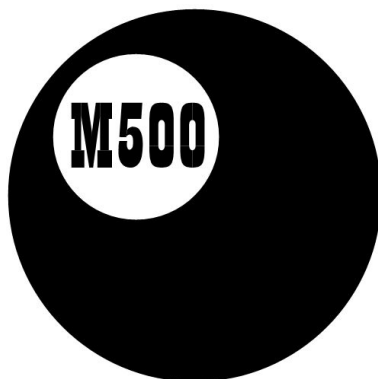


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# M500 207

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## The M500 Society and Officers

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**The M500 Society** is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: [www.m500.org.uk](http://www.m500.org.uk).

**The magazine M500** is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

**MOUTHS** is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

**The September Weekend** is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

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**Advice to authors.** We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail.

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# A revolutionary view of space

**Dennis Morris**

## Introduction

The representation of complex numbers as positions in a 2-dimensional Euclidean space begins with Wallis (1616–1703) in 1673. It was subsequently stated by Wessel (1745–1818) in 1798 and again by Gauss (1777–1855) in 1799. It was given again by Argand (1768–1822) in 1806, and the representation subsequently became known as the Argand diagram. In such times, Euclidean space was seen as being the only real type of space and complex numbers were seen as being associated with the Euclidean geometry of two dimensions. It was inevitable that mathematicians would seek some higher kind of complex numbers associated with 3-dimensional Euclidean space, and it was expected that such numbers would exist.

Throughout all these times, mathematicians assumed without question that 2-dimensional Euclidean space is a subspace of 3-dimensional Euclidean space. From this assumption it follows that the 2-dimensional complex numbers are a subalgebra of the higher 3-dimensional kind of complex numbers, and enormous effort was expended in trying to expand the complex numbers upwards into higher dimensions. These efforts were encouraged by some small, but tantalizing, success and some almost success. In 1813, Servois (1768–1847) proposed a 3-dimensional kind of complex numbers that nearly worked, and in 1843 Hamilton (1805–1865) invented the 4-dimensional quaternions.

The quaternions work, they include the 2-dimensional complex numbers as a subalgebra, but they are not an algebraic field. They are only a division algebra; they lack multiplicative commutativity. However, it was ‘immediately obvious’ to mathematicians of the day that higher-dimensional kinds of complex numbers would be multiplicatively non-commutative because rotations within 3-dimensional Euclidean space are non-commutative. It remains ‘immediately obvious’ to mainstream mathematical thought today. It is nonsense, and it is a confusing misuse of the word commutative to mean two entirely different things. Quaternions do not even have a polar form. However, tantalizingly, quaternions can be used, clumsily, to calculate rotations in 3-dimensional Euclidean space. By such tantalizations, are mathematicians tempted to believe the ‘immediately obvious’.

Unfortunately, developing higher-dimensional kinds of complex numbers seemed too hard, and so, in the middle of the 19<sup>th</sup> century, mathematicians developed a new approach to representing Euclidean space. They

invented vectors and turned their attention to describing higher-dimensional Euclidean spaces by the use of vectors. Vector algebra underlies huge areas, possibly all, of our understanding of the physical universe. It has been very successful.

Your author now wishes to reveal to the world the higher-dimensional forms of complex numbers.

### Matrix representation of algebras

An algebra is a linear space (vector space) together with an operation of multiplication. Matrices are linear transformations. If the multiplication operation of an algebra is matrix multiplication, then that algebra can be represented as operations on matrices. Multiplication of complex numbers is matrix multiplication. (The  $\hat{i} = \sqrt{-1}$  thing is just a diversion.)

### Complex numbers by matrices

The matrix representation of complex numbers is that they are the set of matrices of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

These matrices maintain their form under multiplication, inversion, and addition. They are, of course, multiplicatively commutative, and thus they form a field. The conjugate of a complex-number matrix is the adjoint matrix (the inverse multiplied by the determinant). The adjoint is the conjugate in all such algebras. The norm of the algebra (the modulus) is the square root of the determinant:  $a^2 + b^2$ , and this is half of the definition of 2-dimensional Euclidean space.

Because the submatrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \right\}.$$

are multiplicatively commutative, we can exponentiate the matrix and split it to get the polar form<sup>†</sup>

$$\begin{aligned} \exp\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) &= \exp\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) \exp\left(\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}. \end{aligned}$$

Since  $\det(\exp(A)) = \exp(\text{trace}(A))$ , where  $A$  is a matrix, the determinant of the trigonometric matrix is 1. This is the origin of the identity

$\cos^2 \theta + \sin^2 \theta = 1$ . The  $\{\cos, \sin\}$  functions are the other half of the definition of 2-dimensional Euclidean space. The polar form is a complex number, and so we can write

$$\det \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \det \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = \text{norm}^2.$$

The immediately above expression is important. It says that the determinant of a complex-number matrix is the determinant of the length matrix in the polar form and this is the square of the metric of the space that is associated with complex numbers—2-dimensional Euclidean space. Notice how this algebra picks its own norm.

### Real numbers by matrices

In the case of 1 by 1 matrices we have the algebra of real numbers. This algebra has polar form

$$\exp([a]) = \exp([a]) \exp([0]) = [r] [1].$$

The determinant of the length matrix is the 1<sup>st</sup> power of the metric of 1-dimensional space and 1 is the 1-dimensional trigonometric function.

### Hyperbolic complex numbers

Of all the 2 by 2 matrix forms, there are only two that satisfy the requirements to be an algebraic field. One of these is the complex-number matrix form; the other of these is

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} : |a| > |b|,$$

where we have had to impose the restriction shown. These matrices have polar form

$$\begin{aligned} \exp \begin{pmatrix} a & b \\ b & a \end{pmatrix} &= \exp \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \exp \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \\ &= \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{bmatrix}. \end{aligned}$$

We need no restrictions in polar form. The trigonometric functions do that automatically. This is a general phenomenon applying to all complex-number type algebras. The  $\{\cosh, \sinh\}$  functions together with the determinant

$$\text{norm}^2 = \det \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a^2 - b^2$$

define 2-dimensional hyperbolic space.

