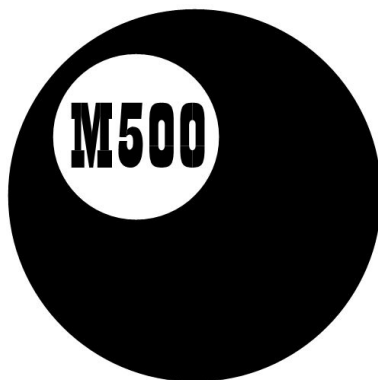
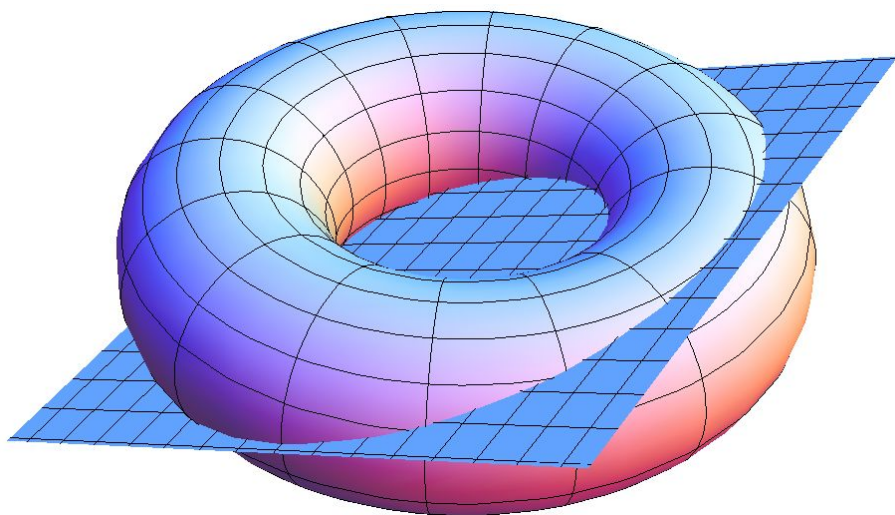


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M500 204



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

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Machin's formula

Bryan Orman

A turning point in the history of the calculation of π occurred with the discovery, by Scotsman James Gregory in 1671, of the arctangent series

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$$

Three years later Gottfried Wilhelm Leibniz independently found the same arctangent series and published it along with an important special case, $x = 1$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

A nice looking result but completely useless when it comes to actually determining a decimal approximation to π . Indeed, 300 terms are required for two decimal place accuracy!

In 1706, Englishman John Machin used the formula

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$

and Gregory's arctangent series to calculate π to 100 digits. Machin's formula is particularly useful since $\arctan 1/5$ has a very simple series and $\arctan 1/239$ converges rapidly.

The mathematical literature has many formulae for $\pi/4$; one of Euler's is

$$\frac{\pi}{4} = 5 \arctan \frac{1}{7} + 2 \arctan \frac{3}{79},$$

which is not as efficient as Machin's formula. Are there any other Machin type formulae? That is, formulae of the kind

$$\frac{\pi}{4} = 4 \arctan \frac{1}{B} + \arctan \frac{1}{C}.$$

One way of examining this formula is to split it into two coupled equations:

$$\frac{\pi}{4} = 2 \arctan \frac{P}{Q} + \arctan \frac{1}{C}, \quad \arctan \frac{P}{Q} = 2 \arctan \frac{1}{B}.$$

Solving the first of these for P/Q gives

$$\frac{P}{Q} = \frac{\sqrt{2(C^2 + 1)} - (C + 1)}{C - 1},$$

and requiring that P and Q be integers, write $\sqrt{2(C^2 + 1)} = 2N$, so that $C^2 - 2N^2 = -1$.

To solve this equation note that the even convergents p_{2k}/q_{2k} of the continued fraction representation of $\sqrt{2}$ satisfy

$$p_{2k}^2 - 2q_{2k}^2 = -1$$

(Burton [1, page 325]). The p_n and q_n are defined by $p_n + q_n\sqrt{2} = (1 + \sqrt{2})(p_{n-1} + q_{n-1}\sqrt{2})$ with $p_0 + q_0\sqrt{2} = 1 + \sqrt{2}$, giving

$$\begin{aligned} p_n &= p_{n-1} + 2q_{n-1}, & p_0 &= q_0 = 1. \\ q_n &= p_{n-1} + q_{n-1}, \end{aligned}$$

Alternatively, $p_n + q_n\sqrt{2} = (1 + \sqrt{2})^{n+1}$. Also note that $p_n^2 + p_{n+1}^2 = 2q_{2n+2}$ and $q_n^2 + q_{n+1}^2 = q_{2n+2}$.

With the identifications $C = p_{2k}$ and $N = q_{2k}$ we have

$$\frac{P}{Q} = \frac{2q_{2k} - p_{2k} - 1}{p_{2k} - 1}.$$

Then

$$\frac{\pi}{4} = 2 \arctan \left(\frac{2q_{2k} - p_{2k} - 1}{p_{2k} - 1} \right) + \arctan \frac{1}{p_{2k}}.$$

Further examination of the properties of p_{2k} and q_{2k} yields four identities:

$$\begin{aligned} 2 \arctan \frac{q_{2k-1}}{q_{2k}} + \arctan \frac{1}{p_{4k}} &= \frac{\pi}{4}, \\ 2 \arctan \frac{p_{2k-2}}{p_{2k-1}} + \arctan \frac{1}{p_{4k-2}} &= \frac{\pi}{4}, \\ 2 \arctan \frac{p_{2k-1}}{p_{2k}} + \arctan \frac{-1}{p_{4k}} &= \frac{\pi}{4}, \\ 2 \arctan \frac{q_{2k-2}}{q_{2k-1}} + \arctan \frac{-1}{p_{4k-2}} &= \frac{\pi}{4}. \end{aligned}$$

Note that the ratio P/Q reduces to the ratio of either consecutive numerators or consecutive denominators of the convergents of $\sqrt{2}$. Indeed, in the limit these identities become

$$2 \arctan(\sqrt{2} - 1) = \frac{\pi}{4}$$

and P/Q is a convergent of $\sqrt{2} - 1$.

We now examine the second of the coupled equations.

Solving for B gives

$$B = \frac{Q + \sqrt{P^2 + Q^2}}{P}$$

and so we need $\sqrt{P^2 + Q^2}$ to be an integer. There are two cases to consider since $P^2 + Q^2$ is either $p_n^2 + p_{n+1}^2$ or $q_n^2 + q_{n+1}^2$, $n = 0, 1, 2, \dots$

Let $p_n^2 + p_{n+1}^2 = u^2$, u an integer. Now p_n is odd for all n since $p_n = p_{n-1} + 2q_{n-1}$ and $p_0 = 1$. So $p_n^2 \equiv 1 \pmod{4}$ for all n , giving $u^2 \equiv 2 \pmod{4}$. But $u^2 \equiv 0$ or $1 \pmod{4}$; hence no solutions in this case.

