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The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

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Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail.

Is there a Ramanujan problem?

Sebastian Hayes

Most readers of M500 have probably heard the story of Ramanujan's irruption on the Cambridge mathematical scene. In January 1913, G. H. Hardy, one of England's leading pure mathematicians, received a bulky letter from a poor clerk in Madras who had three times failed to get into an Indian University. The letter consisted of ten handwritten pages full of weird theorems and included the claim that the writer, Srinivasa Ramanujan by name, had in his hands 'an expression for the number of prime numbers less than N which very nearly approximates to the real result, the error being negligible.'

Hardy at first thought the letter ('the most extraordinary I received in my life') was some sort of a practical joke but by the end of the day he and Littlewood decided no one they knew had the expertise to perpetrate a mathematical hoax on this scale. Hardy eventually arranged for Ramanujan to come to Cambridge (without having to pass any examinations, of course), sponsoring him a year or so later for election to the Royal Society. So far this reads like a fairy tale but there is a sad ending. Ramanujan didn't take to English weather, cooking and stiff upper lipness, tried to commit suicide once in the London Underground and contracted tuberculosis which, after his return to India, killed him. Those interested in his fascinating life story would be recommended to read *The Man Who Knew Infinity* by Robert Kanigel (Scribners 1991). This book is very good indeed on the cultural and social environment at the time both in South India and in Cambridge but could do with an appendix going into Ramanujan's discoveries in depth and assessing their importance today. (There is mathematics in the book but not on a level readers of M500 would find of much interest.)

Ramanujan received no more than the equivalent of A-level formal training in mathematics. Even in his maturer years—he died before he was thirty-five—he gave very few proofs and those he did give left much to be desired. His reputation has fluctuated but for all that seems to be standing up pretty well; it even underwent something of a renaissance when his last Notebook was discovered in the eighties since some saw in it anticipations of string theory. (Not that Ramanujan was at all interested in physics or indeed any applied mathematics.)

The early twentieth century was the era of rigour. The recommended path to mathematical discovery was meticulous observation followed by informed conjecture followed by strict proof. Hardy himself disliked loose thinking in mathematicians rather in the way in which the Victorians disliked loose morals in women. And a stone's throw away from Hardy's rooms, Bertrand Russell was busily reducing the whole of mathematics to logic.

Keen observation (of numbers) accounts for some of Ramanujan's results such as those contained in his first published paper on Bernoulli numbers. In the days when the PC was not even a pipe dream Ramanujan spent a lot of time trawling through seas of integers, exactly the sort of drudgery many Western mathematicians at the time rather looked down on. (*)

However, one can't see observation alone producing

$$1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\dots$$
$$=\frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})(1-q^{14})(1-q^{16})(1-q^{19})\dots},$$

or

$$1-5\left(\frac{1}{2}\right)^3+9\left(\frac{1\cdot 3}{2\cdot 4}\right)^3-13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3+\dots\ =\ \frac{2}{\pi},$$

or

$$1+9\left(\frac{1}{4}\right)^4+17\left(\frac{1\cdot 5}{4\cdot 8}\right)^4-25\left(\frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}\right)^4+\dots\ =\ \frac{2\sqrt{2}}{\sqrt{\pi}\,\Gamma(3/4)^2},$$

where $\Gamma(x)$, the *gamma function*, is defined by $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$.

Is there some sort of mystery about Ramanujan as a mathematician? On the face of it, yes. There have been countless writers and painters who were largely self-taught (the Brontes, Van Gogh, even Leonardo da Vinci) but rather few world-class mathematicians. As a Platonist, Hardy did not think it at all unreasonable to believe that the theorems of higher mathematics, the great ones at any rate, had in some sense always existed and always would exist. But to attain a glimpse of the Sacred Grail required years of hard work, university training and self-discipline—there was no royal road to analysis. Here, however, was a fellow who claimed to receive formulae for hypergeometric series in dreams and who attributed his mathematical achievements to his family's tutelary goddess, Namagiri. This was Rider Haggard, or worse.

If one takes a formalist point of view, mathematics is invention—at least in principle. (In practice, however, students of mathematics are *never* invited to devise their own symbolic systems as, for example, artists are invited to choose their own subjects for paintings.) But to class Ramanujan's theorems as inventions rather than discoveries does not really advance us very much. Why him and not someone else? It gets one nowhere to talk of 'mathematical instinct', 'uncanny judgment' and so forth. What exactly *is* 'mathematical instinct' and can it be developed? Very little seems to have been written on this important topic.

Should we / can we, then, believe that Ramanujan was a sort of mathematical Joan of Arc who received instructions from above? This is an explanation of sorts and seems to have been the one Ramanujan preferred. But there are difficulties—principally the fact that Ramanujan was not always right. His claim to have in his hands a formula giving the distribution of the primes turned out to be mistaken. (It has apparently since then been shown that no such formula can exist.) Of course, there is no reason why a goddess should not err on certain technical points but it is suspicious that the slips made by Ramanujan (or his source) were precisely the ones to be expected from someone not fully *au courant* with the very latest research into the divergence of infinite series.

I personally don't have the sort of trouble Hardy and Russell (or, today, people like Martin Gardner) have with the idea that some people can tune in 'directly' to sources of knowledge most of us can't, though I interpret the phenomenon more in terms of Jungian 'group minds' or 'collective memories' than in terms of goddesses and spirits. On the other hand, I *do* have a deal of trouble with mathematical Platonism. The latter made good sense in the days when people viewed God as the Supreme Mathematician (as Kepler and Newton did) but is hardly plausible today. The consensus amongst physicists is that the world we live in is not the result of intelligence and planning but just happened. To be sure, mathematics has proved to be a useful tool for investigating the cosmos but that doesn't mean the cosmos is mathematical—is a cat mathematical? According to the multiverse theory there's nothing special about the values of G or the fine structure constant. We just happen to be in a universe where they have the values they do and that's all there is to it. And if there's something beyond and behind all possible and actual universes, the Matrix to end all matrixes, well, my guess is that neither language nor concept nor shape nor number is of any help here. Mathematics deals with the measurable and whatever ultimate reality is it's not measurable—that's my feeling anyway.

So how do I explain Ramanujan? As someone who believes that the origins of mathematics lie in our perceptions of the everyday physical world, I must admit Ramanujan worries me a bit. Because of the terseness of his results and his air of absolute conviction he does, at first glance, sound like someone who has a window on a higher reality, a strictly mathematical one, and that all he has to do is to transcribe what he sees. Apparently, in at least some cases, Ramanujan really did wake up and write down straight off complicated formulae he had received in dreams.

But then again I find it credible that a fair bit of reasoning can and does go on 'unconsciously', not just in the case of Ramanujan, but in everyone's case. The result is flashed through to the conscious without the intermediate steps and appears as an 'intuition', 'hunch', sudden conviction. Rationalists get hunches like everyone else but they tend to screen them out like porn on the Internet. Or else they pretend they arrived at the result deductively—it may be that at the end of the day writers don't actually use the unconscious any more than scientists and mathematicians do, it is simply more acceptable (culturally correct) for writers to admit it. In the 17th century Ramanujan would not have appeared so strange: even Descartes, the founder of rationalism, claimed to have been visited by the Angel of Truth.