### Higher dimensions

Choose a finite Abelian group (any you like)—the order 3 cyclic group,  $C_3$ , will do for this demonstration. Write down the Cayley table of this group—using  $A$  to denote the identity.

	$A$	$B$	$C$
$A$	$A$	$B$	$C$
$B$	$B$	$C$	$A$
$C$	$C$	$A$	$B$

Rearrange this table into a (standard) form in which the elements in the first row are in alphabetical order and the identities appear on the leading diagonal.

$A$	$B$	$C$
$C$	$A$	$B$
$B$	$C$	$A$

Copy this into a matrix:

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} : |a| > \phi(|b| + |c|).$$

You now have the matrix form of a 3-dimensional complex-number like algebra. It is a field. We have had to impose restrictions in the Cartesian form— $\phi$  is the golden ratio associated with the Fibonacci numbers. In the polar form no restrictions are necessary. In normal notation this algebra has the multiplicative relations

$$\hat{p}\hat{q} = \hat{q}\hat{p} = +1, \quad \hat{q}^2 = \hat{p}, \quad \hat{p}^2 = \hat{q}, \quad \hat{p}^3 = \hat{q}^3 = +1.$$

The polar form of this algebra is

$$\begin{aligned} \exp\left(\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}\right) &= \exp\left(\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}\right) \exp\left(\begin{bmatrix} 0 & b & 0 \\ 0 & 0 & b \\ b & 0 & 0 \end{bmatrix}\right) \exp\left(\begin{bmatrix} 0 & 0 & c \\ c & 0 & 0 \\ 0 & c & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{bmatrix} \begin{bmatrix} \text{AH}_3(\lambda) & \text{BH}_3(\lambda) & \text{CH}_3(\lambda) \\ \text{CH}_3(\lambda) & \text{AH}_3(\lambda) & \text{BH}_3(\lambda) \\ \text{BH}_3(\lambda) & \text{CH}_3(\lambda) & \text{AH}_3(\lambda) \end{bmatrix} \begin{bmatrix} \text{AH}_3(\psi) & \text{CH}_3(\psi) & \text{BH}_3(\psi) \\ \text{BH}_3(\psi) & \text{AH}_3(\psi) & \text{CH}_3(\psi) \\ \text{CH}_3(\psi) & \text{BH}_3(\psi) & \text{AH}_3(\psi) \end{bmatrix}. \end{aligned}$$

The norm of this algebra is the cube root of the determinant. Also  $\{\text{AH}_3, \text{BH}_3, \text{CH}_3\}$  is one of the two sets of the 3-dimensional trigonometric functions—your author has named these functions the 3-dimensional hyper-trig functions. There are two sets of  $n$  functions associated with  $n$ -dimensional space for all positive integral  $n$ . They are  $n$ -way splittings of the exponential series:

$$\begin{aligned}\text{AH}_3(x) &= \frac{x^0}{0!} + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \dots, \\ \text{BH}_3(x) &= \frac{x^1}{1!} + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \dots, \\ \text{CH}_3(x) &= \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \frac{x^{11}}{11!} + \dots\end{aligned}$$

The other 3-dimensional hyper-trig functions are

$$\begin{aligned}\text{AE}_3(x) &= \frac{x^0}{0!} - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \dots, \\ \text{BE}_3(x) &= \frac{x^1}{1!} - \frac{x^4}{4!} + \frac{x^7}{7!} - \frac{x^{10}}{10!} + \dots, \\ \text{CE}_3(x) &= \frac{x^2}{2!} - \frac{x^5}{5!} + \frac{x^8}{8!} - \frac{x^{11}}{11!} + \dots\end{aligned}$$

These hyper-trig functions, together with the cube root of the determinant define a type of 3-dimensional space.

There are three other 3-dimensional algebras.

If you choose a non-Abelian group, then you will get a division algebra (watch out for restrictions) like the quaternions. Such algebras are not commutative and thus do not have a polar form.

### The nature of space

The  $n$ -dimensional spaces associated with the  $n$ -dimensional complex numbers are very different from the vector spaces we are used to dealing with. Vectors were invented by mathematicians. Complex numbers of all dimensions are already in mathematics and needed only to be discovered. Your author is currently investigating these newly discovered spaces but already has some results.

(1) In vector spaces, we move up a dimension by the addition of an extra term to the metric. In complex-number spaces, one moves up a dimension by multiplication.

(2) The metric of vector spaces is always a square root. The metric of an  $n$ -dimensional CN space is an  $n^{\text{th}}$  root.

(3) The rotation matrices of vector spaces are the same in two dimensions as the CN spaces (as are the metrics); however, in higher dimensions, the rotation matrices of CN spaces contain the hyper-trig functions and appear as angle matrices in the polar form of the algebra.

(4) In vector spaces, 2-dimensional space is a subspace of 3-dimensional space. The 2-dimensional complex numbers are not subalgebras of the 3-dimensional complex numbers because there is no order-2 group that is a subgroup of an order-3 group. Thus 2-dimensional CN space is not a subspace of 3-dimensional CN space.

(5) The 4-dimensional hyper-trig functions are 4-way splittings of the exponential series. We have

$$\begin{aligned} \text{AH}_4(x) - \text{CH}_4(x) &= \cos x, \\ \text{BH}_4(x) - \text{DH}_4(x) &= \sin x, \\ \text{AH}_4(x) + \text{CH}_4(x) &= \cosh x, \\ \text{BH}_4(x) + \text{DH}_4(x) &= \sinh x. \end{aligned}$$

Thus, we can assemble 2-dimensional space from bits of 4-dimensional H-type space. So 2-dimensional space is really 4-dimensional space—and upwards we go.

