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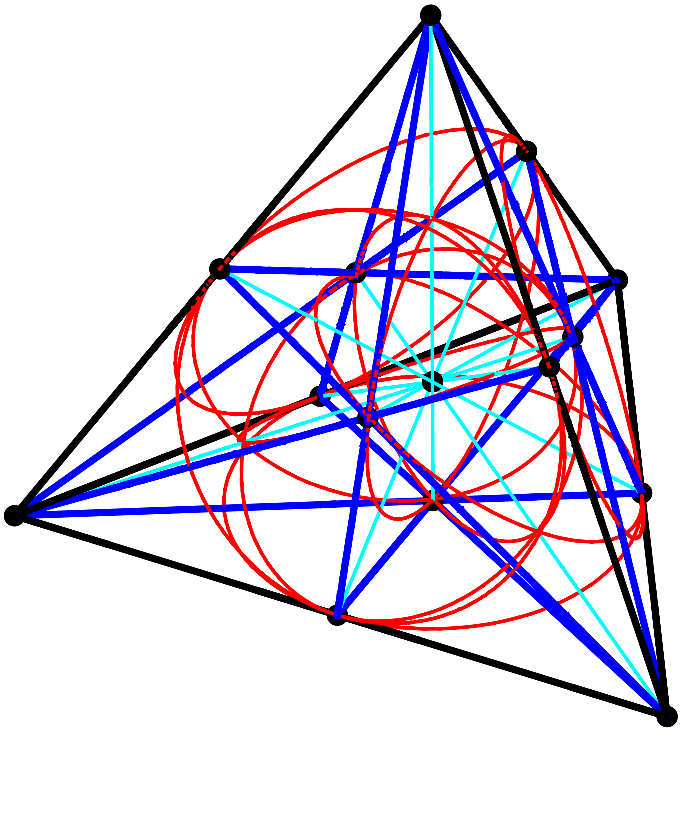

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A problem in geometric probability

Robin Marks

The following problem was considered and solved by H. G. Wendel in 1962 in a paper that was refereed by the famous geometer H. S. M. Coxeter.

> Let N points be scattered at random on the surface of the unit sphere in n-space. Evaluate $p_{n,N}$, the probability that all the points lie on some hemisphere.

Wendel showed (without using any calculus) that

$$p_{n,N} \;=\; 2^{-N+1}\sum_{k=0}^{n-1}\binom{N-1}{k},$$

where $\binom{N-1}{k}$ is the number of ways of choosing k objects from $N-1$ objects. In other words, in general $p_{n,N}$ is the sum of n entries on a row of Pascal's triangle divided by a power of 2. For example, in the case of an ordinary 3-sphere with 5 random points scattered on the surface, the probability that all the points lie on some hemisphere is $(1+4+6)/16 = 11/16$.

Wendel concluded his paper as follows: '... the form of the result shows that $p_{n,N}$ equals the probability that in tossing an honest coin repeatedly, the nth 'head' occurs on or after the Nth toss. But it does not seem possible to find an isomorphism between coin-tossing and the given problem that would make the result immediate.'

In this article we re-analyse the problem. We shall discover that there is an isomorphism with coin-tossing, and that the result applies to more than points on the surface of an n-sphere. In fact it applies to points almost anywhere in n-space. We shall derive an alternative expression,

$$p_{n,N} \;=\; 1-\sum_{i=n}^{N-1} 2^{-i}\binom{i-1}{n-1}.$$

Call the jth point on an n-dimensional sphere $r_{n,j}$. First consider the 2-dimensional cases, $n=2$, with two points which are diametrically opposite each other. Draw the tangents to the circle at these points (Figure 1).

This is a special case. Note that a semicircle with $r_{2,1}$ as an endpoint does not, quite, contain $r_{2,2}$. Also, the tangent lines are parallel hence do not meet in Euclidean space.

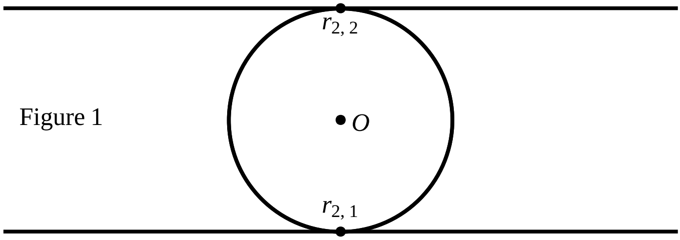


Figure 1

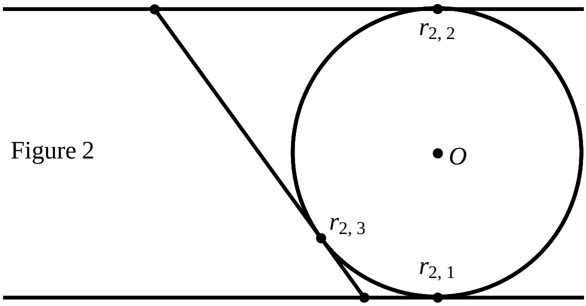


Figure 2

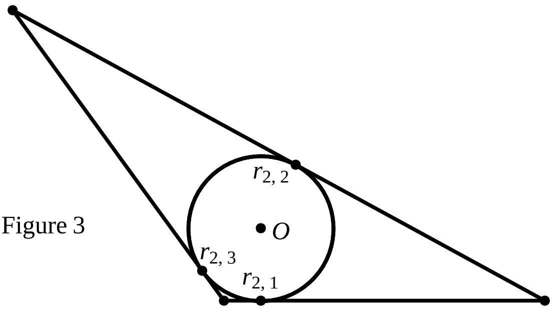


Figure 3

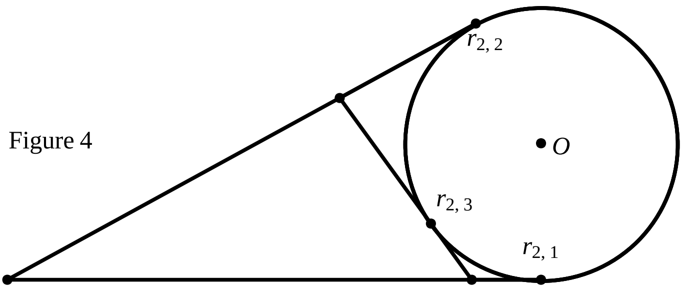


Figure 4

Add a third point $r_{2,3}$ and its tangent (Figure 2). Again the three points do not (quite) fall on one semicircle. If we move the point $r_{2,2}$ on Figure 2 clockwise a little round the circle, we find that (a) the three points still do not fall on one semicircle and (b) the three tangents form a triangle which encloses the origin (Figure 3).

On the other hand, if we move the point $r_{2,2}$ anticlockwise a little, we find that (a) the three points *do* fall on one semicircle and (b) the triangle of tangents does not enclose the origin (Figure 4).

If we exclude cases such as in Figure 2, where tangent lines are parallel, we can see that three points on a circle do not lie on one semicircle iff the origin is enclosed by the triangle formed by the three tangents at those points. Thus, counting the number of ways in which the origin can be enclosed will give us the number of ways in which N points do not lie on one hemisphere.

Call $E(n, N)$ the probability that the origin of an n-sphere is enclosed by N tangent lines (or tangent hyperplanes in higher-dimensional cases); (E stands for 'enclosed'). This is related to $P_{n,N}$ by the equation $E(n, N) = 1 - P_{n,N}$.

In $n = 2$ dimensions we exclude cases where tangent lines are parallel. In technical terms we insist that the tangents to $r_{2,j}$ are linearly independent in sets of two. In general, we insist that the tangent hyperplanes to $r_{n,j}$ are linearly independent in sets of n. This means that each subset of n tangent hyperplanes meet at a point.

The following illustrates the method we will use to choose surface points $r_{n,j}$. Suppose we are working in $n = 2$ dimensions with $N = 3$ points. Step 1: choose the first tangent line. We call this the 'baseline' and will portray it as a horizontal tangent line below the circle. Construct a first line through the origin, parallel to the baseline. Choose, by any means, two different lines through the origin (in two dimensions, different lines through the origin will always be linearly independent). Step 2: Choose a second line through the origin. Toss a coin to choose one of the two tangents parallel to this second line. Label the chosen tangent where it touches the circle. Step 3: As for step 2. Finally remove the construction lines through the origin. Diagrams 5a to 5c show the four equally likely configurations. The baseline is drawn as a thicker line and the non-chosen tangent lines are shown as dashed lines.