Apart from this, quite a lot can be explained by Ramanujan's idiosyncratic working habits. In India at any rate—where he seems to have done most or all of his creative work—he did his mathematics with chalk and slate (because he found paper too expensive). He rubbed out with his elbow as he progressed and only noted down the final result. So he probably couldn't remember the intermediate steps by the time he'd finished and had no means of checking. Maybe he even covered his tracks on purpose—we don't really want a conjuror to reveal the secret, do we? Indian mathematics never was too much concerned with proving things anyway—there is the famous example of the 'proof' of Pythagoras' theorem by way of a diagram with the caption *'Behold!'*

So, all things considered, I'm not sure there is any unresolvable mystery. To most commentators Ramanujan was born not made and he came through despite his environment. I find this questionable. Ramanujan was lucky to be born in the right place and at the right time. India was, at the end of the nineteenth century, a country looking in two directions. It was still immersed in mysticism, the occult, philosophic and religious speculation. But at the same time it had an advanced educational system modelled on the British, and was by then sending a few gifted people to Oxford and Cambridge. The rational plus the irrational (or supra-rational), this is a heady and treacherous mixture but it suits certain types of minds perfectly. Kepler, astrologer and astronomer, mystic and painstaking observer, was a child of a similar place and time. The dangers of irrationalism have been trumpeted in our ears for two centuries already, but there are perhaps equally grave dangers lurking in rationalism. 'They are most rational and most insane,' wrote the Victorian poet Thompson about his 'progressive' co-citizens. And Hardy of all people wrote of one of his contemporaries 'Bromwich would have had a happier life and been a greater mathematician if his mind had worked with less precision.'

Much has been said about Ramanujan's lack of adequate mathematical training. But I am more of the opinion that it was, on the contrary, very suitable—for him. He was given about as much as he needed to get going, namely groundings in most areas, including calculus (still little taught in schools). He didn't make it to university but he did get to know several eminent Indian savants and even his immediate superior in his office was an excellent mathematician. So Ramanujan had people he could talk to about mathematics, and it was in many respects an advantage that they did not know more than they did—for they would have put him off following down certain pathways.

Mathematics has, in the last two centuries, become a matter of solving and proving the problems and theorems bequeathed by the previous generation. It has become grimly serious and has largely ceased to be the carefree exploration of virgin territory that it was in the time of Fermat and Euler. Ramanujan was not a prover nor even especially a problem solver—problems are selected by other people—he was an explorer. In his youth, after giving up the idea of getting into college, he spent five largely happy years supported by his poor parents doing nothing much except sitting on a wooden bench in the sun in front of his house working at mathematics, *his* choice of mathematics. After his excursion into Europe he returned to this mode of life in his last years, exploring peculiar things of his invention he called 'mock theta functions'. The best thing to do with such a person is to let him get on with it and have someone check up on his results from time to time.

In this era of 'education, education, education' it is worth pointing out that, though lack of knowledge renders people impotent, too much knowledge available at the drop of a hat makes them lazy, blasé and unimaginative. It is indeed sometimes salutary to be deprived of knowledge. if Pascal's father had not forbidden him to study geometry, he would not have got off to such a good start by rediscovering whole chunks of Euclid unaided. Ramanujan kicked off with an out-of-date pot-boiler, Carr's *Synopsis* (**), which is apparently all formulae and no proof. The author was an enthusiast for his subject, however, and managed to communicate this to his readers. According to Kanigel, the book has a certain flow and movement. (I'd like to read it myself and am pretty sure I'd get a lot out of it.)

Now you can't teach 'exploration' but it can be encouraged. In mathematics it practically never is. The last thing a born explorer wants to hear is the worldly wisdom of the (apparently historical) director of the American Patent Office who recommended that the Office be closed since 'everything worth discovering has already been discovered'. But this is not that far from the message you get from the pure mathematical fraternity. Probably the most exciting mathematical event in the last twenty-five years has been the discovery of fractals. But they were turned up by an explorer of mathematics in the employ of IBM, almost an amateur—I gather that even today the pure mathematical fraternity does not accept Mandelbrot as being part of the club. And it all came out of looking into a simple function known for three centuries.

On the basis of Kanigel's book, I don't think I am able to subscribe to the conventional wisdom that 'if only Ramanujan had had the proper training what a great mathematician he would have been!' It is an open question whether even Hardy had, in the last analysis, a good or a bad influence on him since Hardy tried, with little success, to inject 'rigour' into his methods. As far as I can make out from Kanigel's book, Ramanujan ceased strictly original work on arriving at Cambridge and only took it up again after his return to India. It seems clear that very different abilities are required for the explorative mathematician and the deductive one and it is going against the grain to expect one person to have both at once. In the 18th century the algorists maybe had too much of a field day, but the pendulum has swung alarmingly to the opposite extreme.

There is something delightfully swashbuckling about Ramanujan as a mathematician—as Hardy himself admitted, even his very failures are glorious. In a sense, who cares whether he was right or wrong? The game's the thing. Although Ramanujan's claim that he had a function giving the distribution of the primes fails for very large numbers, it is for all that a staggering achievement. 'Of the first nine million numbers, 602,489 are prime. Ramanujan's formula gave a figure off by just 53—closer than the canonical version of the prime number theorem.' (Kanigel, op. cit.) This really is David against Goliath: on the one hand a hundred or more years of research from the cream of the West's pure mathematicians with all the data available and on the other a man sitting on a wooden bench with a slate and a piece of chalk. If he'd done nothing else, the man deserves a name in the history books—and this was one of his errors!

(*) The title of Kanigel's book, *The Man Who Knew Infinity* is a misnomer and would be more applicable to a biography of Cantor. To my knowledge Ramanujan never showed any interest in transfinite ordinals and, when he came to England, does not seem to have even heard of set theory. 'The Man who Knew and Loved Ordinary Numbers' would have been a more suitable, but less eye-catching, title for a biography of Ramanujan.

(**) G. S. Carr, *A Synopsis of Elementary Results [in] Pure Mathematics: Containing Propositions, Formulæ, and Methods of Analysis, with Abridged Demonstrations. Supplemented by an Index to the Papers on Pure Mathematics which are to be found in the principal Journals and Transactions of learned Societies, both English and Foreign, of the present century*, Cambridge 1856.