(6) Although I have not shown it, the higher-dimensional CN spaces are ‘folded up’ in the lower-dimensional CN spaces (string theory?). Perhaps the universe began as an unfolding of space which stopped when it got to prime numbered dimensions.

(7) The metrics of CN spaces are a mixture of hyperbolic and Euclidean metrics multiplicatively entwined. This is reminiscent of space–time rather than space alone.

The above is but a brief résumé of a 60-chapter work that your author has produced. The work has not yet been peer reviewed, and the reader should bear this in mind.

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† Technically, it is nonsense to write  $e^A$  unless  $A$  is a 1 by 1 matrix.

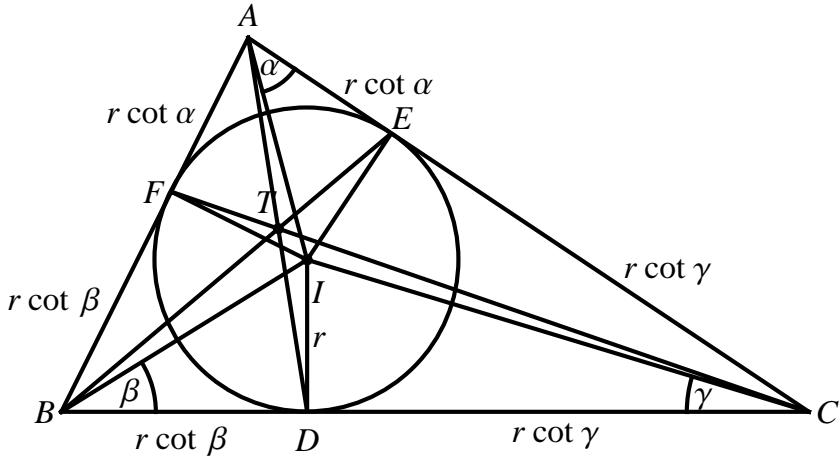
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## Solution 204.6 – A triangle property

John Bull

Given a triangle  $ABC$  with in-circle centre  $I$  and radius  $r$  meeting  $AB$ ,  $BC$ , and  $CA$  at  $F$ ,  $D$  and  $E$  respectively. Prove that  $AD$ ,  $BE$  and  $CF$  are concurrent at  $T$ . A *short* proof is requested.



Let the angles at  $A$ ,  $B$  and  $C$  be  $2\alpha$ ,  $2\beta$  and  $2\gamma$  respectively. Then  $AF = AE = r \cot \alpha$ ,  $BF = BD = r \cot \beta$ ,  $CD = CE = r \cot \gamma$ , and

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{r \cot \alpha}{r \cot \beta} \cdot \frac{r \cot \beta}{r \cot \gamma} \cdot \frac{r \cot \gamma}{r \cot \alpha} = 1.$$

Hence by the converse of the theorem of Ceva,  $AD$ ,  $BE$  and  $CF$  are concurrent at  $T$ .

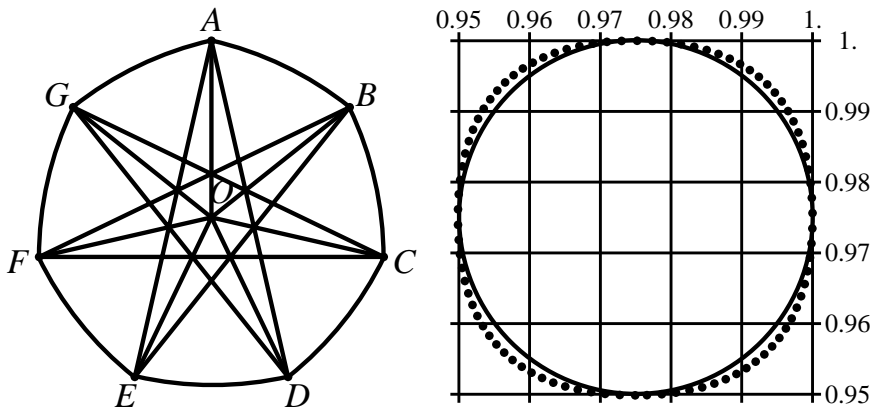
Proofs of Ceva's theorem can be found in a many books and on the Internet, even with music, graphics and brilliant animation. However, the neatest method uses the modern Fundamental Theorem of Affine Geometry—see *Geometry* by David Brannan, Matthew Esplen and Jeremy Gray (CUP 1999), p. 75.

## Solution 203.5 – 50p in a corner

What is the locus of the centre of a 50p piece lying flat in the  $(x, y)$ -plane such that its edge is in contact with both the positive  $x$ -axis and the positive  $y$ -axis?

### Bill Purvis

We are to determine the path traced by the centre of a 50p coin rotated in a square corner. The shape of a 50p coin is given by constructing a circle, then fix seven points around the circumference, equally spaced. Then, taking each point in turn as centre, draw an arc between the two points opposite. The seven arcs so constructed define the outline of the coin. We show this on the left of the diagram, below. I have shown the lines joining each point to the two opposite points, as well as lines joining each of the points to the centre.



In order to simplify the calculations, I have taken the radius of the circumcircle to be 1 unit, and all the calculations that follow will use this convention. We will concentrate on the path determined by 1/7th of a revolution, as there is obviously 7-fold rotational symmetry.

We begin by considering the ‘triangle’ determined by the topmost point and the two lower points:  $ADE$ . The side  $AD$  can be determined using the sine rule:

$$\frac{AD}{\sin AOD} = \frac{AO}{\sin OAD},$$

which, since  $AO = 1$ ,  $AOD = 6\pi/7$ , and  $OAD = \pi/14$ , gives a value of

$AD = 1.9499$ . The curved side  $DE$ , can now be determined as  $AD \cdot DAE = 0.8751$ .