Figure 5a

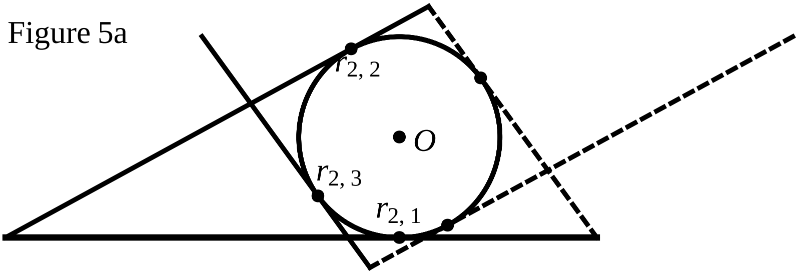


Figure 5b

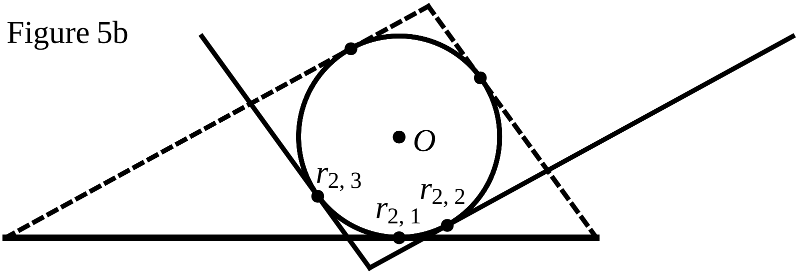


Figure 5c

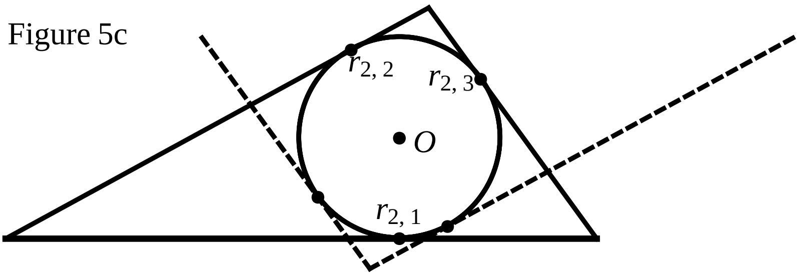
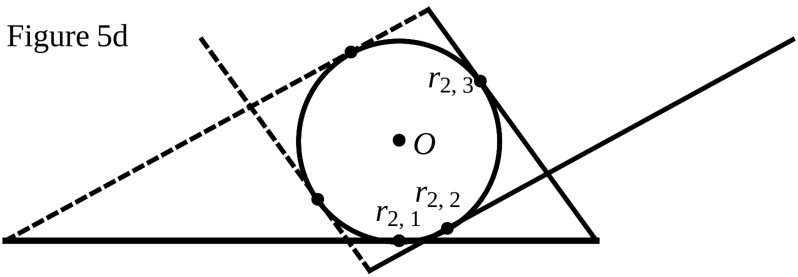


Figure 5d



The two coin tosses give $2 \cdot 2 = 4$ equally likely outcomes. In only one of these, diagram 5c, is the origin enclosed by the three tangents, giving probability of enclosure $E(2,3) = 1/(2 \cdot 2) = 1/4$.

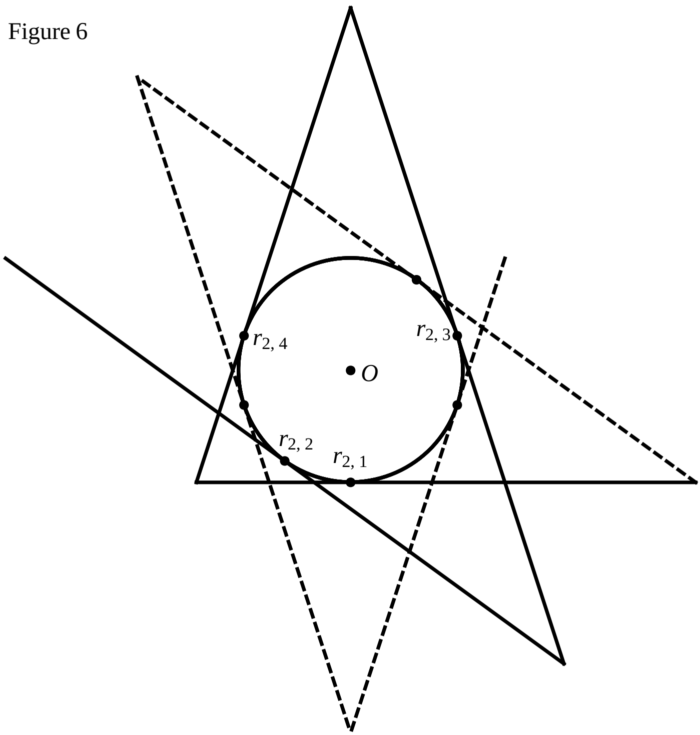
To see this in another way, start with the origin enclosed (Figure 5c). There is one way to do this. Reverse both coin-toss decisions (one 1 way) or reverse one decision (two ways). Each of these 'releases' the origin from enclosure. Thus the probability of enclosure $E(2,3) = 1/(1+2+1) = 1/4$.

We cannot enclose the origin with only one or two tangents; so $E(2,1) = E(2,2) = 0$.

Case $(2,4)$: $N = 4$ points on a 2-sphere. Proceed as previously but after Step 3 add 'Step 4: Take a fourth line through the origin. Construct the two parallel tangents. Toss a coin to choose one of these.'

Figure 6 shows an example in which steps 3 and 4 have been 'successful' in conjunction with step 1, in helping to enclose the origin. That is, coin tosses 2 and 3 have been successful but not 1.

Figure 6



The crucial point is that *any* two tangent lines, together with the baseline, will form a triangle, and if the coin tosses are favourable, this triangle will enclose the origin. If we find for example that tangent lines 1, 2 and 4 enclose the origin, and tangent lines 1, 2 and 3 do not enclose it, we must have tossed the coin three times to achieve enclosure. Thus this outcome occurs with probability $(1/2)^3 = 1/8$. Similarly tangent lines 1, 3 and 4 enclose the origin with probability 1/8. Tangent lines 1, 2 and 3 enclose the origin in only two coin tosses, hence with probability $(1/2)^2 = 1/4$ as already calculated. Thus the total chance of enclosing the origin is $E(2,4) = (1/2)^2 + 2 \cdot (1/2)^3 = 1/2$. Hence $P_{2,4} = 1 - E(2,4) = 1/2$.

Case $(2,5)$: Five points on a 2-sphere. We proceed similarly but if the origin is not enclosed after three tosses, we toss the coin a fourth time, giving additional chances of enclosing the origin with sets of tangent planes $\{1, y, 5\}$ with a probability of $(1/2)^4$ each, where y is chosen from $\{2, 3, 4\}$. There are $\binom{3}{1}$ extra ways to enclose O compared with the $N = 4$ case. Thus the total chance is $E(2,5) = (1/2)^2 + \binom{2}{1} \cdot (1/2)^3 + \binom{3}{1} \cdot (1/2)^4 = 11/16$. Hence $P_{2,5} = 1 - E(2,5) = 5/16$.

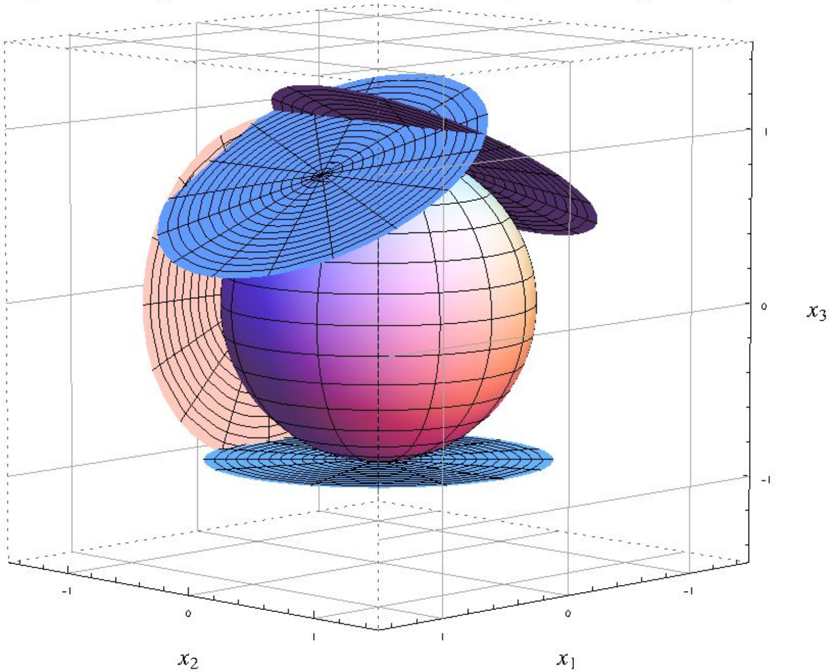
Case $(2,N)$: N points on a 2-sphere. We proceed similarly but if the origin is not enclosed after $N-2$ tosses we toss the coin an $(N-1)$th time, giving an additional chance of enclosing the origin at step $\{1, y, N\}, y \in \{2, 3, 4, \ldots, N-1\}$ with a probability of $(1/2)^{N-1}$ each. Thus the total chance is $E(2,N) = (1/2)^2 + 2 \cdot (1/2)^3 + \cdots + (N-2) \cdot (1/2)^{N-1}$.