Let $q_n^2 + q_{n+1}^2 = v^2$, v an integer. Since $q_n^2 + q_{n+1}^2 = q_{2n+2}$ and $p_{2n+2}^2 - 2q_{2n+2}^2 = (-1)^{2n+3}$, we require $p_{2n+2}^2 + 2v^4 = -1$. What are the solutions of $w^2 - 2v^4 = -1$? There are only two: $w = 1, v = 1$ and $w = 239, v = 13$ (Mordel [2, page 271]).

Hence $p_6 = 239$ and $q_6 = 13^2 = 169$ corresponding to $q_2^2 + q_3^2 = 5^2 + 12^2 = 169 = q_6$. And

$$B = \frac{q_3 + \sqrt{q_2^2 + q_3^2}}{q_2} = 5.$$

Machin's formula follows and there are no others of that type.

References

- [1] David M. Burton, *Elementary Number Theory*, Allen and Bacon, 1980.
 - [2] L. J. Mordell, *Diophantine Equations*, Academic Press, 1969.
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A question that came up at this year's M500 Winter Weekend concerned the calculation of the N th digit of π . Do we need to compute all the previous digits as well? Surprisingly, the answer is 'no' if you are prepared to work in binary. In this case

$$\pi \approx 11.00100100001111110110101010001000100001011101000110000100011.$$

The computation is possible by a formula of David Bailey, Peter Borwein and Simon Plouffe:

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right).$$

This also works well for hexadecimal:

$$\pi \approx 3.243f6a8885a308d313198a2e03707344a4093822299f31d0082efa98ec4e.$$

Hailstones

Ron Potkin

This article is concerned with the Collatz problem, sometimes called ‘ $3n+1$ ’ or ‘hailstones’. It has been known about for the last fifty years and has been given a variety of names.

The problem is based on the following very simple sequence.

- (a) Take any positive integer.
- (b) If it is even, divide by 2.
- (c) Stop if you reach 1.
- (d) If it is odd, multiply by 3 and add 1.
- (e) Using the new number, return to step (b).

It is conjectured that the sequence will always terminate at 1. Here are two examples using 13 and 23:

- (i) 13, 40, 20, 10, 5, 16, 8, 4, 2, 1;
- (ii) 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1.

The problem seems very simple but it has proved intractable. For numbers up to 26, the sequences end in a few steps but when you reach the number 27 it becomes more complicated. The sequence fluctuates up and down but eventually reaches 1 after 111 steps. At some point the sequence reaches over 9000.

There is a related function which is identical in all respects except that in step (d), 1 is subtracted instead of being added; i.e. $n \rightarrow 3n - 1$. In this case, some sequences do not end in 1. For example, 5 reduces to 7 and 7 reduces to 5. There is also a second longer loop: 17–25–37–55–41–61–91–17. There may be others.

At first glance, one would be forgiven for thinking that the answer is ‘obvious’, but try to explain why the ‘27’ sequence is so long. Isn’t it possible that there is a similar sequence that grows forever and, if there are loops within the $3n - 1$ sequence, why shouldn’t the same apply to $3n + 1$? A $3n + 1$ sequence can be of any length. The question is: will it always end in 1? This article does not solve the conjecture; it merely highlights areas of interest and, hopefully, gives structure to the problem.

A modification. Below, the algorithm has been revised without losing generality.

- (a) Take any positive odd integer greater than 1.
- (b) Multiply it by 3, add 1 and divide the result by 2.
- (c) If the new number is even, divide by 2 and go to step (c).

- (d) If the result is 1 then stop.
- (e) Using the new number, go to step (b).

Step (a). A number of the form $x \cdot 2^n$ (where x is a positive odd integer) will eventually reduce to x ; so we can ignore even integers as the start of a sequence.

Step (b). Applying the function $3x + 1$ to an odd integer will result in an even number that we can divide by 2.

Step (c). Remove all powers of 2 from the number.

Repeating 13 and 23 with this revision gives

- (i) 13, 5, 1;
- (ii) 23, 35, 53, 5, 1.

They have been considerably shortened, making the sequences more manageable. The sequence starting with 27 is reduced to 41 steps.

Definitions. (1) The new step (b) can be defined as a function: $q(x) = (3x + 1)/2$. We will be able to iterate this in some instances, e.g. $q(7) = 11$, $q^2(7) = 17$ and $q^3(7) = 26$.

(2) Steps (b) and (c) can be defined as a function $r(x) = (3x + 1)/2^n$ where the variable n reduces $3x + 1$ to an odd number.

(3) There is no easy way to handle step (c) using normal mathematical operations so we will define it using the equation $x = a2^n$ (where a is an odd integer) from which we obtain

$$\begin{aligned} P(x) &= a; & \text{e.g. } P(12) &= 3 \text{ and } P(7) = 7; \\ N(x) &= n; & \text{e.g. } N(12) &= 2 \text{ and } N(7) = 0. \end{aligned}$$

It is not the intention to burden the reader with too many functions. The q function is used frequently. The P and N functions are used as a pair for indexing as described later. The r function is only used once (in Theorem 1). With regard to P and N , there is also a similar set related to $x = a3^n$ but these have been avoided.

(4) A subsequence is defined as the set of integers in a sequence ending with an even number. So, for example, $\{7, 11, 17, 26\}$ is a subsequence. This can be obtained by iterating the function q as above.

Theorem 1. Integers of the form $4x + 1$ need not head sequences because $r(4x + 1) = r(x)$, where x is an odd positive integer.

Proof. Since x is odd, we can express it as $2b + 1$ and so we have

$$r(x) = r(2b + 1) = (6b + 4)/2 = 3b + 2$$

and

$$r(4x + 1) = r(8b + 5) = (24b + 16)/2^3 = 3b + 2. \quad \square$$

Note that $r(x) = q(x) \equiv 2 \pmod{3}$. This is developed later. It follows that having eliminated even numbers and $4x + 1$; we need only consider sequences starting with 3, 7, 11, 15, . . .

Theorem 2. Let $f(a, n) = a2^{n+1} + 2^n - 1$. Then $q(f(a, n)) = b2^n + 2^{n-1} - 1$ for $n > 0$.

Proof. Multiply by 3 and add 1 to get $3a2^{n+1} + 3 \cdot 2^n - 2$. Divide by 2 and rearrange to get $3a2^n + 3 \cdot 2^{n-1} - 1 = b2^n + 2^{n-1} - 1$, where $b = 3a + 1$. □

The significance of the variable b will be seen later.

The effect of this can be seen in the following table where, alongside the subsequence starting with 31, the binary equivalents and the values of the variables a and n are listed.

