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The Cauchy–Riemann equations for quaternions

Dennis Morris

The matrix form of a quaternion is

$$\mathcal{H} = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}.$$

A quaternion function is a map from a quaternion to a quaternion of the form

$$f\left(\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}\right)$$

$$\mapsto \begin{bmatrix} u(a,b,c,d) & -v(a,b,c,d) & -s(a,b,c,d) & -t(a,b,c,d) \\ v(a,b,c,d) & u(a,b,c,d) & -t(a,b,c,d) & s(a,b,c,d) \\ s(a,b,c,d) & t(a,b,c,d) & u(a,b,c,d) & -v(a,b,c,d) \\ t(a,b,c,d) & -s(a,b,c,d) & v(a,b,c,d) & u(a,b,c,d) \end{bmatrix}.$$

We expand the function matrix, omitting the '(a, b, c, d)' for ease of presentation:

$$f = \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix} + \begin{bmatrix} 0 & -v & 0 & 0 \\ v & 0 & 0 & 0 \\ 0 & 0 & 0 & -v \\ 0 & 0 & v & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & -s & 0 \\ 0 & 0 & 0 & s \\ s & 0 & 0 & 0 \\ 0 & -s & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -t \\ 0 & 0 & -t & 0 \\ 0 & t & 0 & 0 \\ t & 0 & 0 & 0 \end{bmatrix}. \qquad (1)$$

Now define four matrices with entries 0, -1 and 1 according to the coefficients of the elements occurring in (1),

$$\mathcal{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathcal{J} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathcal{K} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad \mathcal{L} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and if $\mathcal{M}$ is any 4×4 matrix and x is a scalar, we denote by $\mathcal{M}x$ the result of right-multiplying each element of $\mathcal{M}$ by x. Thus (1) becomes

$$f = \mathcal{I}u + \mathcal{J}v + \mathcal{K}s + \mathcal{L}t.$$

Also we have

$$\mathcal{J} \cdot (\mathcal{I}v) = \mathcal{J}v, \quad \mathcal{K} \cdot (\mathcal{I}s) = \mathcal{K}s \quad \text{and} \quad \mathcal{L} \cdot (\mathcal{I}t) = \mathcal{L}t.$$

In these three expressions, the leftmost matrix is a constant and the middle matrix is of the form that is isomorphic to the real numbers. Hence, by substituting the LHS of the above three expressions into the expanded function matrix, we are effectively dealing with functions of real numbers.

First, we differentiate with respect to $\mathcal{I}a$, which, being of the matrix form that is isomorphic to the real numbers, is, itself, effectively a real variable: a.

We are differentiating $f\left(\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}\right)$ with respect to $\mathcal{I}a$ at the point $\mathcal{X} = \begin{bmatrix} \alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ \alpha_2 & \alpha_1 & -\alpha_4 & \alpha_3 \\ \alpha_3 & \alpha_4 & \alpha_1 & -\alpha_2 \\ \alpha_4 & -\alpha_3 & \alpha_2 & \alpha_1 \end{bmatrix}$. Let

$$\mathcal{A} = \begin{bmatrix} a & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ \alpha_2 & a & -\alpha_4 & \alpha_3 \\ \alpha_3 & \alpha_4 & a & -\alpha_2 \\ \alpha_4 & -\alpha_3 & \alpha_2 & a \end{bmatrix},$$

$$\begin{aligned} u(a) &= u(a, \alpha_2, \alpha_3, \alpha_4), & u(\alpha) &= u(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \\ v(a) &= v(a, \alpha_2, \alpha_3, \alpha_4), & v(\alpha) &= v(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \\ s(a) &= s(a, \alpha_2, \alpha_3, \alpha_4), & s(\alpha) &= s(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \\ t(a) &= t(a, \alpha_2, \alpha_3, \alpha_4), & t(\alpha) &= t(\alpha_1, \alpha_2, \alpha_3, \alpha_4). \end{aligned}$$

Then we have

$$\begin{aligned}
f'(\mathcal{X}) &= \lim_{\mathcal{A}\to\mathcal{X}} \frac{\begin{bmatrix} u(a) & -v(a) & -s(a) & -t(a) \\ v(a) & u(a) & -t(a) & s(a) \\ s(a) & t(a) & u(a) & -v(a) \\ t(a) & -s(a) & v(a) & u(a) \end{bmatrix} - \begin{bmatrix} u(\alpha) & -v(\alpha) & -s(\alpha) & -t(\alpha) \\ v(\alpha) & u(\alpha) & -t(\alpha) & s(\alpha) \\ s(\alpha) & t(\alpha) & u(\alpha) & -v(\alpha) \\ t(\alpha) & -s(\alpha) & v(\alpha) & u(\alpha) \end{bmatrix}}{\mathcal{A}-\mathcal{X}} \\
&= \lim_{\mathcal{I}a\to\mathcal{I}\alpha_1} \left(\frac{\mathcal{I}u(a)-\mathcal{I}u(\alpha)}{\mathcal{I}a-\mathcal{I}\alpha_1} + \frac{\mathcal{J}v(a)-\mathcal{J}v(\alpha)}{\mathcal{I}a-\mathcal{I}\alpha_1} \right. \\
&\qquad \left. \frac{\mathcal{K}s(a)-\mathcal{K}s(\alpha)}{\mathcal{I}a-\mathcal{I}\alpha_1} + \frac{\mathcal{L}t(a)-\mathcal{L}t(\alpha)}{\mathcal{I}a-\mathcal{I}\alpha_1} \right) \\
&= \lim_{\mathcal{I}a\to\mathcal{I}\alpha_1} \left(\frac{\mathcal{I}u(a)-\mathcal{I}u(\alpha)}{\mathcal{I}a-\mathcal{I}\alpha_1} + \mathcal{J}\cdot\frac{\mathcal{I}v(a)-\mathcal{I}v(\alpha)}{\mathcal{I}a-\mathcal{I}\alpha_1} \right. \\
&\qquad \left. \mathcal{K}\cdot\frac{\mathcal{I}s(a)-\mathcal{I}s(\alpha)}{\mathcal{I}a-\mathcal{I}\alpha_1} + \mathcal{L}\cdot\frac{\mathcal{I}t(a)-\mathcal{I}t(\alpha)}{\mathcal{I}a-\mathcal{I}\alpha_1} \right) \\
&= \left(\mathcal{I}\,\frac{\partial u}{\partial a}\right) + \mathcal{J}\cdot\left(\mathcal{I}\,\frac{\partial v}{\partial a}\right) + \mathcal{K}\cdot\left(\mathcal{I}\,\frac{\partial s}{\partial a}\right) + \mathcal{L}\cdot\left(\mathcal{I}\,\frac{\partial t}{\partial a}\right).
\end{aligned}$$

Therefore

$$\frac{\partial f}{\partial a} = \begin{bmatrix} \frac{\partial u}{\partial a} & -\frac{\partial v}{\partial a} & -\frac{\partial s}{\partial a} & -\frac{\partial t}{\partial a} \\ \frac{\partial v}{\partial a} & \frac{\partial u}{\partial a} & -\frac{\partial t}{\partial a} & \frac{\partial s}{\partial a} \\ \frac{\partial s}{\partial a} & \frac{\partial t}{\partial a} & \frac{\partial u}{\partial a} & -\frac{\partial v}{\partial a} \\ \frac{\partial t}{\partial a} & -\frac{\partial s}{\partial a} & \frac{\partial v}{\partial a} & \frac{\partial u}{\partial a} \end{bmatrix}.$$

Secondly, we differentiate with respect to $\mathcal{J}b$, which is a matrix form that is not isomorphic to the real numbers; i.e. it is not a real variable. In due course, we will have to make adjustments.