To summarize: The chance of enclosing the origin with N tangent lines, $E(2,N)$, is the same as the chance, in up to $N-1$ coin-tosses, of making at least $n = 2$ correct binary choices. This is the same chance as the chance that $n = 2$ or more heads will occur in up to $N-1$ tosses. Thus $1 - E(2,N)$ is the chance that $n = 2$ or more heads will occur *on or after* the Nth toss. That is, $1 - E(2,N) = P_{2,N}$ is the chance that 'the nth "head" occurs on or after the Nth toss' (quoting Wendel). So we have explained Wendel's statement about coin-tossing, at least in the 2-dimensional case.

Case $n = 3$: The 3-sphere. We use a similar argument. We cannot enclose any 3-dimensional space with only one, two or three tangent planes, so $E(3,1) = E(3,2) = E(3,3) = 0$. We need $n + 1 = 4$ planes so that a tetrahedron can be formed. To explain this, each plane can be represented by an equation in three variables x_1, x_2 and x_3, of the form $a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 = b_1$, where $a_{1,i}$ and b_1 are constants. The $a_{1,i}$ give the slope of the plane and b_1 displaces the plane so that it does not pass through O. In a set of three linearly independent equations the solution is a single point in 3-space. In a set of four such equations there is a solution point for each subset of three equations; so there are $\binom{4}{3} = 4$ solution points. These make up the four vertices of a tetrahedron, which has the capability of enclosing O.

First consider a case where $n = 3$ tangent planes are linearly *dependent*, each of the three touching the sphere on the same great circle. We show the 'baseplane' at the lower pole in Figure 7.

Figure 7: A 3−sphere with four tangent planes not quite enclosing the origin

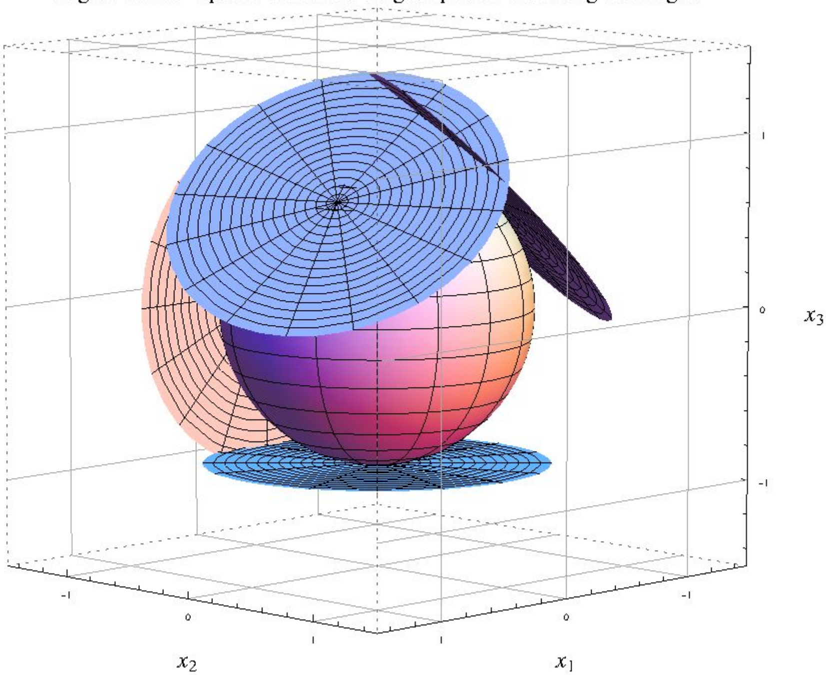


In Figure 7, the baseplane and the two planes above it are three planes that, if extended, would never meet at a point; they are linearly *dependent*. (Only a circular patch of each plane is shown so as not to obscure the view.) The origin is 'not quite' enclosed by the four planes. However, if we move the second plane or the third plane so that the contact point with the sphere moves away from the fourth plane, the origin becomes enclosed as shown in Figure 8.

Apart from the baseplane, we can consider each of the three planes to be one of a pair of equally likely parallel planes on either side of the sphere. Out of the $2 \times 2 \times 2$ combinations there is only one that encloses the origin.

To put it another way, O is 'released' from enclosure by reversing the coin-toss decision at any of the steps 1, 2 or 3 (three ways) or at any two of these steps (three ways) or at all three steps (one way). Thus the probability of enclosure $E(3,4) = 1/(1+3+3+1) = 1/8$.

Figure 8: A 3−sphere with four tangent planes enclosing the origin



Case $(3,5)$: $N = 5$ points on a 3-sphere. We argue as before. Step 1: Make a baseplane and make a parallel copy passing through O. Choose another $N-1 = 4$ planes though O, all different and not parallel to the baseplane. Step 2: Take the second plane through the origin. Construct the two parallel tangent planes. Toss a coin to choose one of these. And so on for steps 3, 4 and 5.

Now *any* three of the randomly chosen tangent planes, together with the baseplane, will form a tetrahedron, and if the coin tosses are favourable this tetrahedron will enclose O. If we find for example that the set of tangent planes $\{1,2,3,5\}$ encloses O, and the set $\{1,2,3,4\}$ does not enclose it, we must have tossed the coin four times to achieve enclosure. Thus this outcome occurs with probability $(1/2)^4 = 1/16$. Similarly, sets $\{1,2,4,5\}$

and $\{1,3,4,5\}$ enclose O with probability $(1/2)^4$. The set $\{1,2,3,4\}$ encloses O with probability $(1/2)^3$. Thus the total chance of enclosing the origin is $E(3,5) = (1/2)^3 + 3 \cdot (1/2)^4 = 5/16$. Hence $P_{3,5} = 1 - E(3,5) = 11/16$.

Case (n,N): N points on an n-sphere. Looking at the previous expression we can see that it can be written more generally as follows:

$$E(n,N) = \sum_{i=n}^{N-1} 2^{-i} \binom{i-1}{n-1}.$$

This is the general expression we have derived for the probability of enclosing O with N tangent hyperplanes in n dimensions. The following table gives some particular values of $E(n,N)$. Subtract from 1 to get $P_{n,N}$.

n	N							
	1	2	3	4	5	6	7	8
1	0	1/2	3/4	7/8	15/16	31/32	63/64	127/128
2	0	0	1/4	1/2	11/16	13/16	57/64	15/16
3	0	0	0	1/8	5/16	1/2	21/32	99/128
4	0	0	0	0	1/16	3/16	11/32	1/2
5	0	0	0	0	0	1/32	7/64	29/128

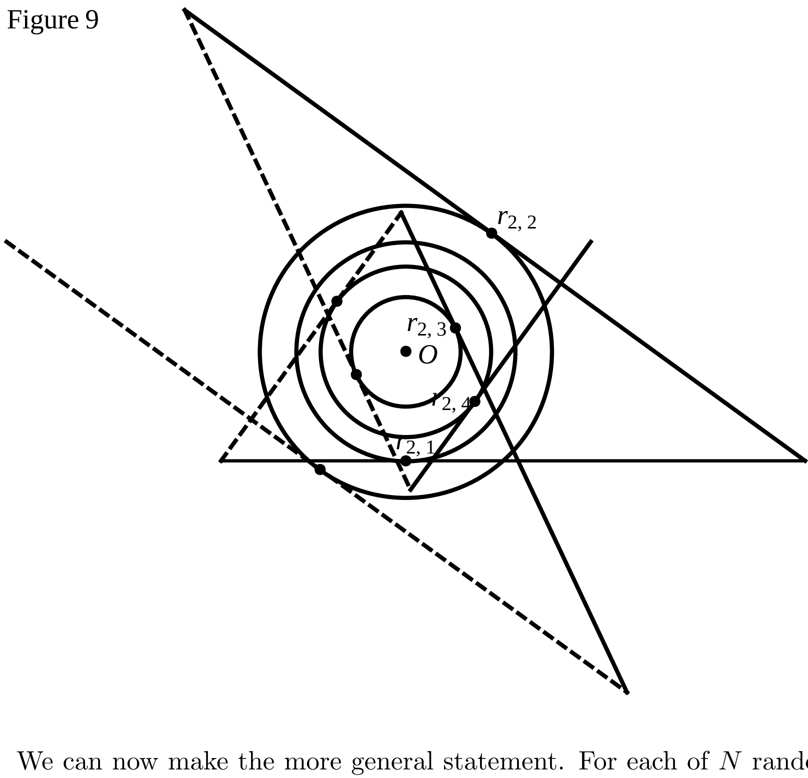
Now consider the $(n = 1)$-dimensional case; 1-space is just a line. Carry out the procedure as before. The results are interesting. Each 'hyperplane' is just a point. Step 1: The 'baseplane' is a point, let us say the point -1. Copy this point to the origin and put another $N - 1$ points at the origin. Their equations $(x = 0)$ satisfy 'linear dependence in sets of 1'. Step 2: Toss a coin to choose whether to copy the second point to the point $+1$ or to -1. If $+1$ is chosen then O has been enclosed. If -1 is chosen it hasn't. And so on for steps 3, 4, 5, ..., N.