	Decimal	Binary	Variable a	Variable n
	31	11111	0	5
$q(31)$	47	101111	1	4
$q(47)$	71	1000111	4	3
$q(71)$	107	1101011	13	2
$q(107)$	161	10100001	40	1
$q(161)$	242	11110010	121	0

The variables are easily obtained. For example, using $q(71) = 107$ as an example, $a = ((107 + 1)/2^2 - 1)/2 = 13$. Using our notation, this is $(P(107 + 1) - 1)/2$; $n = N(107 + 1) = 2$.

Variable n will decrease by 1 at each iteration. We can make its value as great as we wish. If we set it to (say) 10, we know that the sequence will be at least 10 steps in length. If we set it to 10 billion, it will be at least 10 billion steps. In other words, we could create a sequence of arbitrary length.

Table $a2^{n+1} + 2^n - 1$. The following table (M) is derived from the function $f(a, n)$. It is only an extract but it can be extended easily using a spreadsheet. The variable a runs down the first column from 0 to 15. The variable n runs across the second row from 0 to 5. The first row gives an indication of how the end of each number in that column appears in binary. The numbers in bold track the sequence 7–11–17–26–13–20–10–5–8–4–2–1.

Binary	0	01	011	0111	01111	011111
a	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	0	1	3	7	15	31
1	2	5	11	23	47	95
2	4	9	19	39	79	159
3	6	13	27	55	111	223
4	8	17	35	71	143	287
5	10	21	43	87	175	351
6	12	25	51	103	207	415
7	14	29	59	119	239	479
8	16	33	67	135	271	543
9	18	37	75	151	303	607
10	20	41	83	167	335	671
11	22	45	91	183	367	735
12	24	49	99	199	399	799
13	26	53	107	215	431	863
14	28	57	115	231	463	927
15	30	61	123	247	495	991

(1) The values increase from 7 to 11 to 17 to 26 whilst at each step the row number (a) increases to $3a + 1$ and the column number (n) decreases by 1.

(2) When n reaches 0, the row number is 13. Move to that number, which is in row 3, column 1.

(3) The next number is $q(13) = 20$ and it is in row 10, column 0. Move to that number at row 5, column 0.

(4) Observe that 5 is odd, so move to that number in row 1, column 1.

(5) Observe that $q(5) = 8$ is in row 4, column 0. This time we move to row 2 and then row 1 in column 0 at which point we stop.

Explanation. There are three actions.

Action 1: $q(M_{a,n})$ reduces to $M_{3a+1,n-1}$, where $n > 0$. This shows that for any value of x , $q(x)$ will only lie on the rows 1, 4, 7, \dots , $3a + 1$. Relate this to Theorem 2:

$$q(a2^{n+1} + 2^n - 1) = b2^n + 2^{n-1} - 1,$$

where $b = 3a + 1$.

Action 2: If a is even and $n = 0$, the next integer in the sequence after $M_{a,0}$ is $M_{a/2,0}$. Continue moving up column 0 until we reach 1 (the

target) or we reach an odd-numbered row. It is equivalent to step (c) of the algorithm.

Action 3: If a is odd and $n = 0$, we select the integer a at $M_{p,q}$. Thus $p = (P(a+1) - 1)/2$ and $q = N(a+1)$.

I have called action 3 ‘bouncing back’ because, having travelled diagonally down to and then up column 0, we are bounced into a new column in the table.

Theorem 3. Integers in rows $3a + 1$ are of the form $3x + 2$.

Proof. $f(3a+1, n) = (3a+1)2^{n+1} + 2^n - 1 = 3(a2^{n+1} + 2^n - 1) + 2$. \square

The effect is that in any subsequence, all numbers after the start equal $3x + 2$. Following similar arguments, it can be shown that integers in the remaining two rows constantly switch between $3x + 1$ and $3x$ according to whether a and n are odd or even.

So we have three types of numbers.

2 (mod 3): Apart from the starter, subsequences will contain only this type. The q function will never result in 1 (mod 3) or 0 (mod 3).

1 (mod 3): These numbers only arise as a result of division by 2 in column 0 and ‘bouncing back’ into a new column.

0 (mod 3): These numbers will never occur in a sequence other than as a starter. They can never occur in a $3a + 1$ row.

Conjecture. Earlier we concluded that only numbers of the form $4x + 3$ need start a sequence. Now consider the three forms of integer.

First, since all numbers of the form $3x+2$ can be derived from a previous number in the sequence there is no need to consider it as a starter. In other words, there is a number that will reduce to 2 (mod 3). So it would seem that $3x + 2$ need not be a starter.

Second, although $3x + 1$ will never occur within a subsequence, it will occur somewhere within column 0 because, as we ascend dividing by two, 2 (mod 3) can lead to 1 (mod 3). It follows that $3x + 1$ can be derived from $3x + 2$ on a $3a + 1$ row and ‘bounced back’. This is not proved, but if true then $3x + 1$ need not be a starter.

This leaves $3x$, which can never hit a $3a+1$ row and consequently, unless it is a starter, will never appear in a sequence. So only $3x$ need be a starter.

By resolving $3x$ and $4x+3$ we can reduce our list further to 3 (mod 12); that is, 3, 15, 27, 39, 51,

Unanswered questions. *Loops.* Are loops possible? Is it possible to bounce back from column 0 with a number that is already in the sequence? If duplication does occur, it can only be caused by a bounced back number. No such duplication is known; if it did occur, the conjecture is disproved. Can we show that this is not possible?

Inversion. Can we invert the function and, starting from 1, derive any positive integer? What are the obstacles? Is it entirely hit or miss, or are there rules by which we can get back to any number? Of course, if the answer is positive then the conjecture is solved.

Loose ends. Yes, there is a loose end. We know that the q function always results in $3x + 2$. But we also know that $3x + 2$ can occur as we travel up column 0. Is there redundancy here? Why ‘bounce’ it back when we already know the result of q ? The number 5 is of this type. The only sequence ‘passing through’ 5 is started by 3 (3, 10, 5, 16, 8, 4, 2, 1). All other sequences including 5, and that is the majority of them, are as a result of it ‘bouncing back’. So does that mean that all sequences apart from this one are redundant?

Finally. We know that sufficient iterations of an odd number will eventually hit column 0. The formula is $((3/2)^n(x+1)-1)/2$, where n is such that $(x+1)/2^n$ is an odd number. Or, equivalently, $(3^{N(x+1)}P(x+1)-1)/2$. That will get you to a number in column 0. But if it is even, further divisions by 2 will be needed to get an odd number, giving us an even more complicated expression $P(3^{N(x+1)}P(x+1)-1)$. Equally, the root of a number in column 0 may be found. Thus $((2/3)^n(x+1)-1)/2$, where n is such that $(x+1)/3^n$, removes all factors of 3; $x = 3a + 2$, a is odd.

Problem 204.1 – Jigsaw

(i) For which x and y is it possible to make an $x \times y$ jigsaw puzzle which has the same number of boundary pieces as interior pieces?

(ii) Estimate how long it takes to complete a jigsaw puzzle of n pieces, giving the answer as an order-of-magnitude approximation as n tends to infinity. Is the time determined by the difficulty of locating each piece?