We are differentiating $f\left(\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}\right)$ with respect to $\mathcal{J}b$ at the point $\mathcal{X}$. Let $\mathcal{B} = \begin{bmatrix} \alpha_1 & -b & -\alpha_3 & -\alpha_4 \\ b & \alpha_1 & -\alpha_4 & \alpha_3 \\ \alpha_3 & \alpha_4 & \alpha_1 & -b \\ \alpha_4 & -\alpha_3 & b & \alpha_1 \end{bmatrix}$, $u(b) = u(\alpha_1, b, \alpha_3, \alpha_4)$, $v(b) = v(\alpha_1, b, \alpha_3, \alpha_4)$, $s(b) = s(\alpha_1, b, \alpha_3, \alpha_4)$ and $t(b) = t(\alpha_1, b, \alpha_3, \alpha_4)$.

Then we have

$$\begin{aligned} f'(\mathcal{X}) &= \lim_{\mathcal{B}\to\mathcal{X}} \frac{\begin{bmatrix} u(b) & -v(b) & -s(b) & -t(b) \\ v(b) & u(b) & -t(b) & s(b) \\ s(b) & t(b) & u(b) & -v(b) \\ t(b) & -s(b) & v(b) & u(b) \end{bmatrix} - \begin{bmatrix} u(\alpha) & -v(\alpha) & -s(\alpha) & -t(\alpha) \\ v(\alpha) & u(\alpha) & -t(\alpha) & s(\alpha) \\ s(\alpha) & t(\alpha) & u(\alpha) & -v(\alpha) \\ t(\alpha) & -s(\alpha) & v(\alpha) & u(\alpha) \end{bmatrix}}{\mathcal{B}-\mathcal{X}} \\ &= \lim_{\mathcal{J}b\to\mathcal{J}\alpha_2} \left(\frac{\mathcal{I}u(b)-\mathcal{I}u(\alpha)}{\mathcal{J}b-\mathcal{J}\alpha_2} + \frac{\mathcal{J}v(b)-\mathcal{J}v(\alpha)}{\mathcal{J}b-\mathcal{J}\alpha_2} \right. \\ &\qquad \left. \frac{\mathcal{K}s(b)-\mathcal{K}s(\alpha)}{\mathcal{J}b-\mathcal{J}\alpha_2} + \frac{\mathcal{L}t(b)-\mathcal{L}t(\alpha)}{\mathcal{J}b-\mathcal{J}\alpha_2} \right). \end{aligned}$$

Thus far, we are preparing to differentiate with respect to $\mathcal{J}b$. We cannot do this. However, $\mathcal{J}b = \mathcal{J}\cdot(\mathcal{I}b)$, and we can differentiate with respect to $\mathcal{I}b$ because this matrix form is isomorphic to the real numbers and we can treat b as a real variable. Hence, since $\mathcal{J}$ is a constant,

$$\begin{aligned} f'(\mathcal{X}) &= \lim_{\mathcal{J}b\to\mathcal{J}\alpha_2} \left(\frac{1}{\mathcal{J}}\ \frac{\mathcal{I}u(b)-\mathcal{I}u(\alpha)}{\mathcal{I}b-\mathcal{I}\alpha_2} + \mathcal{J}\left(\frac{1}{\mathcal{J}}\ \frac{\mathcal{I}v(b)-\mathcal{I}v(\alpha)}{\mathcal{I}b-\mathcal{I}\alpha_2} \right) \right. \\ &\qquad \left. \mathcal{K}\left(\frac{1}{\mathcal{J}}\ \frac{\mathcal{I}s(b)-\mathcal{I}s(\alpha)}{\mathcal{I}b-\mathcal{I}\alpha_2} \right) + \mathcal{L}\left(\frac{1}{\mathcal{J}}\ \frac{\mathcal{I}t(b)-\mathcal{I}t(\alpha)}{\mathcal{I}b-\mathcal{I}\alpha_2} \right) \right) \\ &= \frac{1}{\mathcal{J}}\left(\mathcal{I}\,\frac{\partial u}{\partial b}\right) + \mathcal{J}\frac{1}{\mathcal{J}}\left(\mathcal{I}\,\frac{\partial v}{\partial b}\right) + \mathcal{K}\frac{1}{\mathcal{J}}\left(\mathcal{I}\,\frac{\partial s}{\partial b}\right) + \mathcal{L}\frac{1}{\mathcal{J}}\left(\mathcal{I}\,\frac{\partial t}{\partial b}\right). \end{aligned}$$

We opt to multiply out these matrices in the order that we present them above. They are non-commutative, and the consequences of multiplying them in the reverse order will be discussed at the end.

Recalling that

$$\mathcal{J}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

we obtain the formula

$$\frac{\partial f}{\partial b} = \begin{bmatrix} \frac{\partial v}{\partial b} & \frac{\partial u}{\partial b} & \frac{\partial t}{\partial b} & -\frac{\partial s}{\partial b} \\ -\frac{\partial u}{\partial b} & \frac{\partial v}{\partial b} & -\frac{\partial s}{\partial b} & -\frac{\partial t}{\partial b} \\ -\frac{\partial t}{\partial b} & \frac{\partial s}{\partial b} & \frac{\partial v}{\partial b} & \frac{\partial u}{\partial b} \\ \frac{\partial s}{\partial b} & \frac{\partial t}{\partial b} & -\frac{\partial u}{\partial b} & \frac{\partial v}{\partial b} \end{bmatrix}.$$

In a similar manner, we differentiate $f\left(\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}\right)$ with respect to $\mathcal{K}c = \mathcal{K}\cdot(\mathcal{I}c)$ at the point $\mathcal{X}$ to get $\partial f/\partial c$ and with respect to $\mathcal{L}d = \mathcal{L}\cdot(\mathcal{I}d)$ at the point $\mathcal{X}$ to get $\partial f/\partial d$.