Here some more formulae taken from Ramanujan's 'Dear Sir, I beg to introduce myself' letter to Hardy. You are welcome to have a go at verifying them.

$$\int_0^\infty \frac{1+(\frac{x}{b+1})^2}{1+(\frac{x}{a})^2}\cdot\frac{1+(\frac{x}{b+2})^2}{1+(\frac{x}{a+1})^2}\cdots dx \;=\; \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(a+\frac{1}{2})\Gamma(b+1)\Gamma(b-a+\frac{1}{2})}{\Gamma(a)\Gamma(b+\frac{1}{2})\Gamma(b-a+1)}.$$

$$\frac{1}{1+}\;\frac{e^{-2\pi}}{1+}\;\frac{e^{-4\pi}}{1+\ldots} \;=\; \left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2}\right)e^{2\pi/5}.$$

$$\frac{1}{1+}\,\frac{e^{-2\pi\sqrt{5}}}{1+}\,\frac{e^{-4\pi\sqrt{5}}}{1+\ldots} \;=\; \left(\frac{\sqrt{5}}{1+\sqrt[5]{5^{3/4}\left(\frac{\sqrt{5}-1}{2}\right)^{5/2}-1}}-\frac{\sqrt{5}+1}{2}\right)e^{2\pi/\sqrt{5}}.$$

If $\alpha\beta=\pi^2$, then

$$\alpha^{-1/4}\left(1+4\alpha\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}dx\right) \;=\; \beta^{-1/4}\left(1+4\beta\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}dx\right).$$

If $u=\dfrac{x}{1+}\dfrac{x^5}{1+}\dfrac{x^{10}}{1+}\dfrac{x^{15}}{1+\ldots}$ and $v=\dfrac{x^{1/2}}{1+}\dfrac{x}{1+}\dfrac{x^2}{1+}\dfrac{x^3}{1+\ldots}$, then

$$v^5 \;=\; u\,\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}.$$

If $F(k) = 1 + \left(\frac{1}{2}\right)^2 k + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^2 + \ldots$ and $F(1-k) = \sqrt{210}\,F(k)$, then

$$\begin{aligned} k &= \left(\sqrt{2}-1\right)^4 \left(2-\sqrt{3}\right)^2 \left(8-3\sqrt{7}\right)^2 \left(\sqrt{7}-\sqrt{6}\right)^4 \left(\sqrt{10}-3\right)^4 \\ &\quad \times \left(4-\sqrt{15}\right)^4 \left(\sqrt{15}-\sqrt{14}\right)^2 \left(6-\sqrt{35}\right)^2. \end{aligned}$$

$$\frac{1^{13}}{e^{2\pi}-1} + \frac{2^{13}}{e^{4\pi}-1} + \frac{3^{13}}{e^{6\pi}-1} + \ldots \;=\; \frac{1}{24}.$$

$$\frac{\coth\pi}{1^7} + \frac{\coth 2\pi}{2^7} + \frac{\coth 3\pi}{3^7} + \ldots \;=\; \frac{19\pi^7}{56700}.$$

$$\begin{aligned} &\frac{1}{1^3}\cdot\frac{1}{2^1} + \frac{1}{2^3}\cdot\frac{1}{2^2} + \frac{1}{3^3}\cdot\frac{1}{2^3} + \frac{1}{4^3}\cdot\frac{1}{2^4} + \ldots \\ &\quad = \frac{1}{6}(\log 2)^3 - \frac{\pi^2}{12}\log 2 + \frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \ldots. \end{aligned}$$

$$\frac{2}{3}\int_0^1 \frac{\tan^{-1} x}{x}dx - \int_0^{2-\sqrt{3}} \frac{\tan^{-1} x}{x} \;=\; \frac{\pi}{12}\log(2+\sqrt{3}).$$

$$\begin{aligned} &\frac{\log 1}{\sqrt{1}} - \frac{\log 3}{\sqrt{3}} + \frac{\log 5}{\sqrt{5}} - \frac{\log 7}{\sqrt{7}} + \ldots \\ &\quad = \left(\frac{1}{4}\pi - \frac{1}{2}\gamma - \frac{1}{2}\log 2\pi\right)\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}} + \ldots\right), \end{aligned}$$

where $\gamma \approx 0.5772156649$ is Euler's constant.

$$\frac{4}{x+}\;\frac{1^2}{2x+}\;\frac{3^2}{2x+}\;\frac{5^2}{2x+}\;\frac{7^2}{2x+\ldots} \;=\; \left(\frac{\Gamma\left(\frac{x+1}{4}\right)}{\Gamma\left(\frac{x+3}{4}\right)}\right)^2.$$

Also you might like to speculate as to what practical use, if any, can be made of the following:

$$1+2+3+4+5+6+\ldots \;=\; -\tfrac{1}{12},$$

$$1^3+2^3+3^3+4^3+5^3+6^3+\ldots \;=\; -\tfrac{1}{120}$$

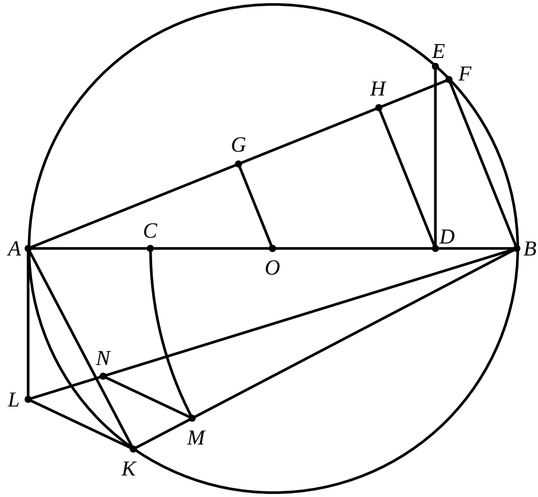
and

$$1-2+3-4+5-6+\ldots \;=\; \tfrac{1}{4}.$$

Problem 202.1 – Squaring the circle

S. Ramanujan

In the diagram on the right, $|OA| = |OB| = 1$, $|OC| = \frac{1}{2}$, $|OD| = \frac{2}{3}$ and $|BM| = |BC| = \frac{3}{2}$; DE is perpendicular to AB and $|BF| = |DE|$; OG, DH and BF are parallel; AL is perpendicular to AB, $|AL| = |GH|$, $|AK| = |AG|$ and MN is parallel to KL.



Construct the diagram using ruler and compasses only.

What is $|BN|$?

Problem 202.2 – Five spheres

ADF

Four spheres of radius a are arranged so that their centres are at the vertices of a regular tetrahedron. Each sphere touches the other three. A fifth sphere of radius 1 is in the middle of the structure and it touches each of the other four spheres. What is a?

Can you generalize to n spheres of equal radius surrounding a sphere of radius 1?

Problem 202.3 – The puzzled hotelier

Ian Bruce Adamson

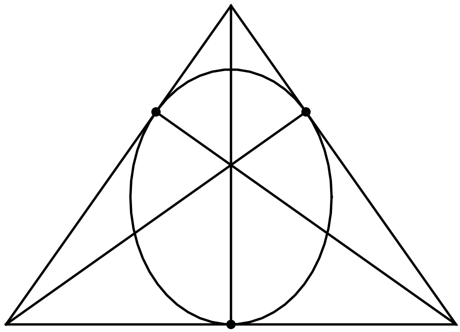
An hotelier told me that the rooms on the first floor had consecutive 3-digit numbers, beginning with 1, starting at 101. “They were,” he said, “off four corridors forming a square and ordered so that the sums of pairs of numbers of adjacent rooms were all primes.”