Assume that the pictures are of reasonably constant squareness and complexity, and that the pieces are of approximately constant size, say $2\text{ cm} \times 2\text{ cm}$, regardless of the value of n . Assume also that you have only one transportation device (say a helicopter) for moving pieces from the box to their correct places. (But note that a helicopter has a limited range; so you will need to consider how to create fuel dumps.)

Slicing a torus in half

Tony Forbes

What is the intersection of a torus and a plane that passes through its centre of symmetry?

In the previous issue we set the problem as a practical exercise involving doughnuts and a knife ('Mathematics in the kitchen – IV'). Now it is time to do some theory.

One way of constructing a torus is to take a circle of radius r and rotate its centre around the circumference of a circle of radius R . If we assume that the torus is lying flat, parallel to the (x, y) -plane and with the centre of symmetry at $(0, 0, 0)$, then it can be represented by

$$\mathcal{T} = \{((R + r \cos \theta) \cos \phi, (R + r \cos \theta) \sin \phi, r \sin \theta) : 0 \leq \phi, \theta < 2\pi\}. \quad (1)$$

Here, $(r \cos \theta \cos \phi, r \cos \theta \sin \phi, r \sin \theta)$ is a circle of radius r , centre at the origin, in a plane at angle ϕ to the (x, z) -plane. To make the torus we translate it to the point $(R \cos \phi, R \sin \phi, 0)$.

Also we can represent the slicing plane by the set

$$\mathcal{P} = \{(x, y, x \tan \alpha) : x, y \in \mathbb{R}\},$$

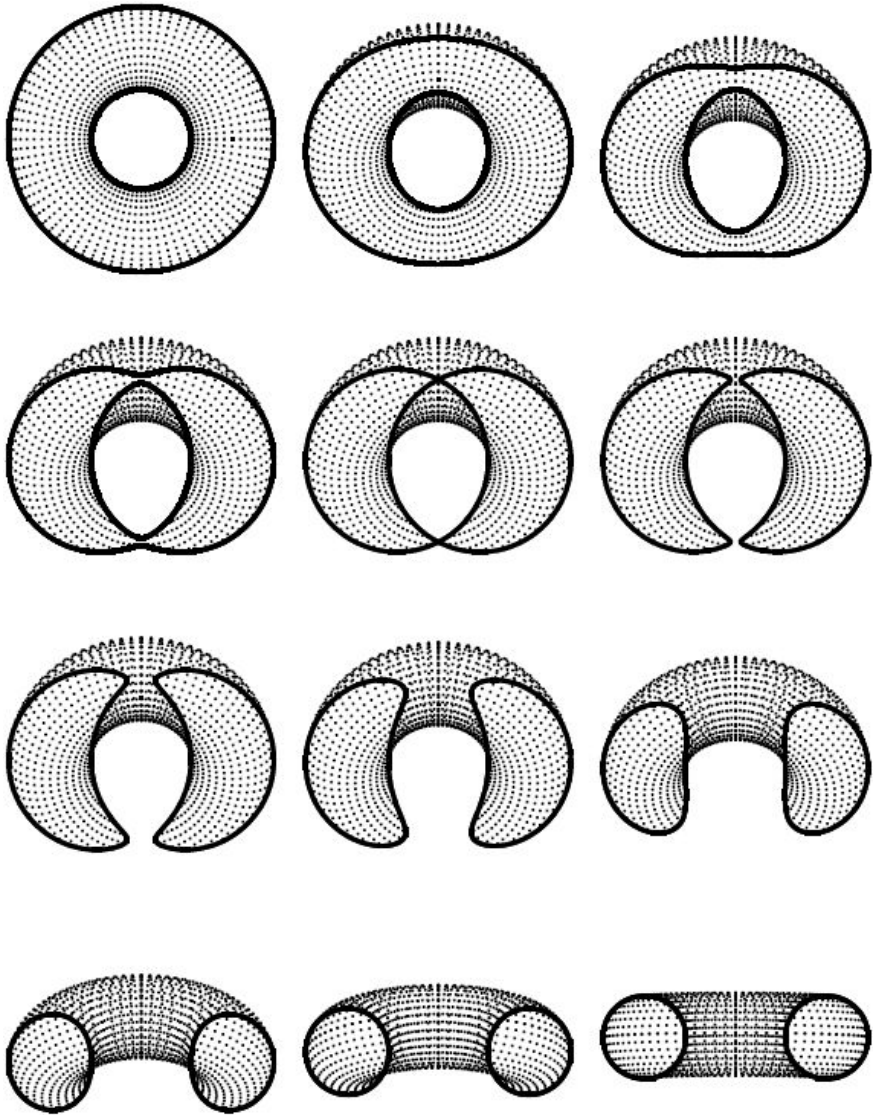
which passes through the y -axis and makes an angle α to the x -axis.

Therefore $\mathcal{T} \cap \mathcal{P}$ is given by $r \sin \theta = (R + r \cos \theta) \cos \phi \tan \alpha$, which simplifies slightly to

$$\cos \phi = \frac{r \cot \alpha \sin \theta}{R + r \cos \theta}. \quad (2)$$

Since the left-hand side must lie in $[-1, 1]$, the range of θ for which (2) is valid is restricted if $\cot \alpha$ is large. In particular, (2) does not make sense when $\alpha = 0$; however, it is clear that in this case the intersection consists of two concentric circles, the inner rim and the outer rim of the torus, of radii $R - r$ and $R + r$.

We present some results on the next page. Reading left to right, top to bottom, we show the intersection as a pair of black curves for α equal to 0° , 20° , 26° ; 26.9° , $\arcsin(r/R) \approx 27.03569^\circ$, 27.1° ; 27.5° , 30° , 40° ; 60° , 74° and 88° . The radii are $r = 1$, $R = 2.2$. The view is from a long way off in the direction of $(-\sin \alpha, 0, \cos \alpha)$. That is, we are looking at the origin from orthogonally above the cutting plane. The x -axis goes up the page and the y -axis to the left. Only the half of the torus that lies below the plane is shown.



Interestingly, the sequence of pictures reveals how the two concentric circles become transformed into two separated circles as α varies continuously from 0° to 90° . The middle picture in the second row is the transition point at which the curves merge and then separate. As you can see, the intersection appears to consist of two circles, centred at $(0, -r, 0)$ and $(0, r, 0)$, each of radius R . We shall now prove that this is really so.

