & \cdot & 3^3 & & & & \end{array}$$

For the HCF, take the lowest power of the primes in the complete columns: 2^2 and 3^2. Multiply them to get $4 \cdot 9 = 36$ as the HCF.

For the LCM, take the highest power of the primes in all the columns, not just the complete columns. Multiply them to get $2^6 \cdot 3^3 \cdot 5 \cdot 11 = 95040$ as the LCM.

Colin Davies

ADF writes — The above method does seem to be the most natural way to get the HCF of two numbers. Unfortunately it can become totally useless if the numbers are large. There is a much faster method, due to Euclid, which is very easy to explain. Suppose we want the HCF of a and b. What you do is repeatedly replace (a, b) by $(b, a \bmod b)$ until you end up with a zero. The HCF is the other number. A simple example: $(360015, 36015) \to (36015, 35880) \to (35880, 135) \to (135, 105) \to (105, 30) \to (30, 15) \to (15, 0)$; the HCF is 15.

I find it quite intriguing that this method usually gives no information about the prime factors of the numbers. To illustrate what I mean, let $a = 3$ 94020 06196 39447 92330 46804 46811 60078 93398 29458 92387 22475 04307 08947 55507 64181 30273 08398 62936 98852 77753 11113 49458 02497, $b = 3$ 94020 06196 39447 92330 46804 48005 17655 67233 17400 56861 65122 75841 35826 02387 09158 68013 02250 74049 82336 01099 13554 78053 53497. Then one can easily compute HCF(a, b) = 627 71017 35386 68076 54900 37295 02911 46996 07048 84196 92145 15621 but I challenge you to obtain the prime factors of a and b.

A little heresy

I am very suspicious of assertions like those made by Colin Davies in his article 'Under the skin' [M500 **197**]. These assertions apply statistical arguments to groups of people and make assumptions like 'uniform mixing' and 'average behaviour'. I am suspicious because people actively try to avoid uniform mixing and average behaviour.

As a very simple example, one could try to work out the probability of someone's car number being the same as their initials but the result you would get would be wildly inaccurate because some people go to a lot of trouble and expense to get a car number matching their initials. Again the chance of a person having two cars with consecutive numbers would be wrong for precisely the same reason.

Choice of marriage partners is an area where people actively avoid uniform mixing. Class distinctions strongly affect choice of marriage partner and I think it is most unlikely that I am a legitimate descendant of William the Conqueror. I would be very proud to believe that I was a descendant of Isaac Newton but unfortunately he died childless.

Colin Davies's conclusions are quite uncheckable and also harmless but there have been cases where statistics have been used to cause severe injustice. A leading specialist in cot deaths claimed that the probability of two cot deaths in one family was the square of the probability of a single cot death in the whole population. This is completely invalid because you can only multiply probabilities of events like this when the events are completely independent whereas two babies in the same family share a lot of genes in common and if the first cot death was caused by a genetic defect there is a much higher than average probability the second baby would inherit it as well. Furthermore, the babies have been looked after by the same people and in the same environment so that if the first death was the result of an infection or a poison in the environment, that would be common as well. Even then, to convince the court, some mothers have had to show a family history of cot deaths!

I personally believe that the current system of drug testing for athletes will produce significant injustices. When the authorities believe that a given substance confers an advantage on an athlete who takes it, they try to devise a test for it. These tests involve feeding the substance to a group of ordinary people (they must be non-athletes because any athlete who took it would be cheating) and then look for unusual levels of various related substances. But of course the whole point of top athletes is that they are not ordinary people and they use their bodies in most unusual ways. To compare them with a group of ordinary people who do not indulge in hard physical exercise is not to compare like with like at all. Pity the unfortunate athlete who is outstanding precisely because his body produces unusual levels of certain chemicals which give him an advantage.

Were I a member of a jury (which won't happen because I am over 65) I would be very reluctant to convict someone purely on DNA evidence. We are told that the DNA of close relatives, brothers, half brothers, first cousins show similarities, sufficient similarities for unknown relatives to be identified. To quote very large numbers as probabilities against two samples being from different people when you haven't checked all the close relatives of the accused seems to me an abuse of statistics.

When I was at college, I specialized in mathematical statistics and I came to the conclusion that to apply these theorems to the real world of awkward people was next to impossible because you could never be certain that the assumptions on which these theorems were based were satisfied. The whole thing was a bit like applying Pythagoras' theorem to all triangles that looked as though they might have a right angle in them. Lies, damned lies and ....