He told me how many rooms there were (on the first floor) and I countered, “There couldn’t have been fewer!” What number did he say?

Solution 199.6 – Inscribed ellipse

(i) Take any triangle and mark a point on each side. When is it possible to draw an ellipse that is tangent to the three sides at the marked points?

(ii) What is the formula for the ellipse when the points are at the bases of the altitudes of a triangle with vertices $(\pm 1/2, -\sqrt{2}/4)$ and $(0, \sqrt{2}/4)$?



Dick Boardman

The condition that allows an ellipse to touch three points on the sides of a triangle is that the lines joining the three points to the opposite vertices must pass through a single point.

To prove this we use Brianchon's theorem, which states: *If a hexagon is circumscribed about an ellipse, the lines joining opposite vertices pass through a single point.*

Let the given triangle have vertices A, B, C and let the given points be P on AB, Q on AC and R on BC. Draw a hexagon with vertices A, P', B, R', C and Q' which can be circumscribed about an ellipse. Then Brianchon's theorem states that the lines $R'A$, $Q'B$ and $P'C$ meet at a single point; call it O.

As we move P' towards P, the points of contact with the ellipse of the lines AP' and BP' move closer together until they merge at P. Similarly, moving Q' towards Q moves the points of contact with the ellipse of AQ' and CQ' closer until they merge at Q. Brianchon's theorem states that R' is on the line AO, where O is now the intersection of QB and PC. If R is on AO then moving R' to R completes the theorem but if R is not on AO there is no possible ellipse which is tangential at P, Q and R

Now to derive the equation of the ellipse in the picture. First move the origin to the base of the triangle, so that the vertices of the triangle are $(-\frac{1}{2}, 0)$, $(0, 1/\sqrt{2})$ and $(\frac{1}{2}, 0)$.

The gradient of the left-hand side is $\sqrt{2}$. The gradient of the normal to this side is therefore $-1/\sqrt{2}$ and the equation of the left-hand side is

$y = \sqrt{2}(x + \frac{1}{2})$. The equation of the altitude which is normal to the left-hand side is $y = (\frac{1}{2} - x)/\sqrt{2}$. The point where these two lines meet, which is the base of the altitude, is $(-\frac{1}{6}, \frac{1}{3}\sqrt{2})$. Hence the three points that the ellipse must pass through are $(0,0)$, $(-\frac{1}{6}, \frac{1}{3}\sqrt{2})$ and $(\frac{1}{6}, \frac{1}{3}\sqrt{2})$.

Consider the general conic with equation

$$x^2 + 2hxy + by^2 + 2gx + 2fy + c \;=\; 0.$$

The conic goes through $(0,0)$; hence $c = 0$. It is symmetrical about the y-axis; hence $h = g = 0$. So we are left with $x^2 + by^2 + 2fy = 0$. Substituting $x = -\frac{1}{6}$ and $y = \sqrt{2}/3$ gives

$$\frac{1}{36} + \frac{2b}{9} + \frac{2\sqrt{2}f}{3} \;=\; 0. \tag{1}$$

The gradient of the tangent at $(-\frac{1}{6}, \sqrt{2}/3)$ is found by differentiating the equation of the conic and solving for dy/dx. It must be $\sqrt{2}$. Thus

$$\frac{1}{2(\sqrt{2}b + 3f)} \;=\; \sqrt{2}. \tag{2}$$

Solving (1) and (2) yields $b = 5/8$ and $f = -\sqrt{2}/8$.

Interest

JRH — Creative interpretation of compound interest from an investment company who are trying to persuade me to buy into their scheme:

> 'Our portfolio rose in value more than 800% over a 20-year period. Almost 40% per year average!'

Call me pedantic, but I reckon that's nearer to 11% per year, which is about what you'd expect. Over 20 years, 40% per year would give you more than 83,000%.

ADF — It seems that I, too, am not above this kind of deception. Here is an extract from a letter I wrote to a credit card company.

> Dear Sirs,
>
> Having just received my statement for October, I am astonished to see a demand for £5.62 interest. I am reluctant to pay because . . . the interest rate is excessive: £5.62 on £374.50 for the period midnight 30th September to 9.30 am 1st October works out at an APR of over 92,000,000 per cent.

Long live Geometry

30° revisited

Rob Evans

Readers of this magazine may remember an interesting geometry problem whose statement first appeared in M500 **189**. That problem was to show that for an arbitrary fixed triangle ABC and an arbitrary fixed point P inside that triangle it is always the case that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ does not exceed 30°. (See Figure 1.) In developing a geometrical solution to this problem it turns out that for the above triangle it is convenient to refer to its positive and negative *Brocard points*.

Let Ω^+ and Ω^- be positive and negative Brocard points of the triangle ABC. Then, by definition, Ω^+ and Ω^- lie inside that triangle. Moreover, by definition, Ω^+ together with its corresponding angle ω^+ satisfy

$$\angle\Omega^+AB \;=\; \angle\Omega^+BC \;=\; \angle\Omega^+CA \;=\; \omega^+.$$

(See Figure 2.) Analogously, by definition, Ω^- together with its corresponding angle ω^- satisfy

$$\angle\Omega^-BA \;=\; \angle\Omega^-CB \;=\; \angle\Omega^-AC \;=\; \omega^-.$$

(See Figure 3.) Note that the existence of Brocard points and their corresponding angles is something which in this article we are going to take on trust. However, I intend soon to write another article that will deal with this important question. On the assumption that for the triangle ABC we have the existence of at least one such +Brocard point/angle pair Ω^+/ω^+ and of at least one such −Brocard point/angle pair Ω^-/ω^-, it is a relatively straightforward matter to prove, from the results of elementary trigonmetry, the following beautiful equation:

$$\cot\omega^\pm \;=\; \cot\angle CAB + \cot\angle ABC + \cot\angle BCA. \qquad (1)$$

Since the cotangent function is one-to-one on the interval $(0,\pi)$, we have as an immediate corollary of this equation that each of ω^+ and ω^- and, consequently, each of Ω^+ and Ω^- is uniquely determined where, moreover, we have that $\omega^+ = \omega^-$. Hence, with regard to the triangle ABC, we are able to refer to its unique positive and negative Brocard points, Ω^+ and Ω^-, respectively, and its unique Brocard angle, $\omega = \omega^+ = \omega^-$.