But wait a minute! We need not restrict ourselves to placing points on the '1-sphere'. That is, at $+1$ and -1. If the coin-toss result is a head we can place the point anywhere on $\mathbb{R}^+$, or for a tail anywhere on $\mathbb{R}^-$. The argument about enclosing the origin remains the same. So if we choose N random points on $\mathbb{R}^+ \setminus \{0\}$, that is, any N random real numbers excluding zero, the probability that there will be both positive and negative numbers in the selection (thus 'enclosing' the origin) is

$$E(1,N) = \sum_{i=1}^{N-1} 2^{-i} \binom{i-1}{0} = \sum_{i=1}^{N-1} 2^{-i}.$$

In fact the distance from the origin does not alter the 'enclosing' argument in any number of dimensions n. We can choose a different radius for each sphere at steps $1, 2, 3, \ldots, N$. This means that the point of contact of the tangent plane and the sphere can be anywhere in n-space (apart from the origin), providing of course that the tangent planes are linearly independent in sets of n. So, for example, in two dimensions if we choose N random points in $\mathbb{R}^2 \setminus \{O\}$, we can draw a circle and a tangent line through each point. The probability of enclosing O by the lines will be $E(2, N)$ as already shown. If O is not enclosed, this must mean that all N points fall on one side of some line passing through the origin. That is, all the points are in a half-space of $\mathbb{R}^2$, with probability $1 - E(2, N) = P_{2,N}$. Figure 9 shows an example with $N = 4$, where all the points are in one half of $\mathbb{R}$.

Figure 9



We can now make the more general statement. For each of N random points in $\mathbb{R}^2 \setminus \{O\}$, construct an n-sphere with the origin O as centre, such that the point is on the surface. Construct the tangent $(n-1)$-dimensional hyperplane at the point. If the hyperplanes are linearly independent in sets of n, then the probability that O is enclosed by any subset of the hyperplanes is $E(n,N) = \sum_{i=n}^{N-1} 2^{-i}\binom{i-1}{n-1}$. Furthermore, if O is not enclosed, all N points fall in a half-space of $\mathbb{R}^2 \setminus \{O\}$ (on one side of some hyperplane passing through the origin), with probability $1 - E(n,N) = P_{n,N}$.

But hang on another minute! Instead of randomly choosing points from anywhere in $\mathbb{R}^2 \setminus \{O\}$, we could choose them from a subset of this space, provided that the subset is symmetrical, that is, for every point P there is an opposite point $-P$. We can then choose between P and $-P$ by tossing a coin. So for example, in $\mathbb{R}^n$, the result applies to a spherical shell of any thickness centred on O and also to an elliptical shell centred on O. In $\mathbb{R}^2$ it applies to a square or hexagon or other symmetrical shape, or in $\mathbb{R}^3$ to a cube or octahedron or dodecahedron or other symmetrical shape.

The equilateral triangle in $\mathbb{R}^2$ and regular tetrahedron in $\mathbb{R}^3$ lack across-the-origin symmetry and I do not know what the expected answer would be.

Applications

1. Select 8 non-synchronized satellites orbiting the earth, each in a different orbit and with a different orbital period. Given that it is equally likely that a satellite is at a point P or at an opposite point $-P$ with respect to the earth's centre, what is the chance that at this moment the N satellites are located directly over points on the earth's surface covering more than one hemisphere? Answer: $E(3,8) = 99/128$.

2. Draw 7 points at random inside an ellipse. What is the probability that all 7 will lie in one half of the ellipse (as delimited by some line drawn through the centre)? Answer: $1 - E(2,7) = 7/64$.

3. Draw 3 points at random inside a rectangle. Join the points to form a triangle. What is the chance that the centre of the rectangle falls inside the triangle? Answer: $E(2,3) = 1/4$.

4. Draw 5 lines at random in $\mathbb{R}^2$. What is the chance that the origin lies inside a bounded region? Answer: $E(2,5) = 11/16$.

Reference: J. G. Wendel 'A problem in geometric probability', *Math. Scand.* **11** (1962), 109–11. A transcript of this paper was recently found at http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/random-cyclic-polygons

Division tests

Dennis Morris

Abstract *This is an article in elementary number theory. It is well known that in number base 10 we can test a number for divisibility by 9 by adding together the digits of that number. Similarly, we can test a number for divisibility by 5 by examining only the final digit of that number. These two division tests in number base 10 are but two aspects of one general division test that is applicable to all integers in all number bases. This article describes that general division test.*

We begin with a statement of the theorem underlying division tests. This is a standard theorem that can be found in any book on elementary number theory. We then go on to demonstrate the theorem by using it to deduce some particular division tests.

Theorem *If $P(x)$ is a polynomial with integral coefficients and a is congruent to b modulo n, then $P(a)$ is congruent to $P(b)$ modulo n.*

The normal way of writing a number is to write it as an abbreviated polynomial.[1] For example: In number base 10, 1089 is an abbreviation of $1 \cdot 10^3 + 0 \cdot 10^2 + 8 \cdot 10^1 + 9 \cdot 10^0$, and in number base 7, 2153 is an abbreviation of $2 \cdot 7^3 + 1 \cdot 7^2 + 5 \cdot 7^1 + 3 \cdot 7^0$. So numbers are represented as polynomials and we can apply the theorem.

We begin in number base 10; this is the same as saying that we begin by setting $a = 10$. We seek a test for divisibility by 9, and so we set $n = 9$. The number we wish to test for divisibility by 9 we denote by $\alpha\beta\gamma\delta$, where each Greek letter represents a digit. In this case, we have chosen a 4-digit number, but the procedure will easily generalize in an obvious way to all numbers. So we have $P(10) = \alpha \cdot 10^3 + \beta \cdot 10^2 + \gamma \cdot 10^1 + \delta \cdot 10^0$. Thus we apply the theorem with $a = 10$, $n = 9$ and $P(10) = \alpha \cdot 10^3 + \beta \cdot 10^2 + \gamma \cdot 10^1 + \delta \cdot 10^0$.

Since $10 \equiv 1 \pmod 9$, we set $b = 1$.[2]

Thus, by the stated theorem, we have that $P(10) \equiv P(1) \pmod 9$.

Therefore the remainder when $\alpha \cdot 10^3 + \beta \cdot 10^2 + \gamma \cdot 10^1 + \delta \cdot 10^0$ is divided by 9 is the same as the remainder when $\alpha \cdot 1^3 + \beta \cdot 1^2 + \gamma \cdot 1^1 + \delta \cdot 1^0$ is divided by 9. Now $\alpha \cdot 1^3 + \beta \cdot 1^2 + \gamma \cdot 1^1 + \delta \cdot 1^0 = \alpha + \beta + \gamma + \delta$, and all we have to do is test $\alpha + \beta + \gamma + \delta$ for divisibility by 9. This is the famous divisibility by 9 test done by adding the digits. To put it obscurely, the division test is: $P(10) \equiv P(1) \pmod 9$.

We demonstrate once more in number base 10. Since we are working in number base 10, we have $a = 10$. We wish to test for divisibility by 8, and so we set $n = 8$. The number we wish to test for divisibility by 8 is the 8-digit number $\alpha\beta\gamma\delta\epsilon\psi\pi\theta$, and so we have $P(10) = \alpha \cdot 10^7 + \beta \cdot 10^6 + \gamma \cdot 10^5 + \delta \cdot 10^4 + \epsilon \cdot 10^3 + \psi \cdot 10^2 + \pi \cdot 10^1 + \theta \cdot 10^0$.

Since $10 \equiv 2 \pmod 8$, we set $b = 2$.

Thus, by the stated theorem, we have that $P(10) \equiv P(2) \pmod 8$.