Thus, including the expressions for $\partial f/\partial a$ and $\partial f/\partial b$, we have

$$\frac{\partial f}{\partial a} = \begin{bmatrix} \frac{\partial u}{\partial a} & -\frac{\partial v}{\partial a} & -\frac{\partial s}{\partial a} & -\frac{\partial t}{\partial a} \\ \frac{\partial v}{\partial a} & \frac{\partial u}{\partial a} & -\frac{\partial t}{\partial a} & \frac{\partial s}{\partial a} \\ \frac{\partial s}{\partial a} & \frac{\partial t}{\partial a} & \frac{\partial u}{\partial a} & -\frac{\partial v}{\partial a} \\ \frac{\partial t}{\partial a} & -\frac{\partial s}{\partial a} & \frac{\partial v}{\partial a} & \frac{\partial u}{\partial a} \end{bmatrix}, \quad \frac{\partial f}{\partial b} = \begin{bmatrix} \frac{\partial v}{\partial b} & \frac{\partial u}{\partial b} & \frac{\partial t}{\partial b} & -\frac{\partial s}{\partial b} \\ -\frac{\partial u}{\partial b} & \frac{\partial v}{\partial b} & -\frac{\partial s}{\partial b} & -\frac{\partial t}{\partial b} \\ -\frac{\partial t}{\partial b} & \frac{\partial s}{\partial b} & \frac{\partial v}{\partial b} & \frac{\partial u}{\partial b} \\ \frac{\partial s}{\partial b} & \frac{\partial t}{\partial b} & -\frac{\partial u}{\partial b} & \frac{\partial v}{\partial b} \end{bmatrix},$$

$$\frac{\partial f}{\partial c} = \begin{bmatrix} \frac{\partial s}{\partial c} & -\frac{\partial t}{\partial c} & \frac{\partial u}{\partial c} & \frac{\partial v}{\partial c} \\ \frac{\partial t}{\partial c} & \frac{\partial s}{\partial c} & \frac{\partial v}{\partial c} & -\frac{\partial u}{\partial c} \\ -\frac{\partial u}{\partial c} & -\frac{\partial v}{\partial c} & \frac{\partial s}{\partial c} & -\frac{\partial t}{\partial c} \\ -\frac{\partial v}{\partial c} & \frac{\partial u}{\partial c} & \frac{\partial t}{\partial c} & \frac{\partial s}{\partial c} \end{bmatrix}, \quad \frac{\partial f}{\partial d} = \begin{bmatrix} \frac{\partial t}{\partial d} & \frac{\partial s}{\partial d} & -\frac{\partial v}{\partial d} & \frac{\partial u}{\partial d} \\ -\frac{\partial s}{\partial d} & \frac{\partial t}{\partial d} & \frac{\partial u}{\partial d} & \frac{\partial v}{\partial d} \\ \frac{\partial v}{\partial d} & -\frac{\partial u}{\partial d} & \frac{\partial t}{\partial d} & \frac{\partial s}{\partial d} \\ -\frac{\partial u}{\partial d} & -\frac{\partial v}{\partial d} & -\frac{\partial s}{\partial d} & \frac{\partial t}{\partial d} \end{bmatrix}.$$

Looking at the four results and putting equal the corresponding elements, we have that the Cauchy–Riemann equations for quaternions are:

$$\begin{aligned} \frac{\partial u}{\partial a} &= \frac{\partial v}{\partial b} = \frac{\partial s}{\partial c} = \frac{\partial t}{\partial d}, \\ \frac{\partial u}{\partial b} &= -\frac{\partial v}{\partial a} = -\frac{\partial t}{\partial c} = \frac{\partial s}{\partial d}, \\ \frac{\partial u}{\partial c} &= -\frac{\partial s}{\partial a} = \frac{\partial t}{\partial b} = -\frac{\partial v}{\partial d}, \\ \frac{\partial u}{\partial d} &= -\frac{\partial t}{\partial a} = -\frac{\partial s}{\partial b} = \frac{\partial v}{\partial c} \end{aligned}$$

of which, the four partial derivatives in the top left-hand corner are the familiar Cauchy–Riemann equations of complex analysis.

We chose to multiply the matrices in the order in which we presented them. If we had consistently reversed this order, we would have found that

$$\frac{\partial f}{\partial a} = \begin{bmatrix} \frac{\partial u}{\partial a} & -\frac{\partial v}{\partial a} & -\frac{\partial s}{\partial a} & -\frac{\partial t}{\partial a} \\ \frac{\partial v}{\partial a} & \frac{\partial u}{\partial a} & -\frac{\partial t}{\partial a} & \frac{\partial s}{\partial a} \\ \frac{\partial s}{\partial a} & \frac{\partial t}{\partial a} & \frac{\partial u}{\partial a} & -\frac{\partial v}{\partial a} \\ \frac{\partial t}{\partial a} & -\frac{\partial s}{\partial a} & \frac{\partial v}{\partial a} & \frac{\partial u}{\partial a} \end{bmatrix}, \quad \frac{\partial f}{\partial b} = \begin{bmatrix} \frac{\partial v}{\partial b} & \frac{\partial u}{\partial b} & -\frac{\partial t}{\partial b} & \frac{\partial s}{\partial b} \\ -\frac{\partial u}{\partial b} & \frac{\partial v}{\partial b} & \frac{\partial s}{\partial b} & \frac{\partial t}{\partial b} \\ \frac{\partial t}{\partial b} & -\frac{\partial s}{\partial b} & \frac{\partial v}{\partial b} & \frac{\partial u}{\partial b} \\ -\frac{\partial s}{\partial b} & -\frac{\partial t}{\partial b} & -\frac{\partial u}{\partial b} & \frac{\partial v}{\partial b} \end{bmatrix},$$

$$\frac{\partial f}{\partial c} = \begin{bmatrix} \frac{\partial s}{\partial c} & \frac{\partial t}{\partial c} & \frac{\partial u}{\partial c} & -\frac{\partial v}{\partial c} \\ -\frac{\partial t}{\partial c} & \frac{\partial s}{\partial c} & -\frac{\partial v}{\partial c} & -\frac{\partial u}{\partial c} \\ -\frac{\partial u}{\partial c} & \frac{\partial v}{\partial c} & \frac{\partial s}{\partial c} & \frac{\partial t}{\partial c} \\ \frac{\partial v}{\partial c} & \frac{\partial u}{\partial c} & -\frac{\partial t}{\partial c} & \frac{\partial s}{\partial c} \end{bmatrix}, \quad \frac{\partial f}{\partial d} = \begin{bmatrix} \frac{\partial t}{\partial d} & -\frac{\partial s}{\partial d} & \frac{\partial v}{\partial d} & \frac{\partial u}{\partial d} \\ \frac{\partial s}{\partial d} & \frac{\partial t}{\partial d} & \frac{\partial u}{\partial d} & -\frac{\partial v}{\partial d} \\ -\frac{\partial v}{\partial d} & -\frac{\partial u}{\partial d} & \frac{\partial t}{\partial d} & -\frac{\partial s}{\partial d} \\ -\frac{\partial u}{\partial d} & \frac{\partial v}{\partial d} & \frac{\partial s}{\partial d} & \frac{\partial t}{\partial d} \end{bmatrix}$$