We now begin to construct the path of the centre of the coin ( $O$ ). First we consider the distance from  $O$  to the  $x$ -axis. The distance from the  $y$ -axis will be similar, but with a phase difference of  $\pi/2$ . There are two distinct segments to the path: that generated when the coin is rotating about one of the points, and that generated when the coin is sliding along the curve. We first consider that of rotation about the point  $E$ .

Let  $\theta$  be the angle between the  $y$ -axis and line  $EO$ . Then the distance from the  $x$ -axis is given by  $y = \cos\theta$ . It will be seen that rotation takes place when  $\pi/14 \leq \theta \leq \pi/14$ . For  $\pi/14 \leq \theta \leq 3\pi/14$ , the coin is sliding with  $y = 1.9499 - \cos(\pi/7 - \theta)$ .

I generated a list of 20 values for  $y$  over the range  $-\pi/14 \leq \theta \leq 3\pi/14$ , which gave the following.

0.974928	0.98393	0.99095	0.995974	0.998993
1.	0.998993	0.995974	0.99095	0.98393
0.974928	0.965926	0.958906	0.953882	0.950863
0.949856	0.950863	0.953882	0.958906	0.965926

The  $x$  coordinate will give the same values but shifted by an angle of  $\pi/14$ , and plotting these points on a scrap of graph paper suggests that the resulting path is a good approximation to a circle. However, a more careful plotting reveals that the plots tend to be outside a circle drawn through the extrema. This is the right-hand diagram, opposite, with a circle of radius 0.025 for comparison.

## Tony Forbes

It is interesting to obtain an exact formula. Imagine the coin rotated by an angle  $\alpha$  from its position in the left-hand figure, opposite. Temporarily placing the origin at the centre of the coin, we ask ourselves where the horizontal and vertical edges of the restraining corner must be. Because of the seven-fold symmetry we can assume that  $-\pi/7 \leq \alpha \leq \pi/7$ .

At this point I strongly recommend that you trace the diagram on to card and cut it out. Denote the length of  $AD$  by  $r = (\sin \frac{6}{7}\pi)/(\sin \frac{1}{14}\pi) = 2 \cos \frac{1}{14}\pi$ .

If  $-\pi/14 \leq \alpha \leq \pi/14$ , then, as you can see by experimenting with the cut-out, the horizontal edge is tangent to the curve  $DE$ , and a little elementary trigonometry shows that the point of contact has  $y$  coordinate  $-r + \cos \alpha$ . See the top right and middle left diagrams on the opposite page. Otherwise the horizontal edge passes through a vertex,  $D$  if  $-\pi/7 \leq \alpha \leq -\pi/14$  (top left diagram), or  $E$  if  $\pi/14 \leq \alpha \leq \pi/7$  (middle right diagram) at  $y$  coordinate  $-\cos(\pi/7 - |\alpha|)$ .

If  $\alpha \geq 0$ , then the vertical edge is tangent to the curve  $FG$  and has  $x$  coordinate  $-r + \cos(\alpha - \pi/14)$  (middle left and middle right diagrams). Otherwise the vertical edge passes through vertex  $F$  at  $x$  coordinate  $-\cos(|\alpha| - \pi/14)$  (top left and top right diagrams).

Moving the origin to the corner and tidying up then gives the following expressions for the coordinates of the centre of the coin.

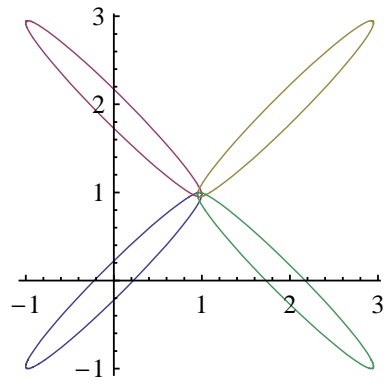
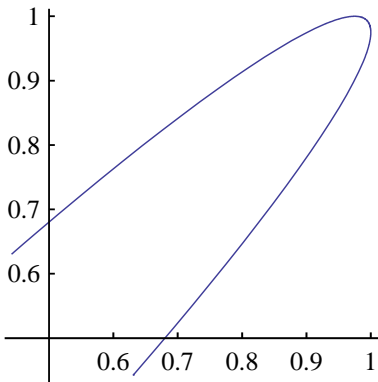
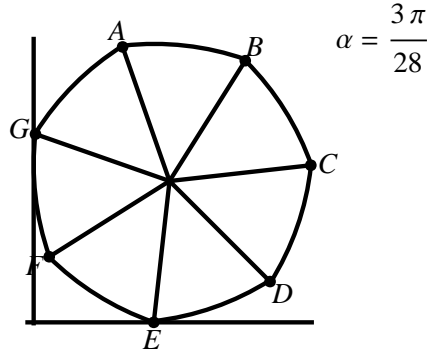
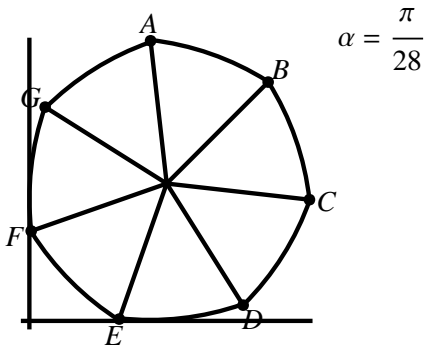
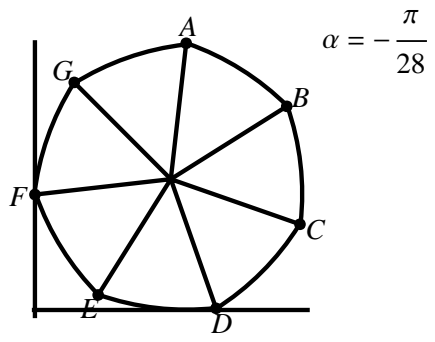
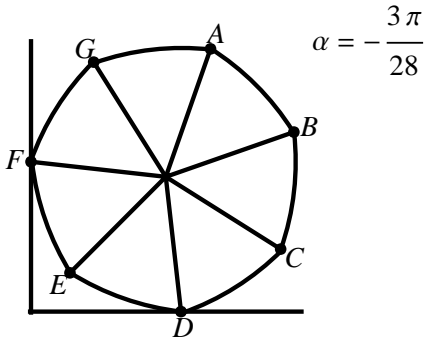
$\alpha$	$x$	$y$
$-\frac{\pi}{7} \leq \alpha \leq -\frac{\pi}{14}$	$\cos\left(\alpha + \frac{\pi}{14}\right)$	$\cos\left(\alpha + \frac{\pi}{7}\right)$
$-\frac{\pi}{14} \leq \alpha \leq 0$	$\cos\left(\alpha + \frac{\pi}{14}\right)$	$r - \cos \alpha$
$0 \leq \alpha \leq \frac{\pi}{14}$	$r - \cos\left(\alpha - \frac{\pi}{14}\right)$	$r - \cos \alpha$
$\frac{\pi}{14} \leq \alpha \leq \frac{\pi}{7}$	$r - \cos\left(\alpha - \frac{\pi}{14}\right)$	$\cos\left(\alpha - \frac{\pi}{7}\right)$