Therefore the remainder when $\alpha \cdot 10^7 + \beta \cdot 10^6 + \gamma \cdot 10^5 + \delta \cdot 10^4 + \epsilon \cdot 10^3 + \psi \cdot 10^2 + \pi \cdot 10^1 + \theta \cdot 10^0$ is divided by 8 is the same as the remainder when $\alpha \cdot 2^7 + \beta \cdot 2^6 + \gamma \cdot 2^5 + \delta \cdot 2^4 + \epsilon \cdot 2^3 + \psi \cdot 2^2 + \pi \cdot 2^1 + \theta \cdot 2^0$ is divided by 8. But all powers of 2 greater than 2^2 are divisible by 8. Hence we only have to consider $\psi \cdot 2^2 + \pi \cdot 2^1 + \theta \cdot 2^0$. By the stated theorem, $\psi \cdot 2^2 + \pi \cdot 2^1 + \theta \cdot 2^0 \equiv \psi \cdot 10^2 + \pi \cdot 10^1 + \theta \cdot 10^0 \pmod 8$. And so: if the last three digits of a number (written in number base 10) are divisible by 8, then the number is divisible by 8. (If the last three digits leave remainder k when divided by 8, then the number will leave remainder k when divided by 8.) This is the well known division test for divisibility by 8 done by inspection of the last three digits.

If we wish to test for divisibility by 5 in number base 10, we set $n = 5$, $a = 10$, and, since $10 \equiv 0 \pmod 5$, we immediately have $b = 0$. By the stated theorem, we have that $P(10) \equiv P(0) \pmod 5$ and the only digit we have to consider for divisibility by 5 is the 0^0 digit – the final digit. (Of course $0^0 = 1$ as does 0!.) A similar situation applies in the case of divisibility by 10. If we wish to test for divisibility by 11 in number-base 10, we set $n = 11$, $a = 10$, and, since $10 \equiv -1 \pmod{11}$, we immediately have $b = -1$. By the stated theorem, we have that $P(10) \equiv P(-1) \pmod{11}$; this is the well known add and subtract alternate digits test for divisibility by 11. Similar tests for division by numbers higher than 11 in number-base 10 obviously exist.

Divisibility tests in number bases other than 10 are found by setting $a = B$ where B is the number base. In general, in all number bases:

> Analogous to divisibility by 9 in number base 10, there is a divisibility by $B - 1$ test based on adding digits because $B \equiv 1 \pmod{B - 1}$.
>
> Analogous to divisibility by 11 in number base 10, there is a divisibility by $B+1$ test based on adding and subtracting alternate digits because $B \equiv -1 \pmod{B + 1}$.

And in all even number bases:

> Analogous to divisibility by 5 in number base 10, there is a divisibility by $B/2$ based on inspection of the final digit because $B \equiv 0 \pmod{B/2}$.

In number base 12, since 12 is congruent to 0 in moduli 2, 3, 4 & 6, divisibility by 2, 3, 4 & 6 can all be tested by inspection of the final digit. Doubtless, the reader could add to this list—prime number bases perhaps.

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[1] Purists would say that a polynomial is nothing more than the coefficient numbers.

[2] We could have chosen $b = -8, 10, 19$ or any other number to which 10 is congruent mod 9, but such choices are of no practical value.

Reference: David M. Burton, *Elementary Number Theory.*

Problem 199.1 – Ellipsoid again

ADF

As we have seen from M500 **197**, it appears that there is no elementary formula for the surface area of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Instead we now ask: Is it possible to find a specific set of parameters $\{a, b, c\}$, $0 < a < b < c < \infty$, for which the surface area is computable exactly?

Problem 199.2 – 30 matches

ADF

(i) Use thirty matches to make a polygon of area 8 square matches.

(ii) Do (i) again but this time with the additional condition that the vertices of the polygon must have integer match coordinates. That is, if there is a vertex at (x, y), then x and y must be integer multiples of a match length.

Some time ago I was listening to a discussion of infinity in Melvyn Bragg's Radio 4 programme *In our time*. One of the contributors said that before the introduction of complex numbers only half the quadratic equations could be solved. Is this statement true? — **David Wild**

A class of arctangent identities

Bryan Orman

One technique for generating examination questions that many examiners employ is illustrated through the following examples.

Establish the identity

$$\arctan\frac{1}{1} \;=\; \arctan\frac{1}{2}+\arctan\frac{1}{3}.$$

This is straightforward.

Now convert this into the following question. Find the positive integers B and C such that

$$\arctan 1 \;=\; \arctan\frac{1}{B}+\arctan\frac{1}{C}.$$

This is a little more difficult although the answer is unique, $B = 2$ and $C = 3$.

We now move on to the next generalization: Find all the positive integer triples $\{A, B, C\}$ such that

$$\arctan\frac{1}{A} \;=\; \arctan\frac{1}{B}+\arctan\frac{1}{C}.$$

Some triples are $\{2, 3, 7\}$, $\{3, 4, 13\}$, $\{3, 5, 8\}$, ..., $\{20, 21, 421\}$. Are there a finite or infinite number of such triples?

My attempt at generating these triples proceeded thus: Elementary manipulation produces $B = A + (1 + A^2)/(C - A)$; so if we're given A, we need to find C such that $C - A$ divides $1 + A^2$. Let $C = A + K$ then the triples are $\{A, A + (1 + A^2)/K, A + K\}$ and are generated by finding the factors of $1 + A^2$. So there are an infinite number of such triples.

Now consider

$$\arctan\frac{1}{A} \;=\; \arctan\frac{1}{B}+\arctan\frac{1}{C}+\arctan\frac{1}{D}.$$

Evidently we can take pairs of connected triples like $\{1, 2, 3\}$ and $\{3, 4, 13\}$ and produce $\{1, 2, 4, 13\}$. So an infinite number of solutions here. Finally, what about

$$\arctan\frac{1}{A} \;=\; N\arctan\frac{1}{B}+M\arctan\frac{1}{C},$$

where N and M are positive integers? This is where I stop to let others investigate!

Weighing tarts

Norman Graham

When I expressed an interest in joining the M500 Society, I was sent a copy of magazine number **193**. I was very interested to read the two references to the 'Twelve tarts' problem because in 1994 I had published a version of this puzzle in *The Actuary*. My proof that $(3^n - 3)/2$ tarts can be resolved in n weighings [below] is not nearly so elegant as that quoted in Ron Potkin's letter.

I demonstrated that that the theoretical maximum of $(3^n-1)/2$ cannot be achieved. If it could, the maximum that could be set aside at the first weighing would be $(3^{n-1}-1)/2$ for the other $n-1$ weighings, leaving 3^{n-1} for the first weighing, which (being odd) is impossible.

However, it can be done if an extra normal tart is available. Using Ron Potkin's notation [M500 **193** 25], call the extra normal tart `N`, with values $222\,C$ and $000\,A$, and suppose there is an extra unknown tart `O` with values $000\,C$ and $222\,A$. Then in every weighing the unknown tart and the normal tart are added to the left and right hand pans, respectively. Of the 27 3-figure numbers in ternary notation, this method would use all except 111, which of course is useless because it represents equal balances at every weighing.

THEOREM. *There are X tarts which look identical, but one has a slightly different weight. Determine the odd tart and whether heavy or light using a simple balance.*

(i) *The maximum value of X for n weighings is $(3^n - 3)/2$.*

(ii) *If an extra normal tart is available, The maximum value for X is $(3^n - 1)/2$.*

LEMMA 1. *If the odd tart is known to be heavy, X is 3^n, and similarly if the odd tart is known to be light.*

Proof. Split the tarts into three groups of 3^{n-1}, then weigh one group against the other to determine which group contains the odd tart: split that group into three and so on. After $n-1$ weighings a group of three will remain, and one more weighing identifies the tart required.

LEMMA 2. *Given a supply of normal tarts, X is $(3^n - 1)/2$.*

Proof. Weigh 3^{n-1} of the tarts against normal tarts. If heavy or light, Lemma 1 applies for $n-1$. If equal, the number of tarts remaining is $(3^n-1)/2 - 3^{n-1} = (3^{n-1}-1)/2$. Hence if true for $n-1$ it is true for n. Clearly it is true for $n=1$, and so by induction is true for all n.