from which we would have derived the different Cauchy–Riemann equations:

$$\begin{aligned} \frac{\partial u}{\partial a} &= \frac{\partial v}{\partial b} = \frac{\partial s}{\partial c} = \frac{\partial t}{\partial d}, \\ \frac{\partial u}{\partial b} &= -\frac{\partial v}{\partial a} = \frac{\partial t}{\partial c} = -\frac{\partial s}{\partial d}, \\ \frac{\partial u}{\partial c} &= -\frac{\partial s}{\partial a} = -\frac{\partial t}{\partial b} = \frac{\partial v}{\partial d}, \\ \frac{\partial u}{\partial d} &= -\frac{\partial t}{\partial a} = \frac{\partial s}{\partial b} = \frac{\partial v}{\partial c} \end{aligned}$$

of which, the four partial derivatives in the top left hand corner are still the familiar Cauchy–Riemann equations of complex analysis.

Solution 202.6 – Prime sum

Show that

$$\sum_{p \text{ prime}} \frac{1}{p^2} = 0.45224\,74200\,41065\,49850\,65433\,64832\,24793\,41732\,\ldots .$$

Basil Thompson

Not a solution; just some observations. How long to calculate a hundred places after the decimal point? For this to be so, the largest prime number must be greater than 10^{50}. How many primes are less than 10^{50}?

One expression for $\pi(n)$, the number of primes less than n, is $\pi(n) \approx n/(\log n)$, and this is an underestimate. Thus

$$\pi(10^{50}) \approx \frac{10^{50}}{\log 10^{50}} \approx 8.7 \cdot 10^{47}.$$

That is, we need to calculate all the $1/p^2$ terms up to at least the first prime greater than 10^{50}, and this requires at least $8.7 \cdot 10^{47}$ calculations.

How long to complete such a calculation? One billion years is equal to about

$$10^9 \cdot 365.25 \cdot 24 \cdot 60 \cdot 60 \approx 3 \cdot 10^{16} \text{ seconds.}$$

Therefore to complete the calculation in this time would require approximately $8.7 \cdot 10^{47}/3 \cdot 10^{16} \approx 3 \cdot 10^{31}$ operations per second. Even if thousands of machines were available, this is clearly impossible. And the problem of deciding which numbers are prime would be even more time consuming.

Is there a simple formula for $\sum 1/p^2$? To get some idea of what is involved let us start with $\sum_{n=1}^{\infty} 1/n^2 = \pi/6$. From this it is easy to deduce that $\sum_q 1/q^2 = \pi/8$, where q represents all the odd numbers. Therefore

$$\sum_{p \text{ prime}} \frac{1}{p^2} < \frac{\pi^2}{8} - 1 + \frac{1}{2^2} = 0.4837\ldots,$$

which at least agrees with the original expression. But $\sum_q 1/q^2$ contains all the primes plus all the other odd prime numbers which are products of prime numbers. In fact,

$$\sum_{p \text{ prime}} = \frac{\pi^2}{8} - 1 + \frac{1}{2^2} - \left(\frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \frac{1}{25^2} + \frac{1}{27^2} + \frac{1}{33^2} + \ldots\right).$$

It might be possible to rearrange terms within the large parentheses into infinite expressions such as $1/3^2 \sum_3^\infty 1/p^2$, $1/5^2 \sum_5^\infty 1/p^2$, ... The $\sum 1/p^2$ terms can be moved to the left-hand side but unfortunately this leaves other terms behind. In any case, the whole equation becomes an infinite number of series each summed to infinity. Is it possible to pull all these series together to obtain a formula for $\sum 1/p^2$?

Tony Forbes

The answer is not to sum the series directly. As Basil Thompson has observed, to obtain 100 decimal places, say, we would need to go at least as far as terms with $p \approx 10^{50}$, and this is clearly out of the question. If you are still not convinced, try making a list of all the primes involved in such a sum.

The trick is to express $\sum_{p \text{ prime}} 1/p^2$ as a series in $\log \zeta(2n)$, where $\zeta(s) = \sum_{m=1}^\infty 1/m^s$ is the Riemann zeta function. This is useful for two reasons; (i) $\log x$ and $\zeta(2n)$ are computable to any reasonable number of decimal places, the latter using Tom Apostol, *Introduction to Analytic Number Theory*, Theorem 12.17:

$$\zeta(2n) \;=\; (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2\,(2n)!},$$

where $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66, \ldots$ are the *Bernoulli numbers*; and (ii) $\log \zeta(2n)$ gets small rapidly. If n is large, $\log \zeta(2n)$ is approximately $1/2^{2n}$.

Start with

$$\zeta(s) \;=\; \prod_{p \text{ prime}} \frac{1}{1 - 1/p^s}.$$

Put $s = 2n$ and take logs,

$$\log \zeta(2n) \;=\; -\sum_{p \text{ prime}} \log\left(1 - \frac{1}{p^{2n}}\right).$$

Expand the right-hand side,

$$\log \zeta(2n) \;=\; \sum_{p \text{ prime}} \left(\frac{1}{p^{2n}} + \frac{1}{2\,p^{4n}} + \frac{1}{3\,p^{6n}} + \ldots\right),$$

and rearrange,

$$\log \zeta(2n) \;=\; \sum_{p \text{ prime}} \frac{1}{p^{2n}} + \sum_{p \text{ prime}} \frac{1}{2\,p^{4n}} + \sum_{p \text{ prime}} \frac{1}{3\,p^{6n}} + \cdots. \qquad (1)$$

Recall the definition of $\mu(n)$, the Möbius function: $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is the product of k distinct primes, and $\mu(n) = 0$ if n is divisible by the square of a prime. Multiply both sides of (1) by $\mu(n)/n$ and sum over n:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(2n) \;=\; \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(\sum_{p \text{ prime}} \sum_{k=1}^{\infty} \frac{1}{k}\, p^{-2kn} \right).$$

This rather complicated expression can then be simplified slightly to get

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(2n) \;=\; \sum_{p \text{ prime}} \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mu(n)}{k\, n\, p^{2kn}} \right).$$

Now the clever bit. Put $r = kn$. Then the double sum over n and k becomes a sum over r and a sum over the divisors n of r:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(2n) \;=\; \sum_{p \text{ prime}} \left(\sum_{r=1}^{\infty} \sum_{n \text{ divides } r} \frac{\mu(n)}{r\, p^{2r}} \right).$$

Swapping sides and rearranging,

$$\sum_{p \text{ prime}} \left(\sum_{r=1}^{\infty} \frac{1}{r\, p^{2r}} \sum_{n \text{ divides } r} \mu(n) \right) \;=\; \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(2n).$$

But the third sum on the left is 1 if $r = 1$ and zero otherwise (Apostol, Theorem 2.1). Hence the second sum on the left vanishes from $r = 2$ onwards, and we just get the bit where $r = 1$:

$$\sum_{p \text{ prime}} \frac{1}{p^2} \;=\; \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(2n),$$

and the right-hand side is a rapidly converging series involving no explicit reference to primes at all.