Adopting the convention whereby the angles $\angle CAB$, $\angle ABC$, $\angle BCA$ are denoted by the symbols A, B, C respectively allows us to rewrite equation

(1) in a more concise form as follows:

$$\cot \omega \ = \ \cot A + \cot B + \cot C.$$

Just how relevant these definitions are to developing a geometrical solution to the original problem was indicated by the solution which appeared in M500 **191**. There the author demonstrated that this problem could be reduced to the new problem of proving that the Brocard angle of a triangle never exceeds 30°. He was able to do this by exploiting the obvious fact (as readers can easily confirm for themselves) that for an arbitrary fixed triangle ABC and an arbitrary fixed point P inside that triangle it is always the case that at least one of the following three inequalities hold. (See Figure 4.)

$$\angle PAB \leq \angle \Omega^{+}AB, \quad \angle PBC \leq \angle \Omega^{+}BC, \quad \angle PCA \leq \angle \Omega^{+}CA.$$

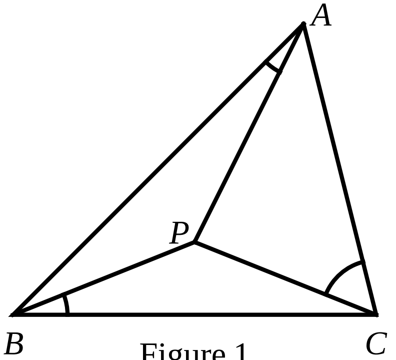


Figure 1

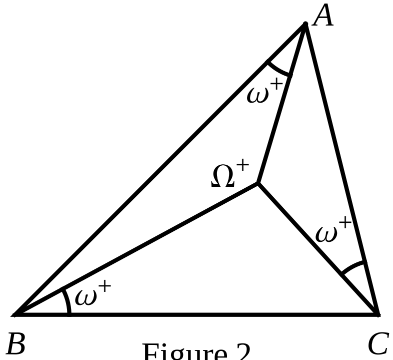


Figure 2

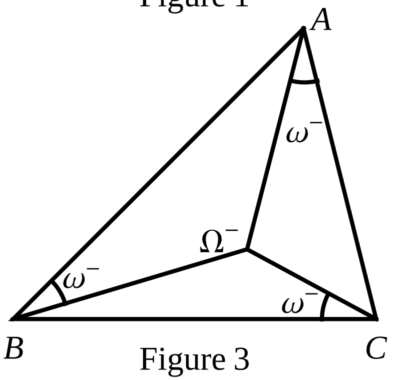


Figure 3

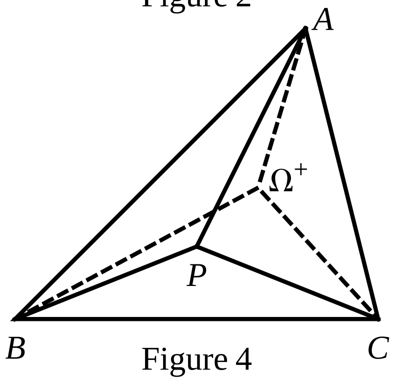


Figure 4

In M500 **195** I tackled this new problem concerning the maximum Brocard angle of a triangle and found my own calculus-based solution. At the time this was the most elementary solution I could come up with. The use of calculus seemed to be unavoidable. However, after much thought and research I am now able to give a proof based on nothing more than a bit of trigonometry and a bit of algebra!

We start with the observation that, by definition, the Brocard points of the triangle ABC lie inside that triangle. Consequently we must have that $0 < \omega < A, B, C$ and hence that $0 < \omega < \min\{A, B, C\}$. But, from simple considerations of arithmetic, $\min\{A, B, C\} \leq \pi/3$. Thus, we must have that $0 < \omega \leq \pi/3$. This (intermediate) result can be exploited to great effect as follows.

The cotangent function is decreasing on the interval $(0, \pi/3]$. Moreover, since $\cot \pi/3 = 1/\sqrt{3} > 0$, on the interval $(0, \pi/3]$ that function is positive. Also the square function is increasing on the interval $[\cot \pi/3, \cot 0)$. Hence the composite function from $(0, \pi)$ into $\mathbb{R}$ which maps x to $\cot^2 x$ is decreasing on the interval $(0, \pi/3]$. But, from our previous (intermediate) result, we know that ω is in that interval. Thus, the inequality $\omega \leq \pi/6$ is equivalent to the inequality $\cot^2 \omega \geq \cot^2 \pi/6 = 3$. So, as a result of this logical equivalence, we know that in order to show that $\omega \leq \pi/6$ it is sufficient to show that $\cot^2 \omega \geq 3$. But, from equation (1) and elementary algebra, we have the following equation:

$$\begin{aligned}\cot^2 \omega &= \cot^2 A + \cot^2 B + \cot^2 C \\ &\quad + 2\,(\cot A \cot B + \cot B \cot C + \cot C \cot A).\end{aligned}$$

Hence, to prove that $\cot^2 \omega \geq 3$ it is sufficient to prove the following two results:

$$\cot^2 A + \cot^2 B + \cot^2 C \;\geq\; 1 \tag{2}$$

and

$$\cot A \cot B + \cot B \cot C + \cot C \cot A \;=\; 1. \tag{3}$$

To prove equation (3) we start from a familiar trigonometric identity, $\tan(A + B) = (\tan A + \tan B)/(1 - \tan A \tan B)$, from which we can easily obtain $\cot(A + B) = (\cot A \cot B - 1)/(\cot A + \cot B)$. But, since $A + B + C = \pi$, it is clear from the definition of the cotangent function that $\cot(A + B) = -\cot C$.

Hence we have $-\cot C = (\cot A \cot B - 1)/(\cot A + \cot B)$ provided that $\cot A + \cot B \neq 0$, from which we can easily obtain (3).

To prove inequality (2), note that from elementary algebra we have

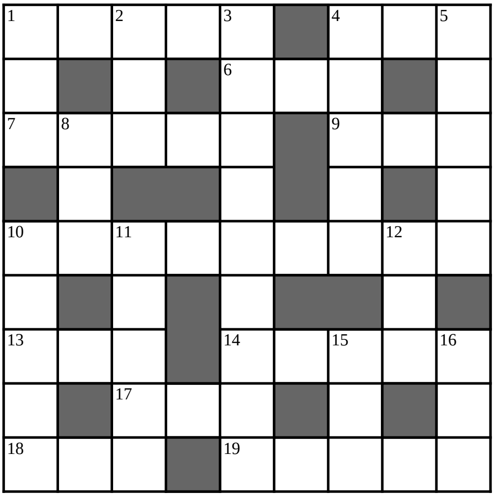
$$\cot^2 A + \cot^2 B + \cot^2 C - (\cot A \cot B + \cot B \cot C + \cot C \cot A)$$
$$= \tfrac{1}{2}\big((\cot A - \cot B)^2 + (\cot B - \cot C)^2 + (\cot C - \cot A)^2\big) \;\geq\; 0.$$

Hence, from result (3), we obtain (2). Q.E.D.

Hence, from the line of argument leading up to the first statement of results (2) and (3), we obtain $\omega \leq \pi/6$; i.e. the Brocard angle of a triangle never exceeds 30°. Q.E.D.