Lemma 3. *Given that the odd tart is either heavy amongst $X/2$ of the tarts* (*group A*) *or light amongst the other $X/2$* (*group B*), *and given a supply of ordinary tarts, $X = 3^n - 1$.*

Proof. On the left of the balance put 3^{n-1} of A and $(3^{n-1} - 1)/2$ of B; on the right, put the remaining $(3^{n-1} - 1)/2$ of A and 3^{n-1} ordinary tarts. If equal, Lemma 1 applies to the omitted 3^{n-1} B tarts. If the left pan is heavy, Lemma 1 applies to the 3^{n-1} A tarts on that side. If the right pan is heavy, the odd tart is either heavy amongst $(3^{n-1} - 1)/2$ A tarts or light amongst $(3^{n-1} - 1)/2$ B tarts, so that the problem is this lemma for $n - 1$. Hence if true for $n - 1$, it is true for n. Clearly it is true for $n = 1$, and so by induction for all n.

Proof of the theorem. Split the $(3^n - 3)/2$ tarts into three groups of $(3^{n-1} - 1)/2$ and weigh one group against another. If equal then Lemma 2 for $n - 1$ can be applied to the third group; otherwise, Lemma 3 for $n - 1$ can be applied to the two groups weighed.

Clearly, n weighings (each light, heavy or equal) gives 3^n items of information. With X tarts (one light or heavy) there are $2X$ possibilities. Hence the theoretical maximum value for X is $(3^n - 1)/2$, since 3^n is odd. This is one more than $(3^n - 3)/2$ proved above. However, $(3^n - 1)/2$ can be achieved with a supply of ordinary tarts (Lemma 2) but not otherwise.

If heavy or light at the first weighing, there are $2 \cdot 3^{n-1}$ items of information, which in theory can be used to determine the answer for 3^{n-1} tarts. This is reduced to $3^{n-1} - 1$ because the first weighing must be done with an even number of tarts. Addition of $(3^{n-1} - 1)/2$ set aside (Lemma 2) gives $X = (3^n - 3)/2$.

ADF writes — I wonder if anyone can offer a solution to that other family of tart weighing problems. This is where you have n tarts but some jam has been transferred from one tart to another, so that there is one light tart, one heavy tart and $n - 2$ normal tarts. For instance, in M500 **188** I asked about 16 tarts in five weighings. **Dick Boardman** and I can do it, but not with easily described weighing instructions, and hence our solution is far too complicated to justify publication. We would be interested if you can solve 'Problem 188.4 – Sixteen tarts' in not too many pages.

And while we're on the subject, you may be amused to learn about M500's fifteen seconds of fame. The filler below the 'Thirteen tarts' article in issue **197** was broadcast by *The News Quiz* (Radio 4) on 4th/5th June 2004.

Getting dressed again

ADF

Some further thoughts about Problem 194.4. Recall that you were asked to calculate the number of ways of putting on a set of clothes chosen from h hats, b bras, p panties, d dresses, s pairs of socks and f pairs of shoes, subject to the following restrictions: (i) you wear one of each type of clothing; (ii) underwear goes on before dress, and sock before shoe; (iii) socks and shoes are paired; (iv) chirality is relevant for shoes (but not socks).

Recall also that at one point in the offered solution (M500 **197**) I asked, 'How many ways are there of splicing together the ordered sets $B =$ (bra, panties, dress) and $F =$ (left sock, right sock, left shoe, right shoe).' The answer I gave went like this.

There are three items in B and five slots created by the elements of F. We could choose to put the bra, panties and dress into three different slots, say, for example, (bra, left sock, right sock, panties, left shoe, dress, right shoe). There are $\binom{5}{3}$ ways of doing this. Or we could put the bra into one slot and the panties and dress into a second slot, or the bra and panties into one slot and the dress into another; that's $2\binom{5}{2}$. Or we could put all three into the same slot: $\binom{5}{1}$. Adding them together makes $\binom{5}{3} + 2\binom{5}{2} + \binom{5}{1} = 35$.

More generally, let $S(a, b)$ denote the number of ways of splicing a sequence of b things into a sequence of a things. Then we have

$$S(a, b) = \sum_{k=1}^{b} \binom{b-1}{k-1}\binom{a+1}{k}$$

and in particular $S(4, 3) = 35$.

However, if we do the computation the other way round, by splicing H into F, we obtain $S(3, 4)$, and when you carry out the summation you should get the same answer, 35. Indeed, generally we ought to have $S(a, b) = S(b, a)$, or, when you express it in terms of binomial coefficients,

$$\sum_{k=1}^{b} \binom{b-1}{k-1}\binom{a+1}{k} = \sum_{k=1}^{a} \binom{a-1}{k-1}\binom{b+1}{k}. \tag{1}$$

I admit that I could not see immediately why (1) should hold. So I fired up MATHEMATICA and it took only a few seconds to respond with

$$S(a, b) = \binom{a+b}{a} = \frac{(a+b)!}{a!\,b!} = \binom{a+b}{b} = S(b, a). \tag{2}$$

If we include the hat, then the number of ways of ordering the eight items of clothing is given by $S(4,3)S(7,1) = 35 \cdot 8 = 280$, and, indeed, it seems that we can look at the original problem in an entirely different way. The number of orderings of the eight items is the same as the number of ways of selecting 1, 3 and 4 things from 8 things, $8!/(1!\,3!\,4!) = 280$. The two ways of looking at the problem give the same answer.

In general, we can make repeated use of (2) to obtain

$$S(a,b)\,S(a+b,c) \;=\; \binom{a+b}{b}\binom{a+b+c}{c} \;=\; \frac{(a+b+c)!}{a!\,b!\,c!},$$

and, even more generally,

$$S(a_1,a_2)\,S(a_1+a_2,a_3)\;\ldots\;S(a_1+a_2+\cdots+a_{n-1},a_n)$$
$$= \frac{(a_1+a_2+\ldots a_n)!}{a_1!\,a_2!\,\ldots\,a_n!}.$$

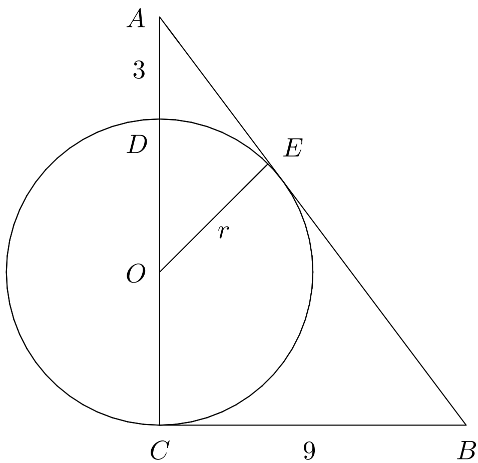
I suspect that it should be possible to prove (2) directly but (at risk of advertising my stupidity) it is not obvious to me how to do it easily. Does anyone have any ideas?

Problem 199.3 – Two tangents

John Reade

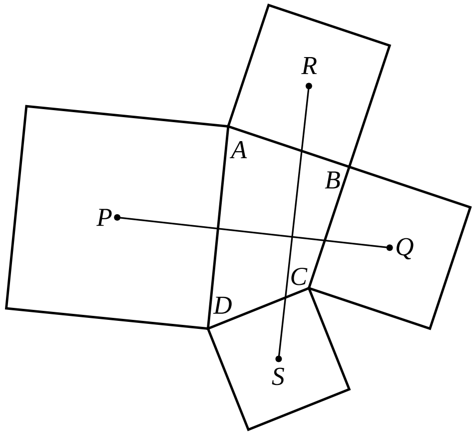
Lines AB and BC are tangents to the circle. The centre of the circle, O, lies on AC. If $BC = 9$ and $AD = 3$, what is the value of r, the radius of the circle?

I am told this problem appeared in the *Daily Mail*. I thought it deserved a wider audience.



Solution 196.2 – Quadrilateral

In any quadrilateral $ABCD$, draw squares on its sides. Join the centres of the squares of opposite sides with PQ and RS. Prove that PQ is perpendicular to RS and that PQ and RS are equal in length.



Dick Boardman

Barbara Lee's problem has a very simple solution using complex numbers. There is another similar problem. If you erect an equilateral triangle on the sides of a general triangle, the centres form an equilateral triangle.

The solution to Barbara's problem is as follows.

Let the four vertices in a complex plane be

$$a = a_x + ia_y, \quad b = b_x + ib_y, \quad c = c_x + ic_y, \quad d = d_x + id_y.$$

Let $\rho = (1 + i)/2$. Multiplying a complex number by ρ will rotate it by 45 degrees and multiply its length by $1/\sqrt{2}$. We have

$$r = a + \rho(b - a), \quad q = b + \rho(c - b),$$

$$s = c + \rho(d - c), \quad p = d + \rho(a - d).$$

Then $p - q - i(r - s) = 0$.

There is a similar solution to the triangle problem.

I often wonder if there is a general result here.