What I find interesting is that you can evaluate a simple infinite series over the primes to great accuracy without knowing any of the actual primes. The mysterious encoding of the primes in the zeta function is presumably what makes this possible.

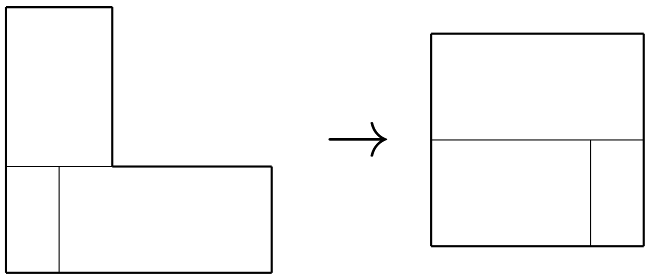
Solution 200.5 – Square with corner missing

> Take two integers, $0 < m < n$. Take an $n \times n$ square of suitable sheet material. Cut out an $m \times m$ square from a corner. Then make two straight-line cuts and rearrange the pieces to make a perfect square. For what values of m and n is this possible?

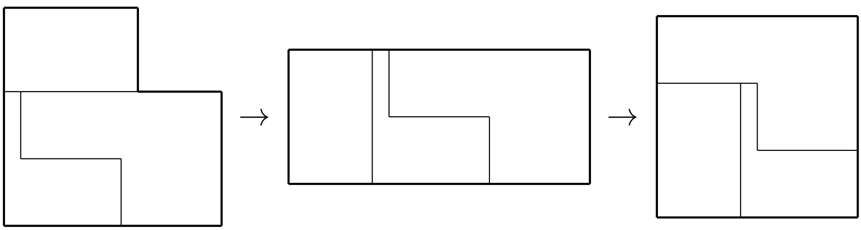
Dick Boardman

As we have seen [M500 **204**, p. 19], $n = 3$ and $m = 1$ will work.

However, there is a series of three-piece dissections of a square with square corner missing. Let n be the side of the large square and m be the side of the missing corner. Cut off an $m \times (n - m)$ rectangle and rotate it to form an $(n - m) \times (n + m)$ rectangle and then use a 'step' technique to convert this to a square. Thus for $n = 5$, $m = 3$, slice off and re-attach a 5×2 rectangle to give an 8×2 rectangle, then halve this to give a 4×4 square.



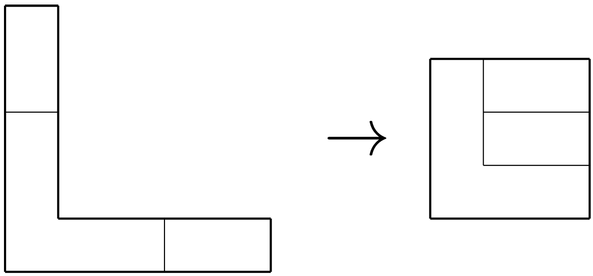
For $n = 13$, $m = 5$, first convert it to an 18×8 rectangle and then use one 'step' to get a 12×12 square.



But this last dissection and others involving more than one step require a slight bending of the rules. To achieve the 'two straight lines' condition, fold the material suitably before making the cuts.

Christopher Ellingham

In addition to Steve Moon's solution in M500 **204** for $n = 3m$, I have a pair of 'Pythagorean' solutions for $n = 5$, $m = 3$ [above], and $n = 5$, $m = 4$, below.



Mark Bastow

We make two straight-line cuts to remove rectangular ends from each leg of the L-shape, and rearrange these to fit within the corner of the L-shape to complete a perfect square of side x. We will then have

$$x^2 \ = \ n^2 - m^2.$$

For the rectangular ends from the two legs of the L-shape to fit together, we must also have

$$n - m \ = \ 2(n - x).$$

We can now eliminate x to obtain $5m = 3n$.

Basil Thompson

We have [as above]

$$x^2 \ = \ n^2 - m^2.$$

For the rectangular ends from the two legs of the L-shape to fit together, we must also have

$$n - x \ = \ 2(n - m).$$

We can now eliminate x to obtain $5m = 4n$.

Solution 204.2 – Surface area of a torus

> Is it always true that a tubular shape made from a cylinder has the same surface area?

Bob Margolis

A distorted cylinder can be generated by drawing a circle at each point of a curve in the plane perpendicular to the tangent to the curve at that point.

Taking the curve α, to be unit speed simplifies some calculations and making the circle unit radius simplifies them further without destroying generality.

Provided the curve does not cause the resulting surface to self-intersect, we can proceed as follows. The usual notation T, N, B is used for the Frenet frame and the Frenet formulae take the form

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N, \end{aligned}$$

and everything is a function of the parameter used to describe the position on the curve (the arc length from an arbitrary starting point in the case of a unit-speed curve).

The parametric description of the surface is

$$\mathbf{x}(u,v) \;=\; \alpha(u) + \cos(v)N + \sin(v)B.$$

Some tedious calculations show that the partial velocities $\mathbf{x}_u$ and $\mathbf{x}_v$, unit normal U and area 2-form A are given by

$$\begin{aligned} \mathbf{x}_u &= (1-\kappa(u)\cos(v))T(u) - \tau(u)\sin(v)N + \tau(u)\cos(v)B, \\ \mathbf{x}_v &= -\sin(v)N + \cos(v)B, \\ U(u,v) &= -\cos(v)N - \sin(v)B, \\ A &= \pm(1-\kappa(u)\cos(v))du\,dv. \end{aligned}$$

For reasons that may be clear later, we choose the negative sign for the area 2-form!

Now, suppose that the original cylinder had length L, so that the area of the cylinder was $2\pi L$.

Integrate the area 2-form over the length of the cylinder (u) and from 0 to 2π (v). The area of the distorted cylinder is

$$\begin{aligned} \text{Area} &= -\int_0^L \int_0^{2\pi} (1-\kappa(u)\cos(v))du\,dv \\ &= \int_0^L \int_0^{2\pi} (1-\kappa(u)\cos(v))dv\,du \\ &= \int_0^L [v-\kappa(u)\sin(v)]_0^{2\pi}\,du \\ &= \int_0^L 2\pi\,du \quad = \quad 2\pi L. \end{aligned}$$

So, the area is preserved under 'reasonable' distortions of the cylinder.