From this table you can see that the centre of the plot in the right-hand figure on page 8 is at  $(\cos \frac{1}{14}\pi, \cos \frac{1}{14}\pi)$  and therefore the radius of the circle passing through the extreme values (which occur when  $\alpha$  is an integer multiple of  $\pi/14$ ) is  $1 - \cos \frac{1}{14}\pi \approx 0.0250721$ .

The curves are actually segments of ellipses. For example, the first function in the table,

$$x = \cos\left(\alpha + \frac{\pi}{14}\right), \quad y = \cos\left(\alpha + \frac{\pi}{7}\right), \quad (1)$$

defines a somewhat elongated ellipse of radii  $\sqrt{2} \cos \frac{1}{28}\pi$  and  $\sqrt{2} \cos \frac{13}{28}\pi$ , centred at  $(0, 0)$  and with its major axis on the line  $y = x$ . A part of this curve is shown in the bottom left diagram on the next page. The other three curves are formed by rotating (1) through integer multiples of  $\pi/2$  about the point  $(\cos \frac{1}{14}\pi, \cos \frac{1}{14}\pi)$ . The bottom right diagram shows all four ellipses complete. The tiny bit in the middle where they collide is the locus of the centre of the 50p piece.



## Solution 203.1 – Circles

Generalize the thing on the cover of M500 **198** (reproduced opposite) so that the  $n$ th ring from the centre has  $k 2^n$  circles in it,  $k = 3, 4, \dots$ . Compute the radius of the limiting circle assuming that the central circle has radius 1. (We have changed the wording slightly—it is less perverse to assign the value 1 to the central circle rather than the limiting circle.)

### Norman Graham

For circles in the  $n$ th ring, denote the radius by  $x_n$  and the tangent from the centre by  $y_n$ , as shown in the diagram opposite. Let

$$t_n = \frac{x_n}{y_n} = \tan \frac{72^\circ}{2^n} \quad \text{and} \quad r_n = \frac{x_n}{x_{n-1}}.$$

Then

$$\left( \frac{x_n}{t_n} - \frac{x_{n-1}}{t_{n-1}} \right)^2 = (y_n - y_{n-1})^2 = (x_{n-1} + x_n)^2 - (x_{n-1} - x_n)^2 = 4x_n x_{n-1}.$$

Therefore

$$r_n^2 - 2r_n \left( \frac{t_n}{t_{n-1}} + 2t_n^2 \right) + \left( \frac{t_n}{t_{n-1}} \right)^2 = 0,$$

$$r_n = \frac{t_n}{t_{n-1}} + 2t_n + 2t_n \sqrt{\frac{t_n}{t_{n-1}} + t_n^2}.$$

(The negative sign is not used as it applies to the small circle.) Thus

$$x_1 = \frac{\sin 36^\circ}{1 - \sin 36^\circ} \approx 1.42590, \quad y_1 = \sqrt{(1 + x_1)^2 - x_1^2} = \sqrt{1 + 2x_1^2} \approx 1.962611.$$

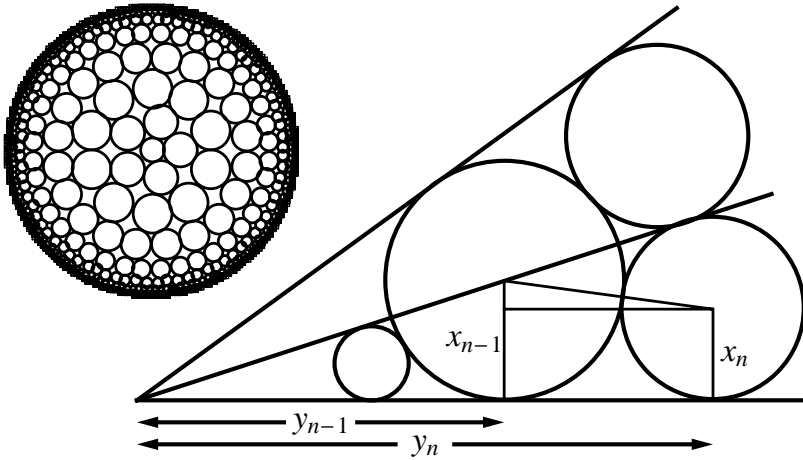
Successive values of  $x_n$  and  $y_n$  can be calculated (preferably by computer!) using  $x_n = r_n x_{n-1}$ ,  $y_n = y_{n-1} + 2\sqrt{x_n x_{n-1}}$ . The number of iterations required to find  $y_\infty$  is reduced by noting that as  $n \rightarrow \infty$ ,

$$t_n \rightarrow 0, \quad \frac{t_n}{t_{n-1}} \rightarrow \frac{1}{2}, \quad r_n \rightarrow \frac{1}{2} \quad \text{and} \quad y_{n+1} \rightarrow \frac{y_n - y_{n-1}}{2}.$$

Therefore

$$y_\infty - y_n \rightarrow (y_n - y_{n-1}) \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right) = y_n - y_{n-1}.$$

For  $k = 5$ , we have the values given in the first table on the next page.