Bryan Orman

Just received my copy of **196** and I'd like to point out that Barbara Lee's Quadrilateral Problem is in fact Aubel's theorem. See page 11 of David Wells, *The Penguin Dictionary of Curious and Interesting Geometry.*

John Spencer

This problem can be approached as a matter of linear algebra, as follows.

Form the points P, R, S, T by rotating each of A, B, C, D through $\pi/2$ about the midpoint of the adjacent line, by the linear transformation

$$F(A,B) \;=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \frac{B-A}{2} + \frac{A+B}{2},$$

so that

$$R = F(A,B),\; Q = F(B,C),\; S = F(C,D) \;\text{ and }\; P = F(D,A).$$

If for ease of notation we designate $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot X$ as X', we can write

$$R \;=\; F(A,B) \;=\; \frac{B' - A' + A + B}{2}$$

and

$$Q = \frac{C' - B' + B + C}{2},\; S = \frac{D' - C' + C + D}{2},\; P = \frac{A' - D' + A + D}{2}.$$

Lines PQ and RS are mutually perpendicular and the same length if one can be rotated onto the other, that is, if

$$P - Q \;=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot (R - S).$$

Expanded,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot (R - S) \;=\; \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot (B' + C' - A' - D' + A + B - C - D).$$

But $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot X' = -X$, so that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot (R - S) \;=\; \frac{-B - C + A + D + A' + B' - C' - D'}{2} \;=\; P - Q.$$

So $PQ \perp RS$ and $\|PQ\| = \|RS\|$ as required.

Solution 196.6 – Pendulum

Show that $\theta(t) = 4\arctan e^{\sqrt{g/L}\,t} - \pi$ is a solution of the differential equation $d^2\theta/dt^2 = -g\sin\theta/L$,

Jim James

The particle equation of motion is given as $d^2\theta/dt^2 = -g\sin\theta/L$, so whenever $\theta = 0$, $d^2\theta/dt^2 = 0$.

The proposed solution function is $\theta = 4\arctan e^{kt} - \pi$, where $k = \sqrt{g/L} > 0$. Using the chain rule with $u = e^{kt}$, we have

$$\frac{d\theta}{dt} = \frac{4}{1+u^2}\,\frac{du}{dt} = \frac{4ke^{kt}}{1+e^{2kt}}.$$

Then

$$\frac{d^2\theta}{dt^2} = \frac{4k^2e^{kt}(1+e^{2kt}) - 4ke^{kt}\cdot 2ke^{2kt}}{(1+e^{2kt})^2} = \frac{4k^2e^{kt}(1-e^{2kt})}{(1+e^{2kt})^2}.$$

The proposed solution function may be written in the form $e^{kt} = \tan(\theta+\pi)/4$; so for $\theta = 0$, $e^{kt} = 1$ and since $k \neq 0$, t must be zero, indicating that this situation corresponds to a single instantaneous occurrence.

But at time $t = 0$, $1 - e^{2kt} = 0$ too, so $d^2\theta/dt^2 = 0$, which is precisely the value given by the equation of motion at $t = 0$. We conclude, therefore, that the proposed function is, indeed, a solution to the equation of motion, as required. Note too, that $d\theta/dt = 2k = 2\sqrt{g/L} \neq 0$.

Comment In the standard analysis of the simple pendulum, as frequently covered in differential equations textbooks, the equation of motion is shown to have no general solution that can be expressed in terms of elementary functions. But the proposed solution function analysed here is so expressed. It cannot, therefore, be a particular solution derived from the general solution for the simple pendulum case.

This fits in nicely with traditional theory, which recognizes the existence of certain differential equations that have rogue solutions, which, like this, have no connection with the general solution. Such anomalous solutions are called singular solutions.

Some academics, however, do not recognize singular solutions; they claim that all solutions that contain no arbitrary constants are particular solutions and that a true general solution must embrace all solutions, including those that were once viewed as being singular (see Tennenbaum & Pollard, *Ordinary Differential Equations* (Harper, 1964), for example).

But this view does not help at all in this instance; the proposed solution function does satisfy the equation of motion at time $t = 0$. But that seems to be all that one can deduce from the data provided. With initial conditions $\theta = 0$, $d\theta/dt \neq 0$, $d^2\theta/dt^2 = 0$, which we have seen must apply for it to be a solution at all, it does not appear to relate to any reasonable physical system either, as implied by Tony's comments. One thing is clear, however, apart from sharing a common equation of motion, it definitely has nothing to do with the classical simple pendulum.

Simon Geard

From the given solution, $\theta(t) = 4\arctan e^{kt} - \pi$, where $k = \sqrt{g/L}$, divide by 4 and take tangents, so that

$$\tan\frac{\theta}{4} = \frac{e^{kt}-1}{e^{kt}+1} = \tanh\frac{kt}{2}.$$

Now differentiate and rearrange:

$$\frac{1}{4}\sec^2\left(\frac{\theta}{4}\right)\frac{d\theta}{dt} = \frac{k}{2}\operatorname{sech}^2\frac{kt}{2},$$

$$\frac{d\theta}{dt} = 2k\cos^2\left(\frac{\theta}{4}\right)\left(1-\tanh^2\frac{kt}{2}\right) = 2k\cos\frac{\theta}{2}.$$

So differentiating again gives

$$\frac{d^2\theta}{dt^2} = -k\sin\frac{\theta}{2}\frac{d\theta}{dt} = -k^2\sin\theta.$$

Tony Forbes writes — We had similar answers from **Basil Thompson** and **Dick Boardman**. There is a brief discussion of the general solution in Henry McKean & Victor Moll, *Elliptic Curves* (CUP). Alternatively, it is possible to get a general solution by developing $\theta(t)$ as a power series. Write $\theta_n(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ and equate coefficients of t^{n-2} for $n \geq 2$. Thus

$$a_n = -\frac{k^2}{n!}\left.\frac{d^{n-2}\sin\theta_{n-2}(t)}{dt^{n-2}}\right|_{t=0}$$

from which a solution of $\theta''(t) = -k^2\sin\theta(t)$ with given $a_0 = \theta(0)$ and $a_1 = \theta'(0)$ may be obtained. For example, $\theta(0) = 0$ and $\theta'(0) = k$ yields

$$k\,t - \frac{(kt)^3}{6} + \frac{(kt)^5}{60} - \frac{13\,(kt)^7}{5040} + \frac{23\,(kt)^9}{51840} - \frac{221\,(kt)^{11}}{2851200} + \frac{12539\,(kt)^{13}}{889574400} - \cdots.$$

A paradoxical dice problem

David Singmaster

Problem proposal. Consider throwing a k-sided die ($k \geq 2$) until a particular pair comes up in order on consecutive throws. It would seem that all pairs would have the same expected time to appear and that for any two pairs, betting on which will appear first should be a fair bet. Paradoxically, this is not true. Find the probability that $(1, 2)$ will occur sooner than $(2, 2)$. Our proof does not make the result intuitively obvious—is there a proof which makes the result seem obvious?

Solution. Let P denote the probability that $(1, 2)$ occurs before $(2, 2)$, and let P_i be the probability of this occurring when we have just thrown an i. Since throwing a value other than 1 or 2 just leaves us where we were, we have $P = P_3 = P_4 = \ldots$. Considering the starting postion, we have

$$P = \frac{(k-2)P}{k} + \frac{P_1}{k} + \frac{P_2}{k}. \tag{1}$$

If we throw a 1, we have

$$P_1 = \frac{(k-2)P}{k} + \frac{P_1}{k} + \frac{1}{k}; \tag{2}$$

and if we throw a 2, we have

$$P_2 = \frac{(k-2)P}{k} + \frac{P_1}{k} + 0. \tag{3}$$

Equations (1)–(3) are readily solved to obtain

$$P = \frac{(k+1)}{2k} = \frac{1}{2} + \frac{1}{2k}.$$

Similar reasoning shows that the expected time to obtain a $(1, 2)$ is k^2 while the expected time to obtain a $(2, 2)$ is $k(k+1) = k^2 + k$. The expected time to get either $(1, 2)$ or $(2, 2)$ is $k(k + 1)/2$. Is there any connection between these expected times and the probability P? Is there any way to make these results more obvious? The above argument is considerably simpler than my initial proof, but is not as elementary as we would like.

When $k = 2$, the process simplifies considerably and one can obtain $P = 3/4$ easily. I presented this on *Puzzle Panel* last year and **Chris Maslanka** realized that the idea could be applied to dice—and has found that most people do not believe it!

Problem 199.4 – Three integers

Find all positive integer triples (a, b, c) such that

$$f(a, b, c) = \frac{a^2}{abc^2 - c^3 + 1}$$

is an integer.