Crossnumber

Tony Forbes



Across

1. $(13 \text{ across})^2$
4. $\sqrt[3]{10 \text{ across}}$
6. $3\sqrt{5 \text{ down}}$
7. $(8 \text{ down})^2$
9. $(\sqrt{16 \text{ down}} - 2)^2$
10. $(4 \text{ across})^3$
13. $\sqrt{1 \text{ across}}$
14. $(1 \text{ down} - 2)^2$
17. $\sqrt{4 \text{ down}}$
18. $4 \text{ across} - 2\,(13 \text{ across})$
19. $(13 \text{ across} - 10)^2$

Down

1. $\sqrt{11 \text{ down}} - 3\sqrt{16 \text{ down}}$
2. $\sqrt{11 \text{ down}} + 6$
3. $(12 \text{ down})^3$
4. $(17 \text{ across})^2$
5. $(6 \text{ across})^2/9$
8. $\sqrt{7 \text{ across}}$
10. $(1 \text{ down} - 16)^2$
11. $(2 \text{ down} - 6)^2$
12. $\sqrt[3]{3 \text{ down}}$
15. $\sqrt{16 \text{ down}} + (\sqrt{9 \text{ across}} - 2)^2$
16. $(\sqrt{4 \text{ across}} - 2)^2$

An application of statistics to family history

Colin Davies

James Davies was a shoemaker, baptized in 1797 at St Mary's Church in Rotherhithe according to church records. His wife was Hannah Leney, born 1798 in New Cross according to the International Genealogical Index and the 1851 census. They married at St Mary's Church in Rotherhithe on 14 July 1817, and they were my great great great grand parents via their son James born 1826. I asked the IGI computer to list all the children of James and Hannah Davies. It came up with a list of twelve, so I got photocopies of the baptismal records or birth certificates of those twelve. These all described James as a shoemaker, and gave the following addresses.

	Baptized	Address of parents
Samuel	Stepney, Jan 1819, St Dunstans	Poplar
Maria	Deptford, May 1820, St Nicholas	Deptford, Pine 13
Elizabeth	Stepney, Dec 1821, St Dunstans	Mile End Old Town
Hannah	Deptford, Jul 1823, St Nicholas	Deptford Green 27

After that, the computer lists the following, all born in Deptford and baptized at St Nicholas: George, James, John, Ann, Emma, William, Susan and George.

This gives the impression that James and Hannah Davies were living in Poplar when their first child was born, Deptford for the second, Mile End Old Town for the third, and Deptford for all the rest. But are James Davies the shoemaker and his wife Hannah of Poplar and Mile End Old Town the same people as James Davies the shoemaker and his wife Hannah of Deptford? How many shoemakers called James Davies would have wives called Hannah? This is the only James and Hannah Davies couple listed by the IGI anywhere in London. That seems a big coincidence if they are not the same people. The distance from Poplar or Mile End to Deptford is not far, but the river has to be crossed.

According to R. L. Vickers writing in *Family Tree Magazine* in 1993, it is estimated that for every shoemaker appearing in a trades directory during the 19th century, there were five or ten others who subcontracted for them, but were not in the directory. Holden's 1811 directory for London lists seven shoemakers called Davies, so it seems there were between 35 and 70 Davies shoemakers in London altogether. The name JAMES occurs 166 times in about 3000 men's names taken from the 15 seemingly most popular surnames in the 1811 directory; that is 1 in 18. If there were 70

Davies shoemakers in London, four were probably called JAMES; if 35, two were called JAMES. Of these, my great great great grandfather lived in Deptford, so there were between three and one other shoemaker(s) called JAMES Davies in London.

The 1811 directory lists no HANNAHs at all in 124 women's names taken from the previous 15 popular surnames, but the 1846 directory lists 9 HANNAHs in 245 women's names; that is 1 in 27. Suppose we use the ratio 1 in 30 for HANNAH in 1811. The probability of one of the other shoemakers called JAMES Davies not marrying a HANNAH is $29/30 = 0.97$, so, if there are three JAMESs, the probability of all three of them not doing so is $0.97^3 = 0.90$. But there may only be one other JAMES, so the probability of the Poplar/Mile End couple being the same as the Deptford couple is between 90 per cent and 97 per cent.

In other words, although I could not prove conclusively that James and Hannah north of the river were the same people as James and Hannah south of the river, I could regard it as highly probable that they were the same.

A criticism made to me of the above is that it is only speculation, and does not prove anything. I answer that by saying that it is useful to know how likely one's speculation is to be correct. A probability of more than 0.9 is encouraging. Bear in mind that there must always be some uncertainty about any family. I heard a West Indian song about a boy trying to do some family history research. The song ended with his mother singing the line:

Your daddy ain't your daddy but your daddy don't know.

This article, being unconventional, has spent some months under editorial consideration. Meanwhile I found a document that had long eluded me: the entry in the 1841 census showing Hannah and James with their children from both sides of the Thames, all living together in Effingham Place.

So they *were* all one family, as my arithmetic strongly suggested!

From my habit of reading old newspapers I found this in *The Times* about three years ago.

> What is significant about the dates August 28th, 888 and February 2nd, 2000?

[**Eddie Kent**]

Existentialism: a philosophy based on the equation $x = 10$.

[**Jeremy Humphries**]

How to solve cubics

Tony Forbes

Going through past issues of M500 I notice that regularly and often we have occasion to find the roots of third degree polynomials. We always just quote the solution, referring to the literature or to computer software for the method of getting there. I find this unsatisfactory. And it is regrettable that cubics are no longer actively solved in schools. So I thought it would be a good idea if M500 were to fill a possible gap in your mathematical education by presenting a short exposition. Surely it can't be that difficult.

The problem is simple to state, 'Solve

$$x^3 + a\,x^2 + b\,x + c = 0 \tag{1}$$

in terms of the coefficients a, b and c.'

When I tried solving (1) I had to give up. It was too difficult! However, after studying solutions produced by MATHEMATICA I was able to invent three expressions which look as if they might be solutions of a cubic, not necessarily (1), but sufficiently like it to be encouraging. We start at the end and work backwards. Let

$$\alpha = 1 + u + v, \qquad \beta = 1 + \rho\,u + \bar{\rho}\,v, \qquad \gamma = 1 + \bar{\rho}\,u + \rho\,v,$$

where

$$u = \sqrt[3]{f + \sqrt{f^2 - g^3}}, \qquad v = \sqrt[3]{f - \sqrt{f^2 - g^3}},$$

$$\rho = \frac{-1 + \sqrt{3}\,i}{2} \qquad \text{and} \qquad \bar{\rho} = \frac{-1 - \sqrt{3}\,i}{2}.$$

Instead of trying to solve (1), we ask ourselves, 'what cubic polynomial has roots α, β and γ?' Clearly the answer is $(X - \alpha)(X - \beta)(X - \gamma)$, which when multiplied out becomes

$$X^3 - (\alpha + \beta + \gamma)X^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)X - \alpha\beta\gamma.$$

Making good use of the equalities $1 + \rho + \bar{\rho} = 0$, $\rho\bar{\rho} = 1$, $\rho^2 = \bar{\rho}$, $\bar{\rho}^2 = \rho$, $uv = g$ and $u^3 + v^3 = 2f$, we can simplify these coefficients considerably:

$$\alpha + \beta + \gamma = 3, \quad \alpha\beta + \alpha\gamma + \beta\gamma = 3 - 3g, \quad \alpha\beta\gamma = 1 + 2f - 3g.$$

Note, by the way, that 1, ρ and $\bar{\rho}$ are the three cube roots of 1.