$n$	$r_n$	$x_n$	$y_n - y_{n-1}$	$y_n$
1		1.425920		1.962611
2	1.141512	1.627705	3.046951	5.009561
3	0.764410	1.244234	2.846223	7.855784
...	...	...	...	...
15	0.500054	0.000470	0.001330	12.258470
16	0.500027	0.000235	0.000665	12.259135

To this level of accuracy (six decimal places),  $y_{16} - y_{15} = \frac{1}{2}(y_{15} - y_{14})$ . Therefore  $y_\infty = y_{16} + (y_{16} - y_{15}) = 12.259800$ . The next table gives the results for various  $k$ .

$k$	$x_1$	$x_2$	$x_3$	$y_\infty$
3	6.464102	12.558515	12.558515	98.643938
4	2.414214	3.359161	2.842404	24.939267
5	1.425920	1.627705	1.244234	12.259800
6	1	1	0.713790	7.854745
7	0.766422	0.696852	0.473438	5.772852
8	0.619914	0.524550	0.343324	4.602403
9	0.519803	0.415764	0.264293	3.867125
10	0.447214	0.341855	0.212270	3.368392

## Solution 205.2 – Ants

Let  $k$  and  $n$ ,  $1 \leq k < n$ , be a fixed integers. There are  $n$  ants,  $A_0, A_1, \dots, A_{n-1}$ , situated at the vertices of a regular  $n$ -gon of side 1 m, arranged in anticlockwise order. At time 0 the ants start walking at speed 1 m/s, ant  $A_i$  always heading in the direction of ant  $A_{i+k \pmod n}$ ,  $i = 0, 1, \dots, n-1$ . When do they meet?

### Ian Adamson

In a regular  $n$ -gon  $\{A_0, A_1, \dots, A_{n-1}\}$ , there exists a unique  $O$  such that all  $OA_i$  are equal,  $0 \leq i \leq n-1$ . Define  $m = \min\{k, n-k\}$ ,  $\theta_m = \text{angle } A_iOA_{i+k \pmod n} = 2m\pi/n$ ,  $0 < m \leq n/2$ .

Clearly  $\sin k\pi/n = \sin m\pi/n$ .

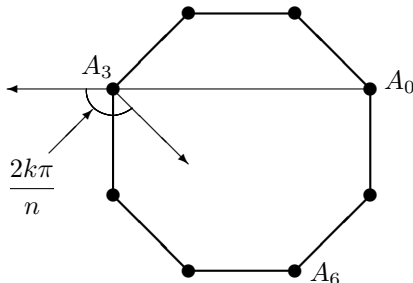
We have  $\sin m\pi/n = A_iA_{i+k \pmod n}/(2OA_i)$ . But  $A_iA_{i+k \pmod n} = 1$  when  $k=1$  (by definition) so  $OA_i = 1/(2 \sin \pi/n)$ . Hence (generally)

$$A_iA_{(i+k) \pmod n} = \frac{\sin m\pi/n}{\sin \pi/n} = l_k \quad (\text{say}).$$

We may think of the ants walking on sides of  $m$   $r$ -grams, clockwise only when  $m < k$  ( $r = n/\gcd(m, n)$ ), of side  $l_k$  each of which subtends an angle of  $2m\pi/n$  at  $O$ .

The component of the velocity of a pursued ant in the direction away from the pursuing ant is  $\cos 2m\pi/n$ ; so the side is decreasing with velocity  $v_{m,n} = 1 - \cos 2m\pi/n = 2 \sin^2 m\pi/n$ . (I am grateful to Professor Leo Moser for this idea.) Hence they (all) meet after time

$$t_{k,n} = \frac{l_k}{v_{m,n}} = \frac{1}{2 \left( \sin \frac{\pi}{n} \right) \left( \sin \frac{m\pi}{n} \right)} = \frac{1}{2} \left( \operatorname{cosec} \frac{\pi}{n} \right) \left( \operatorname{cosec} \frac{m\pi}{n} \right).$$



$$\begin{aligned} n &= 8 \\ k &= 3 \end{aligned}$$



## Solution 205.3 – Reciprocals

Show that if  $1 \leq m < n$ , the following expression cannot be an integer:

$$\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n}.$$

### Ian Adamson

Consider a set  $S$  of at least two consecutive positive integers. Then clearly there exists some  $u \in S$  such that  $\text{ord}_2(u) = r$  and  $\text{ord}_2(v) \leq r$  for all  $v \in S$ . Uniqueness of  $u$  follows since contrariwise existence of  $w$ ,  $\text{ord}_2(w) = r$ ,  $w \neq u$ , implies  $w = 2^r w_1$ ,  $w_1$  is odd as is  $u_1$ , where  $u = 2^r u_1$ . Now the existence of  $w$  means, since  $\text{ord}_2(2^r(w_1 + u_1)/2) \geq r$ , we may choose  $w$  so that  $|u_1 - w_1| > 0$  is as small as we like, so let  $|u_1 - w_1| = 1$  which is impossible as  $u_1, w_1$  are odd; so we have a contradiction.

Suppose now that  $\text{lcm}(S)$  is  $k$ . Also  $\text{ord}_2(k) = r$ ,  $\text{ord}_2(x) < r$ ,  $x \in S \setminus \{u\}$ . Thus

$$\sum_{t \in S} \frac{1}{t} = \frac{1}{k} \left( \sum_{t \in S} \frac{k}{t} \right). \quad (1)$$

Now  $k/u$  is odd and all  $k/x$  are even; so the numerator of (1), say  $N$ , is odd. Hence  $k \nmid N$  which implies the proof.