To fill in the gaps and find the definitions, try Barrett O'Neill: *Elementary Differential Geometry*. By the way, such 'tubes' occurred occasionally in the TMAs and exams for M334/M434, which is now defunct.

Solution 203.6 – Loops

There are n pieces of string. Choose two ends at random and tie them together. Repeat until there are no free ends left. What is the probability of creating a single loop of string?

Dave Wild

Let $P(n)$ be the probability of creating a single loop of string from n pieces of string.

Method 1 – Recurrence relation. A single piece of string forms a single loop so $P(1) = 1$. If there are n pieces of string, where $n > 1$, then after selecting any of the $2n$ ends, a loop will not be created unless the other end of that piece of string is selected. Therefore $P(n) = \dfrac{2n-2}{2n-1}\,P(n-1)$. Using this recurrence relation gives

$$P(2n) \;=\; \frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdot\dots\cdot\frac{4}{5}\cdot\frac{3}{2} \;=\; \frac{2^{2n}}{2n}\cdot\frac{(n!)^2}{(2n)!}$$

for $n = 1, 2, \dots$.

Method 2 – Counting. The total number of ways of selecting the n pairs of ends to be tied is $(2n)!/(2^n\,n!)$. The number of ways of forming n strings into a loop is $(n-1)!\,2^{n-1}$. Dividing these values gives the above answer.

A few approximate probabilities are $P(2) = 0.67$, $P(4) = 0.46$, $P(8) = 0.32$, $P(16) = 0.22$ and $P(32) = 0.16$.

Re: Fibonacci and all that

Sebastian Hayes

Ron Potkin's article [M500 **200** pp 14–17] is fascinating because it reveals—to me at any rate—an unexpected connection between the Golden Section, Fibonacci sequences, Pascal's triangle and the Theory of Equations.

In all likelihood the Golden Section, or Divine Proportion, originated in attempts to solve the problem: *Can there be three numbers $a < b < c$ in continued ratio (i.e. $a : b = b : c$), where $c = a + b$?* Within the terms of Greek mathematics the answer was categorical: *No such numbers exist or can exist.*

Leaping over some twenty-three centuries we arrive at the so-called *Fundamental Theorem of Algebra*, which states that every rational integral function of x of degree n has n (possibly equal) roots and no more, though whether they are real or not depends on certain conditions. Thus $x^2 = x + 1$, the algebraic version of the original problem, has two solutions, namely $p = (1 + \sqrt{5})/2$ and $q = (1 - \sqrt{5})/2$.

What on earth has all this to do with the growth of rabbit populations? Suppose an enclosure and rabbit pairs inside it. At the end of any month n we take the count p_n, of rabbit pairs. Those born within the month that has just ended will be given by the difference $p_n - p_{n-1}$, those born during the previous month by $p_{n-1} - p_{n-2}$ and so on. In Fibonacci's original scheme no rabbit pairs die, we start with a single (presumably newlyborn) pair and there is a maturation period of a full month. Those just born will not reproduce but will maintain themselves, those born the previous month or earlier, given by the count p_{n-1}, will double their numbers. Thus

$$p_{n+1} \;=\; (p_n - p_{n-1}) + 2p_{n-1} \;=\; p_n + p_{n-1}.$$

If we introduce further grading, for example having a pair unproductive for a month, then producing one pair and finally producing two, we obtain

$$p_{n+1} \;=\; (p_n - p_{n-1}) + 2(p_{n-1} - p_{n-2}) + 3p_{n-2} \;=\; p_n + p_{n-1} + p_{n-2},$$

which is the tribonacci sequence, so-called. More generally, we can produce a Fibonacci-style sequence with m terms on the right-hand side by introducing arbitrary (integral) constants $a, b, \ldots$ which indicate varying productivity factors. Sterility, death and even the capacity to infect other rabbits with a mortal disease can be dealt with by judicious choice of constants. For example,

$$p_{n+1} \;=\; 2(p_n - p_{n-1}) + 3(p_{n-1} - p_{n-2}) \;=\; 2p_n + p_{n-1} - 3p_{n-2}$$

models the case where rabbit pairs produce one pair at once, then two pairs in the following month and then die of exhaustion.

Now, from the above we can work out all terms of a 'two-tier' Fibonacci sequence given values for a and b; i.e. $t_1 = 1$, $t_2 = a$, $t_3 = a^2 + b$, $t_4 = a(a^2 + b) + ab, \ldots$. But there appears to be no neat way of characterizing the nth term.

By identifying a and b with $p + a$ and $-pq$ from $x^2 = (p + q)x - pq$ Binet obtained a formula for the original Fibonacci sequence, namely $t_n = (p^n - q^n)/(p - n)$. For $m > 2$ there is no formula as such but the general case is easily recognizable—it is the expansion of the multinomial $(p + q + \ldots)^n$ with all the coefficients reduced to unity.

The general term is $\dfrac{n!}{i!j!\ldots k!}(p^i q^j \ldots r^k)$, where $i + j + \cdots + k = n$ and the various combinations of $i, j, k \ldots$ must be found by trial. (When there are only two variables the multinomial reduces to the binomial.)

1	2	3	4	5	${}^nM_m$
t	s	r	q	p	$\ldots$
1	1	1	1	1	${}^0M_m$
t	$t + s$	$t + s + r$	$t + s + r + q$	$t + s + r + q + p$	${}^1M_m$
t^2	$(t + s)^2$	$(t + s + r)^2$	$(t + s + r + q)^2$	$(t + s + r + q + p)^2$	${}^2M_m$
t^3	$(t + s)^3$	$(t + s + r)^3$	$(t + s + r + q)^3$	$(t + s + r + q + p)^3$	${}^3M_m$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Now reduce all the coefficients to unity. The resulting expression will be a polynomial of degree n in m variables and, in order to emphasize the connection with Pascal's triangle, I refer to it as ${}^nS_m$; ${}^0S_m$ is defined as 1 for all m.

t	s	r	q	p	$\ldots$
${}^0S_1$	${}^0S_2$	${}^0S_3$	${}^0S_4$	${}^0S_5$	$\ldots$
${}^1S_1$	${}^1S_2$	${}^1S_3$	${}^1S_4$	${}^1S_5$	$\ldots$
${}^2S_1$	${}^2S_2$	${}^2S_3$	${}^2S_4$	${}^2S_5$	$\ldots$
${}^3S_1$	${}^3S_2$	${}^3S_3$	${}^3S_4$	${}^3S_5$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

The rule of formation is ${}^0S_m = 1$, ${}^{n+1}S_m = v_m\,{}^nS_m + v_{m-1}\,{}^nS_{m-1} + \ldots v_1\,{}^nS_1 = v_m\,{}^nS_m + {}^{n+1}S_{m-1}$.