Thus α, β and γ are the roots of

$$X^3 - 3X^2 + (3 - 3g)X - (1 + 2f - 3g) \;=\; 0, \tag{2}$$

not quite the same as (1) but that doesn't matter. Substitute $X = -3x/a$. Then $\alpha' = -a\alpha/3$, $\beta' = -a\beta/3$ and $\gamma' = -a\gamma/3$ are the roots of

$$x^3 + ax^2 + \frac{(1-g)\,a^2}{3}\,x + \frac{(1+2f-3g)\,a^3}{27},$$

which, you will agree, looks slightly more like (1). In fact we can make it look exactly like (1) if we choose f and g appropriately,

$$f = 1 - \frac{9b}{2a^2} + \frac{27c}{2a^3}, \quad g = 1 - \frac{3b}{a^2}.$$

Now that these parameters have been determined, α', β' and γ' are defined in terms of the coefficients of (1) and the problem is completely solved.

Well, nearly. If $a = 0$ in (1), the above method won't work. But only a minor adjustment is required. We shift the roots by 1 to the left. That is, we set

$$\alpha_0 = u + v, \quad \beta_0 = \rho\, u + \bar\rho\, v, \quad \gamma_0 = \bar\rho\, u + \rho\, v,$$

so that

$$\alpha_0 + \beta_0 + \gamma_0 = 0, \quad \alpha_0\beta_0 + \alpha_0\gamma_0 + \beta_0\gamma_0 = -3g, \quad \alpha_0\beta_0\gamma_0 = 2f.$$

Therefore α_0, β_0 and γ_0 are the roots of

$$x^3 - 3gx - 2f \;=\; 0,$$

and to complete the solution we put $f = -c/2$ and $g = -b/3$.

The quantity $\Delta = -\frac{4}{27}\,a^6\,(f^2 - g^3)$, involving the thing being square-rooted in the definition of u and v, is usually known as the *discriminant*—it is non-zero if and only if the cubic has three distinct roots. If you have nothing better to do with a pencil and a large sheet of paper, you might like to have a go at proving that

$$\Delta \;=\; (\alpha' - \beta')^2(\alpha' - \gamma')^2(\beta' - \gamma')^2.$$

The formula in terms of the coefficients of (1) is

$$\Delta \;=\; -\,4\,c\,a^3 + b^2\,a^2 + 18\,b\,c\,a - 4\,b^3 - 27\,c^2.$$

For instance, if $a = 2d$, $b = d^2$ and $c = 0$ then $\Delta = 0$, which is reasonable because the equation $x^3 + 2dx^2 + d^2x = 0$ does indeed have a multiple root at $x = -d$. The left-hand side factorizes as $x\,(x+d)^2$.

If $a = 0$, the formula simplifies to

$$\Delta \;=\; -\,4b^3 - 27c^2.$$

Letters to the Editor

Heresy and surnames

Tony,

My computer is not working, and in desperation I have bought a second-hand laptop. I am now in a position to reply to Dick Boardman [A little heresy, M500 **198** 22].

I fully agree with Dick that I am assuming good mixing. I did ask for suggestions for better mathematical models of populations. The difficulties caused by bad mixing are what I had in mind when asking for suggestions. I agree that choice of marriage partners tends to be selected within classes of people, and I am prepared to believe that Dick has no legitimate connection with William the Conqueror. But I never said anything about legitimacy. I am talking about who one's ancestors actually were, not who they ought to have been because of marriage. The English aristocracy certainly left plenty of illegitimate descendants. At home they did things 'because they could' (to quote a modern American aristocrat), and they also had the money to go travelling. So while they were leading armies of virile young men off to hammer the Scots, or to argue about roses or religion or who the king ought to be, they all had plenty of opportunity to spread their paternity far and wide. Isaac Newton was rather a strange man. I think it quite likely that he did not father any children, though I see no way of proving that.

I also agree that a lot of nonsense involving statistics is given in court as 'expert evidence'. I did not understand what the cot death expert meant by the probability of two cot deaths being (say) one in a hundred million. Does that mean deaths per mother? Or deaths per year? Or per person? Or per family? Or what? With 60 million people in the country, I think it quite likely that there would be two cot deaths somewhere in the country over a period of ten years or so.

I would be surprised if anybody I knew won the lottery, but I am never surprised to hear that somebody has won it.

Re: Problem 198.2 – Two students. [Take a random group of 23 or more people, and the odds are better than evens that two of them share a birthday. In a tutorial group of about this size, two students found that they shared the same first name and family name. About how likely is this?]

In 1955 I worked in a sawmill in Mesachie Lake, BC, Canada. The names of all the hourly rate employees were posted above the time clocks. About one third of the employees were named Singh, and they had a separate time clock for themselves. So around the villages near Mesachie Lake I expect it was very common to share first and last names.

For Britain without Scotland, I have a table of surname frequencies.

England as a whole	
Smith	1.37 %
Taylor	0.68 %
Brown	0.57 %
Jones	0.43 %
Johnson	0.48 %
Robinson	0.36 %
Wilson	0.36 %
Wright	0.34 %
Wood	0.33 %
Hall	0.33 %

Wales as a whole	
Jones	13.8 %
Williams	8.91 %[1]
Davies	7.09 %[2]
Thomas	5.70 %
Evans	5.46 %
Roberts	3.69 %
Hughes	2.98 %
Lewis	2.97 %
Morgan	2.63 %
Griffiths	2.58 %

[1] Up to 23% in Caernarfonshire and Anglesey
[2] 22% in South Cardiganshire

Source: *Second Stages in Researching Welsh Ancestry*, Ed. John and Sheila Rowlands (ISBN 186006 066 8). Perhaps someone can calculate probabilities of name sharing from that data.

Colin Davies

Quadratic equations

Dear Tony,

In M500 **199** (page 14), David Wild quoted a radio contributor saying 'before the introduction of complex numbers only half the quadratic equations could be solved.' This brought to mind a problem published by Frederick Mosteller in *Fifty Challenging Probability Problems* (Dover 1965): What is the probability that a random quadratic equation $x^2 + 2bx + c = 0$ has real roots?

Avoiding the philosophical issues, if we assume that the coefficients b and c are taken from finite uniform distributions, it turns out that the probability of real roots tends to 1 (that is, certainty) as the sample space becomes infinite.

Mosteller points out that taking the quadratic as $ax^2 + 2bx + c = 0$ is not the same problem because each of three coefficients is then sampled independently from uniform distributions; that is, one cannot simply divide through by a. He does not offer a solution to this form of the problem; so, as yet, we do not know the outcome.