According to a cutting from *The Hemel Hempstead and Berkhamsted Gazette* sent to me (ADF) by **Colin Davies**, if you can solve this problem under examination conditions, you should find the International Mathematics Olympiad a doddle; and if you happen to satisfy the entry requirements (under 20 years old and still at school), you could seriously consider becoming a contestant. The special case $b = 2$ appeared in the 2003 Olympiad, held in Japan.

To get started, notice that $f(a, b, 1) = a/b$ and $f(a, b, ab) = a^2$. Then a useful strategy is to try out various a, b, c and see if there is a pattern of some sort. The difficult part seems to be proving that you have got all the solutions.

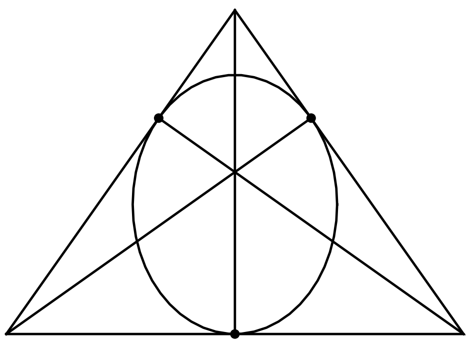
The article, by the way, was mainly about Berkhamsted student Paul Jeffreys, 17, who won a gold medal—Britain's first since 1997. The British team came tenth out of the 82 countries which participated. Bulgaria won.

Problem 199.5 – Inscribed ellipse

ADF

(i) Take any triangle and mark a point on each side. When is it possible to draw an ellipse that is tangent to the three sides at the marked points?

(ii) What is the formula for the ellipse when the points are at the bases of the altitudes of a triangle with vertices $(\pm 1/2, -\sqrt{2}/4)$ and $(0, \sqrt{2}/4)$?



From the diagram, you can see that I have at least an approximate answer for (ii). However, I am not sure I got it exactly right—a second opinion would be appreciated.

What's next?

Jeremy Humphries

I don't know what the genuine terms are, but by analogy from Latin, deviating slightly from Ralph Hancock [M500 **197** 28], I'd say,

11 – undecuplets (from undecim)	16 – sedecuplets
12 – d[u]odecuplets	17 – septendecuplets
13 – tredecuplets	18 – duodevigintuplets
14 – quattuordecuplets	19 – undevigintuplets
15 – quindecuplets	20 – vigintuplets

Next I'm not sure. Twenty-one is unus et viginti, 22 is duo et viginti, etc., so unus-et-vigintuplets, etc? Then ...

28 – duodetrigintuplets (from triginta)
29 – undetrigintuplets
30 – trigintuplets
31 – unus-et-trigintuplets [?]
38 – duodequadragintuplets (from quadraginta)
40 – quadragintuplets
50 – quinquagintuplets (from quinquaginta)
60 – sexagintuplets (from sexaginta)
70 – septuagintuplets (from septuaginta)
80 – octogintuplets (from octoginta)
90 – nonagintuplets (from nonaginta)
100 – centuplets (from centum)

That's enough. No doubt there exists an expert who will put us right.

Then there's this inflected poem *de motore bo* by A. D. Godley. From the references (Corn and High) he was an Oxford man, I assume.

What is it that roareth thus?
Can it be a Motor Bus?
Yes, the smell and hideous hum
Indicat Motorem Bum!
Implet in the Corn and High
Terror me Motoris Bi:
Bo Motori clamitabo
Ne Motore caedar a Bo—
Dative be or Ablative
So thou only let us live:
Whither shall thy victims flee?
Spare us, spare us, Motor Be!
Thus I sang; and still anigh
Came in hordes Motores Bi,
Et complebat omne forum
Copia Motorum Borum.
How shall wretched lives like us
Cincti Bis Motoribus?
Domine, defende nos
Contra hos Motores Bos!

Letters to the Editor

Under the skin

I found Colin Davies's article ('Under the skin', M500 **197**) interesting but a couple of points struck me. Firstly, there may be people who share many chromosomes who are not related closely by recent common ancestors. This is relevant to his third question.

The other point was as to whether there could exist something approaching closed subgroups of the population from which there is little or much reduced genetic exchange. He mentioned geographic separation but one wonders about personality, social niches, familiarity with family behaviour patterns. These might constrain the choice of partner to some extent, even indirectly through parental influence. I think that the only way to answer these questions is through empirical study.

One then has to consider if some genes function synergistically together and that in such cases there would be a considerable advantage if they were in close proximity. To my mind the genetic and family structure could be more interesting although much more difficult to quantify.

Yours sincerely,

David Robertson

Dear Tony,

The proof that I am descended from William I is very interesting. The proof that I am also descended from his second son, Richard (1054–1075 — died without issue in a hunting accident) is even more interesting. I am always prepared to put my fortune on odds of 10^{99} to one in my favour.

The model has a major flaw—many die childless or the line dies out. Families with ten or more children seen to have been fairly common in the past. There was no population explosion so there must have been an appropriate number of childless individuals to keep everything in balance. Nothing like sending a few children off to war or to become priests to keep the population down.

Assuming that on average from ten births, eight are childless while two marry (if necessary) and have ten children, what is the probability of the line continuing after n generations?

On a similar topic we are all descended from 'Eve', a common female ancestor. The proof is simple—each generation has more daughters than mothers. DNA tests show that we are all the same strain.

Best wishes,

John Seldon

Balls

I got the following from **Jeremy Humphries**.

> There's an article in my local paper about the local golf club. The club has bought a load of new golf balls for use on its driving range. There's a photo of a square pyramid built with some of the new balls, and readers are invited to guess how many balls there are in the pyramid. The entry is 50p, all proceeds to go to the local hospice. The photo clearly shows that the pyramid is 33 layers tall.

Our initial reaction was to avoid guesswork altogether and simply plug 33 into the formula $n(n+1)(2n+1)/6$. But this of course works only if the pyramid is solid—that is, nobody has removed any balls from the interior.

This leads to a very interesting question. *What is the minimum number of spheres that you need to build a stable structure which looks from the outside like a square pyramid on a base of dimensions $n \times n$?* Assume that bottom layer is complete and its spheres are securely anchored to the ground, as would be the case if you were stacking cannon balls on a flat patch of soft earth. For instance, the answer for $n = 4$, the first non-trivial case, is 29 because you don't need to fill the central slot in the second layer from the bottom.

Later we discovered that the golf club was not as devious as we had thought. Its pyramid turned out to be complete. Furthermore, Jeremy did indeed submit the sum-of-squares value, 12529, plus 50p and was somewhat embarrassed when the club awarded him the first prize of one hour's tuition with the club pro!

Problem 199.6 – Change

The British currency system has the property that you can always use the so-called *greedy algorithm* to make up an amount of money with the minimum number of currency units (notes or coins). To make £n, select the largest currency unit £m such that $m \le n$ and, if necessary, repeat the process with the remaining £$(n - m)$. For example, £16 = £10 + £5 + £1.

But it doesn't work for all imaginable currency systems. For instance, if the units are \$10, \$8 and \$1, it is clear that \$16 can be done with two units, \$8 + \$8, whereas the greedy algorithm insists on seven units, \$10 + \$1 + \$1 + \$1 + \$1 + \$1 + \$1.

What characterizes those systems where the greedy algorithm always minimizes the number of currency units?

Twenty-five years ago

From M500 60

Steve Murphy — On an M101 programme they showed us that we can tell if a clock with no figures is upside down by examining the relative positions of the hour and minute hands.

Would this still be true if the clock were reflected in a mirror?

What is the effect of identical hour and minute hands, either in the mirror case or the upside down case?

Angus Macdonald — When Bert was just one year younger than Bill was when Ben was half as old as Bill will be three years from now, Ben was twice as old as Bill was when Ben was one third as old as Bert was three years ago. But when Bill was twice as old as Bert, Ben was one quarter as old as Bill was one year ago. If Bert is over fifty, how old are they all.

M500 Winter Weekend 2005

Preliminary announcement

The twenty-fourth M500 Society Winter Weekend will be held on **Friday 7th to Sunday 9th January 2005**.

This is an annual residential weekend to dispel the withdrawal symptoms due to courses finishing in October and not starting again until February. It's an excellent opportunity to get together with friends, old and new, and do some interesting mathematics in a leisurely and congenial atmosphere.

We have a different venue this year:

Trevelyan College, **Durham**.

As this may be a little further afield for some regular attenders, there will also be the option of staying over on Sunday night, and we have a choice of standard or en-suite rooms, with some twin rooms available.

We don't know the theme yet, but we hope to announce that by the time of the September Revision Weekend. Further details will be available with the next magazine, or after September 12th you can send a stamped, addressed envelope for a booking form to **Norma Rosier**. A booking form will also be available at www.m500.org.uk after that date, with details of costs and options available.