Dick Boardman

Re: Problem 195.3 – Doublings

> Given a positive integer n, let $D_f(n)$ denote the sequence $(d_0(n), d_1(n), \ldots, d_f(n))$, where $d_i(n)$ is the number of digits in $2^i n$. How big must f be such that n is uniquely identified by $D_f(n)$?

We seek $f(n)$, where $D_f(n)$ differs from $D_f(m)$ for all $m \neq n$ and $D_{f-1}(n) = D_{f-1}(m)$ for some $m \neq n$.

But $d_i(n) = [1+i\log_{10} 2+\log_{10} n]$. So $d_{i+1}(n)-d_i(n) = 0$ or 1 (because $\log_{10} 2 < 1$, and $d_i(n+1)-d_i(n) = 0$ or 1 (because $\log_{10}(n+1)-\log_{10}(n) = \log_{10}(1+1/n) < 1$).

Now consider a table of $d_i(n)$. We seek $f(n) = \max(i, j)$, where i is the smallest k such that $[k\log_{10} 2 + \log_{10} n] > [k\log_{10} 2 + \log_{10}(n-1)]$ and j is the smallest l such that $[l\log_{10} 2 + \log_{10} n] < [l\log_{10} 2 + \log_{10}(n+1)]$.

Can the problem be further analyzed?

Ian Adamson

What's missing?

Dear Tony,

In part (iii) of the 'What's Missing?' question on page 29 of M500 **194** there is a group of numbers which bear an uncanny resemblance to a list of prime numbers except for the 'missing' number between 337 and 347:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, **?**, 347, 349, 353, 359.

Since no prime number exists between those two numbers the only conclusion that I can draw is that it is a trick question.

Keith Drever

ADF writes — I see what you mean! I realize that our past conduct might have generated a reputation for trickery. However this is perfectly genuine. There is a simply defined mathematical function which yields primes almost all the time. The only exception < 360 is the question mark. It is in fact the sequence of numbers $n \geq 2$ which satisfy

$$2^n \equiv 2 \pmod{n}. \tag{1}$$

According to Fermat's Little Theorem, (1) holds for prime n. The converse is not true, however, but composite solutions of (1) seem to be rare, the only cases up to 50000 being

341, 561, 645, 1105, 1387, 1729, 1905, 2047, 2465, 2701, 2821, 3277, 4033, 4369, 4371, 4681, 5461, 6601, 7957, 8321, 8481, 8911, 10261, 10585, 11305, 12801, 13741, 13747, 13981, 14491, 15709, 15841, 16705, 18705, 18721, 19951, 23001, 23377, 25761, 29341, 30121, 30889, 31417, 31609, 31621, 33153, 34945, 35333, 39865, 41041, 41665, 42799, 46657, 49141, 49981.

We can quickly verify by hand that $2^{341} \equiv 2 \pmod{341}$. Observe that $2^{10} = 1024 = 3 \cdot 341 + 1$; so $2^{10} \equiv 1 \pmod{341}$. Therefore $2^{341} \equiv (2^{10})^{34} \cdot 2 \equiv 2 \pmod{341}$.

A question. Let $\pi(n)$ denote the number of primes not exceeding n, and let $\eta(n)$ denote the number of composite integers $\leq n$ satisfying (1). Is it the case that $\eta(n)/\pi(n) \to 0$ as $n \to \infty$?

Conversion factors

Tony Forbes

Conversion factors are always useful for converting things from one unit to another. Here's a selection of easy-to-remember examples. But beware of using them for precision work—most are only approximate. Any more?

To convert	to	multiply by
pounds	kilograms	$\log \pi/2$
miles	feet	$e^{\sqrt{67}\pi/3}$
years	seconds	$10^7\,\pi$
gallons	litres	$\pi + \sqrt{2}$
ounces	Planck masses ($\sqrt{hc/G}$)	2^{19}
guineas	farthings	$10^3 + 2^3$
price without VAT	price including VAT	$\sinh 1$

Mathematics Revision Weekend 2004

The **30th M500 Society Mathematics Revision Weekend** will be held at **Aston University**, **Birmingham** over **10–12 September 2004**.

Tutorial sessions start at 19.30 on the Friday and finish at 17.00 on the Sunday. We plan to present most OU maths courses. The Weekend is designed to help with revision and exam preparation, and is open to all OU students.

On the Saturday night we have a mathematical guest lecture. After the lecture **Charles Alder** is hosting a disco, and for the less energetic we plan to organize a ceilidh to which you are especially invited to contribute if you can sing or play a musical instrument.

See the Society's web page, www.m500.org.uk, for full details and an application form, or send a stamped, addressed envelope to

Jeremy Humphries, M500 Weekend 2004.

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