Did you know that the first two words are sufficient (and sometimes necessary) to identify a Shakespeare play? See how many you can recognize before you look them up.

As by Boatswain! / Here, Good day, I come I learn In sooth, I wonder Noble patricians, Now, say, Open your So shaken Two households, You do	As I Cease to Hence, home, If music I'll pheeze In Troy, Let fame, Now, fair O for Proceed, Solinus, To sing When shall	Before we Escalus! / My Hung be If you In delivering I thought Nay, but Now is Old John Sir Hugh, Tush, never Who's there?
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## Letter

### Primes and partitions

Tony,

I recently read *The Music of the Primes* by Marcus du Sautoy. This is a good history of mathematicians, but it is written by ‘a populist with staunch faith in the public’s intelligence’, to quote the back cover. So you don’t get much maths as such. However, he gives details of the continuing discussions over Riemann’s hypothesis, about which I knew almost nothing.

Here is one bit of maths he gives. Apparently it is a formula for generating primes, and which was discovered in 1976. He does not say who discovered it.

$$\begin{aligned}
 &(k+2)(1 - [wz + h + j - q]^2 \\
 &\quad - [(gk + 2g + k + 1)(h + j) + h - z]^2 \\
 &\quad - [2n + p + q + z - e]^2 \\
 &\quad - [16(k+1)^3(k+2)(n+1)^2 + 1 - f^2]^2 \\
 &\quad - [e^3(e+2)(a+1)^2 + 1 - o^2]^2 - [(a^2 - 1)y^2 + 1 - x^2]^2 \\
 &\quad - [((a + u^2(u^2 - a))^2 - 1)(n + 4dy)^2 + 1 - (x + cu)^2]^2 \\
 &\quad - [n + l + v - y]^2 - [(a^2 - 1)l^2 + 1 - m^2]^2 \\
 &\quad - [ai + k + 1 - l - i]^2 \\
 &\quad - [p + l(a - n - 1) + b(2an + 2a - n^2 - 2n - 2) - m]^2 \\
 &\quad - [q + y(a - p - 1) + s(2ap + 2a - p^2 - 2p - 2) - x]^2 \\
 &\quad - [16r^2y^4(a^2 - 1) + 1 - u^2]^2 \\
 &\quad - [z + pl(a - p) + t(2ap - p^2 - 1) - pm]^2).
 \end{aligned}$$

I am unable to attach another interesting formula (by Ramanujan) which calculates the number of ways that the elements of a set can be partitioned into subsets. My scanner won’t put it into a WORD format because it can’t cope with the sigmas and the big brackets around fractions.

Having read his book, though, I was able to follow the reasoning behind your prime definition (iv) (M500 200 pp 24–5), and the connection of the zeta function with primes and  $\pi^2/6$ . Without reading the book, I would not have followed it.

**Colin Davies**

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**ADF** — I should point out that the prime generating formula [J. P. Jones, D. Sato, H. Wada and D. Wiens, Diophantine representation of the set of prime numbers, *American Mathematical Monthly*, 1976] involves integer variables and yields primes only when it is *positive*. And that can happen only if all of the [...] terms are zero, in which case the prime is  $k + 2$ . Choosing  $a, b, \dots, z$  at random usually produces something which is large, negative and composite. In fact, we would be interested if anyone could tell us how to choose the parameters to get positive numbers.

I assume that Colin's unscannable formula is the main result in G. H. Hardy and S. Ramanujan, 'Asymptotic formulæ in combinatory analysis', *Proc. London Math. Soc.* **2** XVII (1918), 75–115. Let  $p(n)$  denote the number of partitions of a set of  $n$  elements. Then

$$p(n) = \sum_{q=1}^{\nu} A_q(n) \phi_q(n) + O(n^{-1/4}),$$

where

$$\phi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}} \frac{d}{dn} \left( \frac{\exp\left(\frac{2\pi}{\sqrt{6}q} \sqrt{n - \frac{1}{24}}\right)}{\sqrt{n - \frac{1}{24}}} \right),$$

$$A_q(n) = \sum_{(p)} \omega_{p,q} \exp\left(-\frac{2np\pi i}{q}\right),$$

$$\omega_{p,q} = \begin{cases} \left(\frac{-q}{p}\right) \exp\left[-\left(\frac{2-pq-p}{4} + \frac{(q^2-1)(2p-p'+p^2p')}{12q}\right) \pi i\right], & p \text{ odd,} \\ \left(\frac{-p}{q}\right) \exp\left[-\left(\frac{q-1}{4} + \frac{(q^2-1)(2p-p'+p^2p')}{12q}\right) \pi i\right], & q \text{ odd,} \end{cases}$$

$(a/b)$  is the Legendre/Jacobi symbol,  $p'$  is any positive integer such that  $1 + pp' \equiv 0 \pmod{q}$ ,  $\nu = \alpha\sqrt{n}$  and  $\alpha$  is any positive constant.

That symbol,  $(a/b)$ , which looks like something over something with a redundant pair of brackets, is not a fraction. In the simplest case, where  $b$  is an odd prime and  $a \not\equiv 0 \pmod{b}$ , it is the quadratic residue function:  $(a/b) = 1$  if  $a \equiv x^2 \pmod{b}$  is solvable for  $x$ , and  $(a/b) = -1$  otherwise. Look it up in a number theory book for a full explanation.



## Problem 207.2 – Parts of a partition

A *partition* of  $n$  can be represented as a vector  $(a_1, a_2, \dots, a_n)$ , where the  $a_i$  are defined by

$$n = a_1 + 2a_2 + 3a_3 + \dots + na_n.$$

Thus, for example, in this notation the eleven partitions of 6 are

$$(6, 0, 0, 0, 0), (4, 1, 0, 0, 0), (3, 0, 1, 0, 0), (2, 2, 0, 0, 0), \\ (2, 0, 0, 1, 0), (1, 1, 1, 0, 0), (1, 0, 0, 0, 1), (0, 3, 0, 0, 0), \\ (0, 1, 0, 1, 0), (0, 0, 2, 0, 0) \text{ and } (0, 0, 0, 0, 1).$$

What is the maximum possible number of non-zero elements  $a_i$  in a partition of  $n$ ? How many partitions of  $n$  use this maximum number?