The relevant angle is $\alpha = \arcsin(r/R)$. Substituting into (2) and noting that $\cot \alpha = \sqrt{R^2 - r^2}/r$, we obtain

$$\cos \phi = \frac{\sin \theta \sqrt{R^2 - r^2}}{R + r \cos \theta}.$$

We now substitute this value of $\cos \phi$ into (1) to get the x , y and z coordinates of $\mathcal{T} \cap \mathcal{P}$. Straightaway we have

$$x = \sin \theta \sqrt{R^2 - r^2} \quad \text{and} \quad z = r \sin \theta.$$

For the y coordinate we must be careful. When we substitute $\cos \phi$ into the second coordinate of (1), we need to consider both square roots in the formula $\sin \phi = \pm \sqrt{1 - \cos^2 \phi}$. Thus we obtain two values for y :

$$y^- = R \cos \theta - r \quad \text{and} \quad y^+ = R \cos \theta + r,$$

corresponding to the left-hand and right-hand circles, respectively. Finally we verify that

$$x^2 + (y^- - (-r))^2 + z^2 = R^2 \quad \text{and} \quad x^2 + (y^+ - r)^2 + z^2 = R^2,$$

the correct equations for circles of radius R centred at $(0, \pm r, 0)$.

There is another interesting phenomenon. If you look again at the sequence, you will see that in the second picture there are no points of inflection (places where the radius of curvature changes sign). However, in the third picture the outer curve has acquired four such points symmetrically placed about the x - and y -axes. So there must be a transitional value of α after which four points of inflection appear, and when $R/r = 2.2$ that value lies somewhere between 26° and 26.9° . Similarly, in picture 8 the two points of inflection present in each of the curves—which look rather like satsuma segments—have disappeared when we get to picture 9. So there is yet another special value of α . I leave it for someone else to compute the exact values of these two transitional α s.

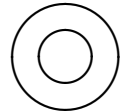
Problem 204.2 – Surface area of a torus

Show that the surface area of a cylinder of length $2\pi R$ and radius r is $4\pi^2 rR$. Show that this formula also holds for a torus with the same parameters r and R , $r < R$, so that $2\pi R$ is the length of the circular axis of the tube and r is its radius. Thus when you construct a torus from a cylinder by joining the two open ends together, the surface area lost by compression on the inside is exactly balanced by the stretching on the outside.

We now ask: Is it always true that a tubular shape made from a cylinder has the same surface area? For instance, what happens if you make an elliptical torus-like object?

Problem 204.3 – Area of an annulus

Keith Drever



An annulus is a disc of radius A with a central hole of radius a . Can you devise a method to find its area by taking only one measurement?

Problem 204.4 – Ones

Show that

$$\frac{11}{10} \cdot \frac{1111}{1110} \cdot \frac{111111}{111110} \cdot \frac{11111111}{11111110} \cdot \dots = 1.101001000100001000001\dots$$

Show that this is true in any number base, not just 10. For example, when the base is 2 we have (using decimal notation)

$$\frac{3}{2} \cdot \frac{15}{14} \cdot \frac{63}{62} \cdot \frac{255}{254} \cdot \dots = \sum_{n=0}^{\infty} 2^{-n(n+1)/2}.$$

Problem 204.5 – Circles

Consider a triangle with vertices A , B and C . Let Q the centre of the incircle. Let Q_A be the centre of the escribed circle that touches AB extended, AC extended and BC between B and C . Let Q_B and Q_C denote the centres of the other two escribed circles defined similarly. Show that

$$|QQ_A||QQ_B||QQ_C| + d(|QQ_A|^2 + |QQ_B|^2 + |QQ_C|^2) = 4d^3,$$

where d is the diameter of the circumcircle.

Problem 204.6 – A triangle property

Dick Boardman

First some terminology. Let A , B and C be the vertices of a triangle. A *median* is a line joining a vertex to the mid-point of the opposite side; an *altitude* is a line joining a vertex to the opposite side (possibly extended) and meeting it at 90° ; a *perpendicular bisector* is a line passing through the mid-point of a side at 90° ; and an *angle bisector* is a line that passes through a vertex splitting the angle there into two equal halves.

As is well known, the following triples of lines have common intersections.

- (i) The medians. The intersection is the triangle's centre of gravity.
- (ii) The altitudes. The intersection is the *orthocentre* of the triangle.
- (iii) The perpendicular bisectors of the sides. The intersection is the centre of the *circumcircle*, the circle that passes through A , B and C .
- (iv) The angle bisectors. The intersection is the centre of the *in-circle*, the unique circle inside the triangle that is tangent to the three sides.

Not so well known (at least to Tony Forbes and me) is that if the in-circle touches the sides of the triangle at D on side BC , at E on AC and at F on AB , then the lines AD , BE and CF meet at a common point, T , say.

Is there a short proof of this theorem?

ADF writes — I thought it would be a good idea to illustrate these interesting concepts. Hence the diagram opposite.

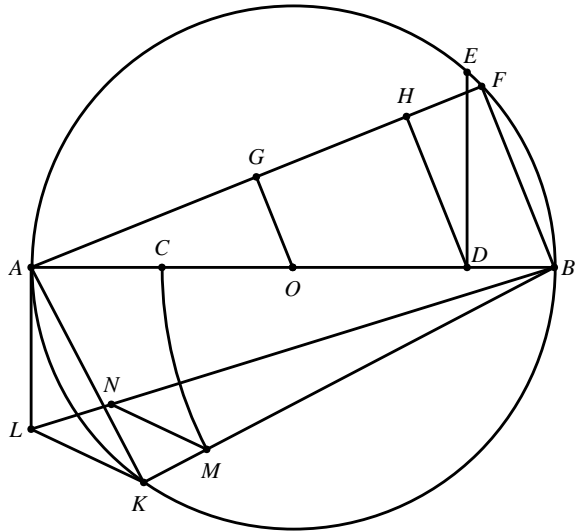
The common intersections are labelled by the following letters directly underneath them: M – medians, O – altitudes, P – perpendicular bisectors of the sides, Q – angle bisectors, and T – AD , BE and CF as defined above.

Observe that P , M and O are collinear. This is the *Euler line*, shown dashed.

Also included in the diagram is the nine-point circle. The centre (labelled N) bisects the Euler line, and the circumference passes through the mid-points of the sides and the bases of the altitudes. The other three points are where it bisects the lines joining the orthocentre to the vertices. Moreover, the nine-point circle touches the in-circle as well as the three exterior circles that make tangents to all three (possibly extended) sides of the triangle.

Solution 202.1 – Squaring the circle

In the diagram on the right, $|OA| = |OB| = 1$, $|OC| = \frac{1}{2}$, $|OD| = \frac{2}{3}$ and $|BM| = |BC| = \frac{3}{2}$; DE is perpendicular to AB and $|BF| = |DE|$; OG , DH and BF are parallel; AL is perpendicular to AB , $|AL| = |GH|$, $|AK| = |AG|$ and MN is parallel to KL .



Construct the diagram and determine $|BN|$.