If now we take the further step of setting $t = s = r = \cdots = 1$, we obtain

t	s	r	q	p
1	1	1	1	1
1	2	3	4	5
1	3	6	10	15
1	4	10	20	35
1	5	15	35	70

.

These are the binomial coefficients displayed somewhat differently.

Prior to reading Ron Potkin's article, I had not realized that we can transform the recursive definition $t_{n+1} = at_n + bt_{n-1}, +ct_{n-2} + \ldots$ into a formula in n by identifying the constants $a, b, c, \ldots$ with $\sum$ roots, $-\sum$ (roots taken in pairs), $\ldots$, $(-1)^{m+1} \prod$ roots, and so on for any number of terms, not just two. This strikes me as a remarkable but also somewhat puzzling discovery because there is *a priori* no reason why a Fibonacci-style sequence of m terms should have anything to do with an m-degree polynomial. In terms of my matrix ${}^nS_m$, it comes down to deriving ${}^nS_{m+1}$ 'column-wise' instead of 'across-wise'; i.e. using only entries ${}^nS_{m+1}$, ${}^{n-1}S_{m+1}$, $\ldots$, ${}^1S_{m+1}$. I wondered whether the result was in fact correct. However, it is easy to show the pattern for $m = 1$, $m = 2$, since

$$ {}^{n+1}S_1 \; = \; t\,{}^nS_1, \qquad {}^{n+1}S_2 \; = \; s\,{}^nS_2 + {}^{n+1}S_1 \; = \; s\,{}^nS_2 + t\,{}^nS_1. $$

But by definition, ${}^nS_1 = {}^nS_2 - s\,{}^{n-1}S_2$; so

$$ {}^{n+1}S_2 \; = \; s\,{}^nS_2 + t({}^nS_2 - s\,{}^{n-1}S_2) \; = \; (s+t)\,{}^nS_2 - st\,{}^{n-1}S_2. $$

From here on it is not too hard to establish the general case by induction using the rule of formation.

This means that, if we so desire, we can 'work the other way' and given a sequence defined by a formula in n, we can define it recursively using polynomial roots though the result is rather like using a pile-driver to crack a nut. Thus the natural numbers themselves, the triangular numbers and so on are given by

	$t_1 = 1$ for all s_r	
s_0:	$t_{n+1} = t_n$	1, 1, 1, 1, 1, ...
s_1:	$t_{n+1} = 2t_n - t_{n-1}$	1, 2, 3, 4, 5, ...
s_2:	$t_{n+1} = 3t_n - 3t_{n-1} + t_{n-2}$	1, 3, 6, 10, 15, ...
s_3:	$t_{n+1} = 4t_n - 6t_{n-1} + 4t_{n-2} - t_{t-3}$	1, 4, 10, 20, 35, ...

Note, however, that in a population model the roots must be real—coefficients which do not produce real roots correspond to an impossible situation, for example when more rabbits die in a single month than are in existence e.g. $p_{n+1} = 2p_n - 5p_{n-1}$. More restrictive still, the terms in a Fibonacci-type sequence must be integers including and above all the first term denoting the initial number of pairs.

The net result is that we do not quite manage to square the circle. Returning to the original Fibonacci problem with $t_1 = 1$, instead of obtaining $t_n^2 = t_{n-1}t_{n+1}$ we get the near miss $t_n^2 = t_{n-1}t_{n+1} \pm 1$. Moreover, the same sort of thing will apply given any positive integral choice of t_1, for we will obtain the ratio of successive terms as an infinite continued fraction:

$$\begin{aligned} t_{n+1} &= at_n + bt_{n-1}, \\ \frac{t_{n+1}}{t_n} &= a + \frac{bt_{n-1}}{t_n} = a + \frac{bt_{n-1}}{at_n + bt_{n-1}} \\ &= a + \cfrac{b}{a + \cfrac{bt_{n-1}}{t_n}} = a + \cfrac{b}{a + \cfrac{b}{a + \dots}}. \end{aligned}$$

Setting $a = b = 1$ gives the elegant continued fraction representation of ϕ discovered by Euler.

Problem 206.1 – Swap sort

Tony Forbes

Let $S(n)$ denote the minimum number of instructions of the form

$$\text{if } A_i < A_j \text{ then interchange } A_i \text{ and } A_j \qquad (1)$$

to guarantee to sort a vector of n numbers whose kth element is A_k.

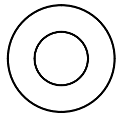
Obviously $S(2) = 1$. Also it is easily seen that the instructions

$$\begin{array}{l} \text{if } A_1 < A_2 \text{ then interchange } A_1 \text{ and } A_2 \\ \text{if } A_2 < A_3 \text{ then interchange } A_2 \text{ and } A_3 \\ \text{if } A_1 < A_2 \text{ then interchange } A_1 \text{ and } A_2 \end{array}$$

are necessary and sufficient for dealing with three numbers. If we write (1) as $[i, j]$, we can express this three-number sort program concisely as [1, 2] [2, 3] [1, 2]. With a little more work we find that [1, 2] [3, 4] [1, 3] [2, 4] [2, 3] is the shortest program that sorts four numbers; thus $S(4) = 5$.

Derive a general formula for $S(n)$. If that's too difficult—and I suspect it is—try and obtain good upper and lower bounds.

Solution 204.3 – Area of an annulus



> An annulus is a disc of radius A with a central hole of radius a. Can you devise a method to find its area by taking only one measurement?

ADF

A popular problem! Sebastian Hayes, John Bull, Martyn Lawrence, Bill Purvis, Hugh McIntyre, Hugh Luxmoore-Peake, Mandy Corbett, Claudia Gioia, Basil Thompson and Steve Moon all wrote in to offer more or less the same solution. Basically, you place your ruler so that it is tangent to the inner circle and measure the distance between the two points where it cuts the outer circle.

Assume that the centre of the annulus is at $(0, 0)$ and the ruler makes contact with the inner circle at $(0, a)$. Then the equation of the outer circle is $x^2 + y^2 = A^2$ and that of the ruler is $y = a$. Solving gives the points of contact as $(\pm\sqrt{A^2 - a^2}, a)$ and the distance between them is $2\sqrt{A^2 - a^2}$ from which the area is obtained by squaring and multiplying by $\pi/4$.

While I was writing this up I had a thought. Why not do it the other way round? Place the ruler at a tangent to the outer circle and see where it meets the inner circle. For this to make sense we would have to work with complex numbers, but the principle is exactly the same as before. The equation of the inner circle is $x^2 + y^2 = a^2$ and that of the ruler is now $y = A$. The contact points are $(\pm\, i\sqrt{A^2 - a^2}, A)$ and the distance between them is $2i\sqrt{A^2 - a^2}$. This time you square and multiply by $-\pi/4$ to get the area.

John Bull is reminded of another problem. He says:

> A string is tied around the entire circumference of the earth. The string is cut and a 1 metre length inserted. If the string now arranged to stand equidistant from the earth all the way round, what would be its height? M500 readers ought to be able to figure this one out for themselves.