## Problem 207.3 – Odds

Let  $Q$  be the set of integers defined by (i)  $1 \in Q$ ; (ii)  $n \in Q \Rightarrow 2n + 1 \in Q$ ; (iii)  $3n \in Q \Rightarrow n \in Q$ .

Either prove that  $Q$  is the set of positive odd integers, or determine the smallest odd integer not in  $Q$ .

## Problem 207.4 – Sextic

Solve

$$500x^6 - 13000x^3 = 77613.$$

## Balls

### Bob Newman

A and B play snooker. They play a complete frame which B wins with the minimum possible score, and B pots only one ball. *What colour is it?*

**ADF** writes — Bob Newman also has the most plausible answer we have heard to that other snooker problem [M500 203, p. 14]. *How can you score 162 in one visit to the table?*

A gives away 159 in fouls. B meanwhile makes no score and commits no fouls (0, 159). B commits a 4-point foul and leaves a free ball (4, 159). A pots the free ball (equivalent to a red) and takes a black (12, 159). A does a 147 clearance (159, 159). The black is re-spotted, A wins the toss and pots the black (166, 159). A has scored 162 in one visit.

## 81 cells revisited

### Tony Forbes

Continuing the problem we set in M500 206, have a look at the sudoku puzzle on this page. (Fill in the blanks to make a Latin square on  $\{1, 2, \dots, 9\}$  with the extra constraint that the nine  $3 \times 3$  boxes also contain  $\{1, 2, \dots, 9\}$ .) Notice that in the starter-digits there is a significant bias towards high numbers. There is no 1. The number 2 is not absent because a valid sudoku puzzle cannot omit two different digits. Exercise for reader—Why? But there is the next best thing: only one 2. And only one 3 and one 4. However, in this example each digit 5, 6,  $\dots$ , 9 appears more than once. We ask: Is this best possible? Can you have a sudoku puzzle with no 1s, and only one each of 2, 3, 4 and 5?

	9							
4			6					
8		6		9		7		
			2					9
			7		6			3
	5					8		
9		7		5				8
							7	
6	8	5			9			

We can represent a sudoku puzzle as a vector  $\mathcal{S} = (S_0, S_1, \dots, S_{80})$ , indexed by  $I = \{0, 1, \dots, 80\}$ , whose elements  $S_i$  are sets of integers. Certain subsets of  $I$  are called *regions*—if we imagine  $\mathcal{S}$  arranged as the familiar  $9 \times 9$  array, a region is precisely the set of indices of a row, column, or  $3 \times 3$  box. For  $J \subseteq I$ , define  $S_J = \bigcup \{S_j : j \in J\}$ . We say that  $\mathcal{S}$  is *inconsistent* if  $S_i$  is empty for some  $i \in I$ , and  $\mathcal{S}$  is *valid* if  $S_I \subseteq \{1, 2, \dots, 9\}$  and for each region  $R$  and each  $n \in \{1, 2, \dots, 9\}$  there is at most one  $i \in R$  such that  $S_i = \{n\}$ .

In its initial state (as published in *The Times*, for example),  $\mathcal{S}$  is valid and there is a set  $H \subseteq I$  such that  $|S_h| = 1$  for  $h \in H$  and  $S_i = \{1, 2, \dots, 9\}$  for  $i \in I \setminus H$ . You then transform  $\mathcal{S}$  until either (i)  $\mathcal{S}$  is inconsistent, or

(ii)  $|S_i| = 1$  for all  $i$ . The transformations must preserve the validity of  $\mathcal{S}$  and they must leave  $S_h$  unchanged for  $h \in H$ . But otherwise you can do anything you like—you can work logically or you can make changes entirely at random. The objective is to achieve (ii), and in a genuine sudoku puzzle this final state (with valid  $\mathcal{S}$ ) is unique.

Let us consider a specific transformation,  $\Phi$ , called the *critical set strategy* and defined as follows.

$\Phi$  If  $R$  is a region and  $P \subseteq R$  such that  $|P| = |S_P|$ ,  
then replace  $S_q$  by  $S_q \setminus S_P$  for each  $q \in R \setminus P$ .

Given an initial  $\mathcal{S}$ , we apply  $\Phi$  repeatedly until  $\mathcal{S}$  is stable and we note the number  $\phi(\mathcal{S})$  of indices  $i$  for which  $|S_i| = 1$ . This leads to a very interesting question: *What values can  $\phi(\mathcal{S})$  take?* There exist  $\mathcal{S}$  for which  $\phi(\mathcal{S}) = 81$ , and it is obvious that 80, 79 and 78 are impossible. But what about others?

Surprisingly,  $\phi(\mathcal{S})$  can be quite large, as in the example below. There are 70 numbers in the array but  $\Phi$  has no effect; you need other methods to complete it. This happens to be the largest in my collection of sudoku puzzles which are closed under  $\Phi$ . In fact I have examples for all values of  $\phi(\mathcal{S})$  from 22 to 70, and I would be particularly interested if someone can create a puzzle with  $\phi(\mathcal{S})$  outside this range.

3	9	5	2	1	6	7	4	8
8	4	7	9	3	5	1	6	2
1	6	2	8	4	7	9	3	5
	3		1	5	8	2	7	
	2	8	3	7	9		5	1
7	5	1	6	2	4	3	8	9
	7	3	5		2	8	1	
5	8		7		1		2	3
2	1	6	4	8	3	5	9	7

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Cover: A 24-digit sudoku puzzle which is closed under the critical set strategy (see p. 20).