Peter Fletcher

This is a straightforward triangle-bashing exercise. We have

$$|BF| = |DE| = \sqrt{1 - \left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$$

and, since angle AFB is 90° , $|AF| = \frac{1}{3}\sqrt{31}$. Hence

$$|AK| = |AG| = \frac{|AF|}{2} = \frac{\sqrt{31}}{6}, \quad |AL| = |GH| = \frac{|AF|}{3} = \frac{\sqrt{31}}{9},$$

$$|BK| = \frac{\sqrt{113}}{6} \quad \text{and} \quad |BL| = \frac{\sqrt{355}}{9}.$$

Then the expressions for $|BK|$ and $|BL|$ together with $|BM| = \frac{3}{2}$ give the answer,

$$|BN| = \frac{|BL||BM|}{|BK|} = \sqrt{\frac{355}{113}} \approx \sqrt{3.141592920}.$$

Therefore the circle has area very nearly equal to $355/113$, the area of a square on BN . Hence the title.

Tony Forbes

From the given solution it is clear how to draw a square of area $355/113$ using just those traditional weapons of math construction, a ruler and a pair of compasses. The original diagram is due to Ramanujan, ‘Squaring the circle’, *Journal of the Indian Mathematical Society* **v** (1913). At the end, Ramanujan observes: ‘If the area of the circle be 140,000 square miles, then BN is greater than the true length by about an inch.’

Of course, there are other ways to construct a line of length $\sqrt{355/113}$, but Ramanujan’s method is interesting because it is particularly economical on the use of paper. Here’s a simpler procedure which does not have that property.

Draw a long line, \mathcal{X} , call it the x -axis and select a point $O = (0, 0)$. Using the compasses set to the unit length, mark off points at integer intervals along \mathcal{X} from $A = (-1, 0)$ to $B = (355, 0)$. Through O draw a line, \mathcal{Y} , perpendicular to \mathcal{X} and mark points at $C = (0, 1)$ and $D = (0, 113)$. Draw a line through C parallel to DB meeting \mathcal{X} at E . Draw a circle with AE as diameter. Let the circle meet \mathcal{Y} at F and G . The required length is $|OF| = |OG|$.

For $|OA| = 1$ and from the construction we have $|OE| = 355/113$. The chord theorem then states that $|OF||OG| = |OA||OE|$.

In general, let $RSTU$ be a convex quadrilateral inscribed in a circle and let the diagonals RT and SU meet at X . The chord theorem asserts that

$$|RX||TX| = |SX||UX|.$$

This follows from **Ptolemy’s theorem**: *The sum of the products of the two pairs of opposite sides of a convex quadrilateral inscribed in a circle is equal to the product of the lengths of the diagonals.*

In our case the quadrilateral is $AFEG$, $|AO| = 1$ and $|OE| = 355/113 = \alpha$, say. Let $|OF| = |OG| = x$. Then

$$2\sqrt{1+x^2}\sqrt{\alpha^2+x^2} = 2x(1+\alpha).$$

Solving gives $x = \sqrt{\alpha}$.

To learn more about Ptolemy, his theorem and related results, I recommend *Topics in Geometry* by Hazel Perfect (Pergamon Press, Bristol, 1963). Alternatively, Jeremy Humphries suggests consulting that woman whose daughter sang this bluesy song,

My mamma done Ptolemy / When I was in pigtails

Coordinate representation of Euclidian algorithm

Sebastian Hayes

Readers will be familiar with the *Euclidian algorithm* for finding the gcd (greatest common divisor) of two integers, so-called because Euclid gives it pride of place in the first of the four books he devotes to number theory. Almost as well known and even more useful is the theorem

If the gcd of two integers a and b is d , then we can find integers x and y such that $ax + by = d$.

By extension we can solve for multiples of d and if $d = 1$, i.e. a and b are relatively prime, we can solve for any integer on the right-hand side. However, if d is not a multiple of the gcd of a and b there are no solutions (in integers).

How do we find such integers? We can get a whole stream of solutions if we can turn up a single pair x, y and these we obtain by ‘working the Euclidian algorithm backwards’. This is rather fiddly, and I have just recently come across a way of setting out the Euclidian algorithm which keeps track of where you are so you get your pair of numbers right off. (For this I am indebted to David Sharpe, the editor of *Mathematical Spectrum*.)

We start off with two numbers whose gcd we wish to find, say 17 and 56. We may consider that we have coordinate axes which are calibrated in an eccentric manner so that one unit along the x axis has 17 parts and one unit along the y axis has 56. On this understanding we can associate positive and negative integers with points. For example, 17 itself will be associated with the point $(1, 0)$ since $1 \cdot 17 + 0 \cdot 56 = 17$ and direct multiples of 17 will be $(m, 0)$; 56 is associated with the point $(0, 1)$. Zero is produced when we have $56 \cdot 17 - 17 \cdot 56 = 0$; i.e. we associate zero with the points $(56, -17)$ and also $(-56, 17)$.

Surprisingly, it is possible to add, subtract and even multiply coordinate values—these are, don’t forget, not proper complex numbers. The working is as follows.

$$\begin{array}{rcl}
 & & 56 \quad (0, 1) \\
 (1, 0) & 17 & \longrightarrow 51 \quad (3, 0) \\
 \hline
 (-9, 3) & 15 & \longleftarrow 5 \quad (-3, 1) \\
 \hline
 (10, -3) & 2 & \longrightarrow 4 \quad (20, -6) \\
 \hline
 (-46, 14) & 2 & \longleftarrow 1 \quad (-23, 7) \\
 \hline
 (56, -17) & 0 &
 \end{array}$$

For example, I write $51(3, 0)$ because $51 = 3 \cdot 17$ and the next line is a subtraction. We then put the result 5 into 17 and we multiply the coordinates by 3. And so on. The last line on the left checks out since we recover our original numbers, as we should do.

If we are interested in the point associated with 1, we have it, namely $(-23, 7)$; i.e. $7 \cdot 56 - 23 \cdot 17 = 1$. Also, we can obtain from this the first point the ‘other way round’; i.e. with $17x > 0$ and $56y < 0$,

$$\begin{aligned} (17 - 10)56 - (56 - 33)17 &= 17 \cdot 56 - 10 \cdot 56 - 56 \cdot 17 + 33 \cdot 17 \\ &= 33 \cdot 17 - 10 \cdot 56 = 1. \end{aligned}$$

So the other point is $(33, -10)$.

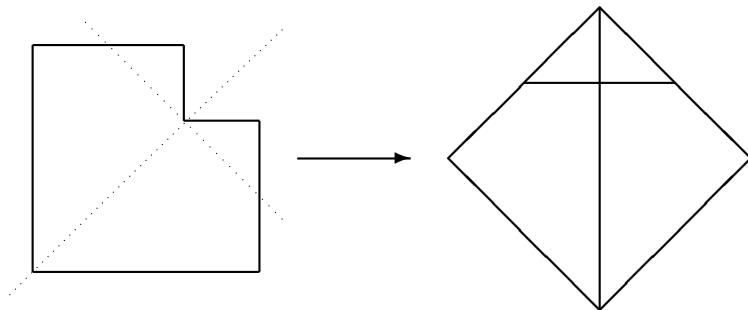
From here it is easy to generalize—finding as many solutions as required, positive and negative.