And I am reminded of *The Chicken from Minsk (and 99 Other Infuriatingly Challenging Brainteasers from the Great Russian Tradition of Maths and Science)* by Yuri B. Chernyak and Robert M. Rose. The string-around-the-world is actually the one referred to in the title, and to give you some idea of the general tone of the book we state their version of the problem here.

> A fibre-optic cable completely encircles the Earth, by chance passing through a new, privately owned chicken farm on the outskirts of Minsk. The chickens refuse to walk or fly over the cable and will only pass under it. Clearly, the cable must be raised off the ground by 1 foot. But for technical reasons, the cable must then be raised 1 foot higher everywhere else, around

> the entire circumference of the Earth. The farmer, exercising his newly acquired individual rights (no more USSR!), refuses to permit the cable to cross his land unless it is raised. The bureaucrat in charge of the project is a holdover from the old days. He maliciously agrees to raise the cable only if the farmer agrees to pay for *all* of the additional cable, at $1 per foot. The farmer agrees, provided that the government will pay for all the supporting structures. How much must the chicken farmer pay?

Judging by the layout and generous use of cartoons, the book is obviously aimed at the person in the street. Nevertheless, it differs from the usual rubbish put out by British publishers, and indeed I was pleasantly surprised to see a considerable number of novel and stimulating problems. Here's one, with the title 'Jack-in-the-Box (An Exercise in Renormalization)'. *What force is required to depress the Jack so that both the Jack and the box will leave the ground?* And another, from the chapter 'Expanding and Contracting Universes'. *Boris has drawn a straight line on a coin. He is convinced that when the coin is heated the line becomes curved because some parts are further from the centre. Marina disagrees. Who is right?*

The book has a significant bias towards applied mathematics, with problems on frames of reference, gravity, harmonic oscillators, mechanics, special relativity, and thermodynamics. Presumably nobody leaves the Russian education system without a reasonable working knowledge of these subjects. But there is also a chapter on 'Geometry and Numbers', including an infinitely nested square root exercise similar to something we did in M500 **190**. And *Monty Python* fans have not been forgotten. One of the 'Warming Up' problems concerns a man who buys a parrot from a pet shop and later returns to complain about it!

Problem 206.2 – 81 cells

Look at the *sudoku* puzzle the front cover of this magazine. In case you have not seen these things before, the object is to complete the array so that each row, column and 3×3 box contains the numbers $1, 2, \ldots, 9$. When you have achieved that we have two more problems.

(i) Observe that the cover example has eight empty regions (three rows, two columns and three boxes). Either prove the non-existence of a sudoku puzzle with nine empty regions, or find one.

(ii) There exist sudoku puzzles with 17 starter digits. Is this best possible? If not, construct a sudoku puzzle with 16 (or fewer) starter digits. Note that in a valid sudoku puzzle the rows and columns cannot be permuted and the solution is unique.

Re: Problem 202.4 – Commas and brackets

> In the Zermelo–Fraenkel scheme for constructing the non-negative integers, $0 = \{\}$ and $n = \{0, 1, 2, \ldots, n-1\}$. How many commas and brackets are there in the expression for n?

ADF

Indeed, it is not too difficult to prove that there are in fact $2^{n-1} - 1$ commas and 2^{n+1} brackets. (Obviously the expression for 0 contains -0.5 commas.) I also discovered—after someone pointed it out to me—that you can easily determine n from its representation. Count the brackets at the end and subtract one. Thus the expression on the front cover of M500 **202**, $\{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}, \ldots, \{\{\}\}\}\}\}\}\}\}\}\}\}\}$, represents 10.

Sebastian Hayes

Readers might be interested to know what the axioms referred to are. In Robert R. Stoll, *Set Theory and Logic*, a book I recommend, they are given as follows. *Set* and *set membership* are taken as 'primitive notions'. Then:

ZF1 (axiom of extension) If a and b are sets and if, for all x, $x \in a$ iff $x \in b$, then $a = b$.

ZF2 (axiom schema of subsets) For any set a there exists a set b such that, for all x, $x \in b$ iff $x \in a$ and $A(x)$, where $A(x)$ is some condition on x which contains no free occurrence of b.

ZF3 (axiom of pairing) If a and b are sets, then there exists a set c such that $a \in c$ and $b \in c$.

ZF4 (axiom of union) For every set c there exists a set a such that if $x \in b$ for some member b of c, then $x \in a$.

ZF5 (axiom of power set) For each set a there exists a set b such that, for all x, if $x \subseteq a$, then $x \in b$.

ZF6 (axiom of infinity) There exists a set a such that $\emptyset \in a$ and, if $x \in a$, then $x \cup x \in a$. (It is this axiom that permits the construction of the natural numbers.)

ZF7 (axiom of choice) For each set a there exists a function f whose domain is the collection of non-empty subsets of a and, for every $b \subseteq a$ with $b \neq \emptyset$, $f(b) \in b$.

These axioms suffice apparently for practically all mathematical purposes though a further two axioms were introduced by Skolem and von Neumann to cope with the 'full-blown theory of transfinite ordinal and cardinal numbers' (Stoll, p. 303).

My own objection to this approach is that it does not really start from first principles. Inevitably it has to assume an 'intuitive' understanding of what 'set' and 'set membership' mean while such notions are, I would claim, ultimately based on our 'intuition'—I would say perception—of 'oneness' and 'plurality' in the actual physical world. In effect we have the natural numbers in our heads already, so it is ultimately futile (though for all that not without interest) to construct them axiomatically. My view is that if we want to go further back, we must look in the direction of *basic physics* and not logic; i.e. ask ourselves whether there can be a 'universe' in which the dichotomy *one/many* is meaningless. Such a 'world' would be radically non-numerical but set theory would not be any help here either.

Problem 206.3 – Odd socks

Norman Graham

Out of n different pairs of socks in a drawer, r socks are removed at random. What is the probability of obtaining d matched pairs? What if two pairs are identical?

M500 Winter Weekend 2006

The twenty-fifth M500 Society Winter Weekend will be held on **Friday 6th to Sunday 8th January 2006** at

NOTTINGHAM UNIVERSITY.

This is an annual residential weekend to dispel the withdrawal symptoms due to courses finishing in October and not starting again until February. It's an excellent opportunity to get together with acquaintances, old and new, and do some interesting mathematics in a friendly and leisurely atmosphere. Ian Harrison is running the event and this year's theme is

Euclidean-style Geometry.

Cost: £180.00 for M500 members, £185.00 for non-members. This includes standard accommodation and all meals from dinner on Friday to lunch on Sunday. For full details and a booking form, send a stamped, addressed envelope to

Diana Maxwell.

Enquiries by email to diana@m500.org.uk. A booking form is also available at www.m500.org.uk.

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