John Bull

Conversion factors

Dear Tony,

In the article on page 23 of M500 **200** entitled 'Conversion factors', the author states that what he dubs 'crackpot numerology' seems to have been the inspiration for Michael Moore's recent film, *Fahrenheit 911*. This is not the case. Moore's film took its title from Ray Bradbury's celebrated novel *Fahrenheit 451*, written in 1953 (and subsequently made into a well-known movie directed by François Truffaut), which concerns a futuristic society demanding of its citizens order and harmony at the expense of individual rights. Books were outlawed in this society, and if any were found they were burned—451 degrees Fahrenheit being the temperature at which books are supposed by the author to ignite. Moore's comment that 911 degrees Fahrenheit was the temperature at which freedom burned explains his adaptation of Bradbury's title to incorporate the 9/11 date of the twin towers attack and his claim that the political aftermath represented an attack on human rights in the US. Bradbury, incidentally, is reported to have been infuriated (incandescent, even?) that Moore should have used his title in this way.

I apologize for offering a comment which some may regard as pedantic and which is in any case totally devoid of any mathematical comment, but it may be salutary to appreciate that, despite the proportion of our time that we spend as OU maths students immersed in the world of numbers, there are just occasionally non-numerological explanations to everyday issues!

Paul Wright

Dear Tony,

I've always found $\sinh 1$ interesting but not particularly useful for computing value added tax. A tenth plus half plus half again usually amazes the salesperson!

Also you could add to your list the following:

$$\text{To convert days to seconds:} \quad \text{multiply by } \frac{10!}{42}.$$

Bit of the *The Hitchhiker's Guide to the Galaxy* here!

Cheers,

Bryan Orman

What's next?

I was interested in Jeremy Humphries's attempt to make the -plets more regular ['What's next', M500 **199**] and I don't see why one shouldn't both smooth this rather messily named series, and extend it—which he has done from 21 in immaculate style conforming to the list of cardinal numbers in *Kennedy's Revised Latin Primer*. Kennedy insists on *octo et nonaginta* for 98 and will not allow *duodecentum*, but that is simply because the word does not appear in Latin literature of the classical period.

Here is a list of extant Latin words ending in *-plex*, ('-fold', as in English 'twofold', 'threefold') based on a complete search of that literature, kindly provided by Mindaugas Strockis, who says, '*Simplex*, *duplex* and *triplex* are used universally (and also *multiplex*, 'manifold'); others appear only in the indicated authors. The words for 11 to 99 are also missing, and most likely no one ever needed them.'

1 *simplex*, 1.5 *sesquiplex* (Cicero), 2 *duplex*, 3 *triplex*, 4 *quadruplex* (Plautus), 5 *quinquiplex* (Martial, Vulgate), 6 *senus* (Seneca Junior, P. Papinius Statius), 7 *septemplex* (Vergil), 8 *octuplex* (Christian Latin, should really be *octoplex*), (9 missing, though *novenarius* is very close in meaning), 10 *decemplex* (Cornelius Nepos, M. Terentius Varro), (11–99 missing), 100 *centuplex* (Plautus).

Not a very satisfactory foundation, and anyway 5, 6 and 7 have been overtaken by words for numbers of babies. But it is nice to know that if Tony Forbes ever finds a group of primes with one and a half members, the elegant word 'sesquiplet' is waiting for him.

Ralph Hancock

Trigonometry

Re: 'More arctangent identities', M500 **201**. To prove the formulae for $\cos(a+b)$ and $\sin(a+b)$ using Euler's formula, $e^{ix} = \cos x + i \sin x$, is a severe case of putting the cart before the horse. Both $\cos a + b$ and $\sin a + b$ are simple geometrical theorems which could easily have been proved by the ancient Greeks. On the other hand, complex numbers and de Moivre's theorem,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta,$$

were not known until Newton's time, and de Moivre's formula was used to prove Euler's formula.

Dick Boardman

An elliptic gardening problem

Dear Tony,

Thank you for the latest issue of M500.

Since you are into ellipses, perhaps you can solve my gardening problem. I made an elliptical shrubbery by the two-sticks-(at the foci)-and-string method. Now I want the ellipse to be a bit bigger, but plants prevent me from putting strings across the existing bed. I'm prepared to settle for the larger ellipse being in the same alignment as the original one, and for the mid-point between the new foci to be the same as the old mid-point.

Best wishes,

Donald Preece

Book received

F. R. Watson

Elementary Mathematics from an Advanced Perspective

Keele University 2004, 121 pages

The book is aimed at teachers and teachers in training. It arises out of courses which have been taught at Keele University, over a period of several years, to intending teachers and those following short in-service courses or working for an MA degree or a diploma. We examine the background to school mathematics, providing a context in which to examine what we teach, and aiming at a deeper understanding than we would expect children to have. A typical example is the statement $- \times - = +$, which generations of pupils have used, with varying degrees of success, despite the rather subtle matters involved. Anyone who has not thought about why this statement should be correct, or exactly why 'you are not allowed to divide by 0', is invited to pause for a moment and consider!

The book is not (directly) concerned with ways of teaching (of negative number, for example) but with the questions '*Why* is it so?', 'What is the status of ...?' For example, $0^0 = 1$, or $x \cdot 0 = 0$; is it a theorem? a convention? or do we say, 'It just *is*.' Instead of doing or performing mathematics, in the way that undergraduates spend (some of) their time, we aim to look *at* mathematics.

The book is available from D. J. Miller, KMEP, Department of Education, Keele University, Staffs ST5 5BG.

Problem 202.4 – Commas and brackets

ADF

In the Zermelo–Fraenkel scheme for constructing the non-negative integers, zero is represented by the empty set, and a number n greater than zero is defined as the set of all non-negative integers less than n. In symbols,

$$0 = \{\}, \qquad n = \{0, 1, 2, \ldots, n-1\}.$$

For example, 3 is $\{\{\},\{\{\}\},\{\{\},\{\{\}\}\}\}$. This is by no means the most compact way of representing numbers—if you don't agree, try writing out 10—but the amazing thing is that it works. You can go on to define the fundamental arithmetical operations, and once you have done that you have the basis of a mechanism for translating any problem in number theory to an equivalent problem in set theory. We won't do that here. Instead we merely ask:

(i) How many commas are there in the expression for n?

(ii) How many brackets?

Problem 202.5 – Interesting equality

ADF

Here is an interesting equality involving rational numbers and a square root:

$$\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right) = \frac{4}{3}\sqrt{\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\left(1-\frac{1}{19^2}\right)}.$$

Are there any others?

Problem 202.6 – Prime sum

ADF

Show that

$$\sum_{p \text{ prime}} \frac{1}{p^2} = 0.45224\,74200\,41065\,49850\,65433\,64832\,24793\,41732\,\ldots.$$

Given time, you could do it by summing the series directly. However, what we are really looking for is a method that can deliver the answer to a hundred decimal places, say, well before the solar system expires.

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