Solution 200.2 – Square with corner missing

Given integers m, n , $0 < m < n$, take an $n \times n$ square of sheet material and cut out an $m \times m$ square from a corner. Then make two straight-line cuts and rearrange the pieces to make a perfect square. For what values of m and n is this possible?

Steve Moon

This is how to do it for integer pairs $m, n = 3m$.



Nobody has sent any other significantly different solutions.

After the menopause, women’s risk of heart disease is the same as men’s. One woman in six dies of it every year. — *Telegraph* 15/7/2003.

[Sent by Peter Fletcher]

Solution 191.2 – LCM

Denote by $[a, b, c]$ the least common multiple of a , b and c . Show that $[a, b, c] \leq (n/3)^3$ for all sufficiently large n , where $a + b + c = n$, $1 \leq a < b < c \leq n$. Investigate the difference between the maximum value of $[a, b, c]$ and $(n/3)^3$.

Norman Graham

In general, if $\sum x_i = n$ for $i = 1, 2, \dots, m$ (n fixed) and all the x_i are positive, the product $x_1 x_2 \dots x_m$ is a maximum when the x_i are all equal to n/m . This follows from the fact that if any two x_i s are replaced by their average, the product is increased, since $\frac{1}{2}(a+b)^2 - ab = \frac{1}{2}(a-b)^2 > 0$.

In the present problem, a , b and c are all different; so it follows that $[a, b, c] \leq abc < N$, where $N = (n/3)^3$. The values of a , b and c to maximize $[a, b, c]$ for selected values of n were obtained by trial and error, and are shown in the following table. We write Δ for $N - [a, b, c]$.

n	N	a	b	c	$[a, b, c]$	Δ	Δ/N	F
6	8	1	2	3	6	2	0.25	6
7	343/27	1	2	4	4	235/27	0.685131	
8	512/27	1	3	4	12	188/27	0.367188	14
9	27	1	3	5	15	12	0.444444	24
10	1000/27	2	3	5	30	190/27	0.19	14
11	1331/27	1	3	7	21	764/27	0.574005	56
12	64	3	4	5	60	4	0.0625	6
13	2197/27	1	5	7	35	1252/27	0.569868	56
14	2744/27	3	4	7	84	476/27	0.173469	26
15	125	3	5	7	105	20	0.16	24
16	4096/27	4	5	7	140	316/27	0.0771484	14
17	4913/27	4	6	7	84	2645/27	0.538368	
18	216	5	6	7	210	6	0.0277778	6
19	6859/27	3	5	11	165	2404/27	0.350488	104
20	8000/27	5	7	8	280	440/27	0.055	14
21	343	5	7	9	315	28	0.0816327	24
22	10648/27	5	8	9	360	928/27	0.0871525	26
23	12167/27	5	7	11	385	1772/27	0.14564	56
25	15625/27	5	9	11	495	2260/27	0.14464	56
35	42875/27	7	13	15	1365	6020/27	0.140408	104

To obtain the answer for any other value of n , express it in the form $18r + n_0$, where n_0 is a value of n other than 7 or 17 from the table. Then

the required values of a , b and c are those for n_0 plus $6r$, and the value of Δ is that for n_0 plus rF , where F is the number in the last column of the table. In fact, $F = 2(a^2 + b^2 + c^2 - ab - ac - bc)$.

For example, take $n = 50 = 18 \cdot 2 + 14$, so that $n_0 = 14$, and $r = 2$. Then $[a, b, c] = [3 + 6 \cdot 2, 4 + 6 \cdot 2, 7 + 6 \cdot 2] = [15, 16, 19] = 4560$ and $\Delta = 476/27 + 26 \cdot 2 = 1880/27$. As a check, we have $N = (50/3)^3 = 125000/27 = 4560 + \Delta$.

A note on divisibility tests

Shena Flower

With reference to the letter from J. V. Budd on p. 22 of M500 **201** concerning a test for divisibility by 7. Since childhood (i.e. 56 years) I have known of the following test.

<i>Cross out the units digit.</i>	3 1 4 1 5 9 2 6 5
<i>Double it and subtract from the remaining number.</i>	$\begin{array}{r} 31415926\cancel{5} \\ \underline{10} \\ 3141591\cancel{6} \end{array}$
<i>Continue until you cannot go further.</i>	$\begin{array}{r} 3141591\cancel{6} \\ \underline{12} \\ 314157\cancel{9} \end{array}$
<i>Test the number that remains for divisibility by 7 by inspection.</i>	$\begin{array}{r} 314157\cancel{9} \\ \underline{18} \\ 31413\cancel{9} \\ \underline{18} \\ 3139\cancel{5} \\ \underline{10} \\ 312\cancel{9} \\ \underline{18} \\ 29\cancel{4} \\ \underline{8} \\ 2\cancel{1} \\ \underline{2} \\ 0 \end{array}$

For example, using the number 314159265, we perform the test as shown on the right.

This procedure is equivalent to subtracting multiples of 21 and since the final number is 0 we know that $314159265 \equiv 0 \pmod{21}$ and is thus divisible by 7.

Similar methods can be used for other primes. For divisibility by 13, apply the same test but multiply by 9 (instead of 2), since $13 \times 7 = 91$. For divisibility by 17, apply the same test but multiply by 5, since $17 \times 3 = 51$.

For which logarithm base b does the equation

$$\sin x = \log_b x$$

have precisely two distinct roots. What about precisely $2k$ distinct roots?

Random quadratic equations

Tony Forbes

In M500 **199**, David Wild wrote about someone on Radio 4 saying that before the introduction of complex numbers only half the quadratic equations could be solved. Of course this wasn't intended to be taken literally. Then John Bull's letter in M500 **202** reported Frederick Mosteller (*Fifty Challenging Probability Problems*) asserting that the probability of a random quadratic equation $x^2 + 2bx + c = 0$ having real roots tends to 1 as the sample space size tends to infinity. However, Mosteller leaves unanswered the case where there are *three* coefficients, a , b and c .

Well, let's see what happens. We wish to calculate $p(X)$, the probability that $ax^2 + bx + c = 0$ has real roots, where $-X \leq a, b, c \leq X$ for some (large) positive number X , and a , b and c are uniformly distributed random variables. As I can think of no good reason to double the x coefficient, I shall insist on the more natural bx rather than $2bx$.

The problem is not without interest. Indeed, we have had a couple of suggestions. One reader assumed that the coefficients were integers and arrived at the value $p(X) = 2/3$ by a clever summation. Another used integration to obtain $p(X) = 11/18$. Being unable to decide which of the two totally different answers was correct, I had to resort to a simulation. I chose 100000 triples (a, b, c) at random from a suitable sample space and counted the number of times when $b^2 \geq 4ac$.

This situation is quite common in mathematical research. If you have a difficult probability problem and if (as with me) probability theory is not amongst your greatest strengths, it is absolutely vital to test your solution. All you need is a decent random number generator, a modicum of computer programming and a computer to run your simulation on. Statistics textbooks refer to such techniques collectively as *Monte Carlo* methods—presumably because of the close association with Formula One motor racing.

Anyway, the result is that $p(X) \approx 0.627$ and moreover it appears to be independent of X . Unfortunately 0.627 is too far removed from both $2/3 \approx 0.667$ and $11/18 \approx 0.611$ for the discrepancy to be dismissed as experimental error.

But now that we know the answer to the question, we can proceed with confidence. It simplifies matters if we can restrict the coefficients a , b and c to non-negative numbers. Thus

$$p(X) = \frac{1}{2} + \frac{q(X)}{2},$$

where $q(X)$ is the probability that $ax^2 + bx + c = 0$ has real roots, assuming $0 \leq a, b, c \leq X$. This is seen by dividing the cube $-X \leq a, b, c \leq X$ into eight sub-cubes. The probability of real roots is 1 in the four sub-cubes where $ac \leq 0$, and in the other four sub-cubes the probabilities are the same (and equal to $q(X)$) by symmetry. All we do now is integrate:

$$q(X) = \frac{1}{X^3} \int_0^X \int_0^X \int_0^X \chi(b^2 - 4ac) da dc db,$$

where $\chi(x) = 1$ if $x \geq 0$, $\chi(x) = 0$ otherwise. Let us fix b and concentrate on the two inner integrals (over a and c). Then

$$q(X) = \frac{1}{X^3} \int_0^X A(X, b) db,$$

where $A(X, b)$ is the area in the (a, c) -plane bounded by the positive a -axis, the positive c -axis, the line $a = X$, the line $c = X$ and the curve $ac = b^2/4$, as in the picture below. By a straightforward calculation we have

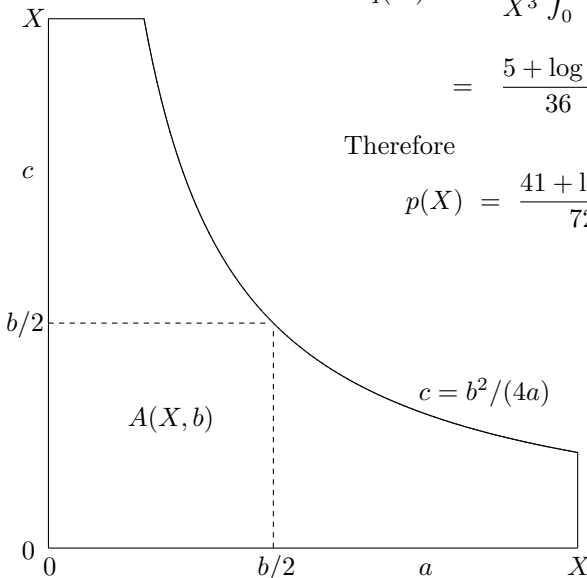
$$A(X, b) = \frac{b^2}{4} + 2 \int_{b/2}^X \frac{b^2}{4a} da = \frac{b^2}{4} + \frac{b^2}{2} \log \frac{2X}{b}.$$

Hence

$$\begin{aligned} q(X) &= \frac{1}{X^3} \int_0^X \left(\frac{b^2}{4} + \frac{b^2}{2} \log \frac{2X}{b} \right) db \\ &= \frac{5 + \log 64}{36} \approx 0.254413. \end{aligned}$$

Therefore

$$p(X) = \frac{41 + \log 64}{72} \approx 0.627207.$$



If you really do want your quadratics to have $2bx$ rather than bx , the computation is slightly easier. The relevant inequality is $b^2 \geq ac$ and the final answer is rational: $7/9$.

Letters to the Editor

Dizziness

Many thanks for M500 201. Even my feeble brain was adequate for enjoyment of the ‘Odds and ends’ statistical gaffes.

I think I can explain the dizzying effect of the two diagrams with circles, which Tony mentioned in ‘An exponential sum’. As any fule kno, our mental image of what we see is a complete artefact, the original image on the retina having been completely deconstructed and reconstructed in a way that makes sense to us.

The very first stage of image processing is to find edges in the data from the retina. An ‘edge’ here is a border between a light area and a dark area—all the early stages of visual processing happen in black and white, and colour is not filled in till considerably later. There are neurons in the visual part of the brain to detect edges at a large range of different angles. However, they all work in terms of straight lines. And they are not very exact about the angle that causes them to fire. So if you look at a diagram of a circle, it causes a good deal of untidy neuron activity as the cells respond to the differently angled black-white transitions of the inside and outside of the circular black line.

If you are just looking at one or two circles, later stages of brain processing will tidy up the impression, substituting from memory a form that you see as a smooth, stable circle. However, if there are a lot of circles—or in the case of the cover picture, a lot of marks that are seen as curves—there is too much chaotic data to tidy up, and some of the original chaos gets through, producing a dancing effect.

The image of 36 circles also presents another problem: the brain is unsure whether or not to interpret the circular tangents as intersecting lines (an image familiar to our ancestors as they swung among the branches) or to stick to the interpretation of a lot of circles. The effect, at least for me, is to make the line look too thin at the point of contact of the circles, although under a magnifying glass I can see that it isn’t. This effect is also probably partly due to the fact, known from the familiar optical illusion tricks, that we see white forms—the insides of the circles—as larger than they are.

The fizzy effect of the cover diagram has been diluted by printing it on blue paper, which reduces the contrast.

Ralph Hancock

Paradoxical dice

Tony,

Re: 'Paradoxical dice', [M500 199 24]. As every backgammon player knows, the chance of throwing a (2,1) is $2/36$ and the chance of throwing a (2,2) is $1/36$. So it does not seem paradoxical that a (2,1) would occur before a (2,2) on average.

Am I understanding the question properly?

Robin Marks

Songs likely to be of interest to mathematicians

Affine Romance

Love Me Tensor

A Random Walk in the Black Forest

Oranges and Lemmas

A Martingale Sang in Berkeley Square

Root 66

Magnetic Moments

This Merely Was Nine

The last one is from *The Reverend Spooner sings Rodgers and Hammerstein*. See page 17 for another. Any more?

Mathematics Revision Weekend 2005

The **31st M500 Society Mathematics Revision Weekend** will be held at **Aston University, Birmingham** over **9–11 September 2005**.

The cost, including standard accommodation and all meals from bed and breakfast Friday to lunch Sunday, is £150. Add £22 for *en suite* facilities. The cost for non-residents is £75. M500 members get a discount of £10. See the Society's web page, www.m500.org.uk, for full details and an application form, or send a stamped, addressed envelope to

Jeremy Humphries, M500 Weekend 2005.

The Weekend is designed to help with revision and exam preparation, and is open to all Open University students. Tutorial sessions start at 19.30 on the Friday and finish at 17.00 on the Sunday. We plan to present most OU mathematics courses.

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