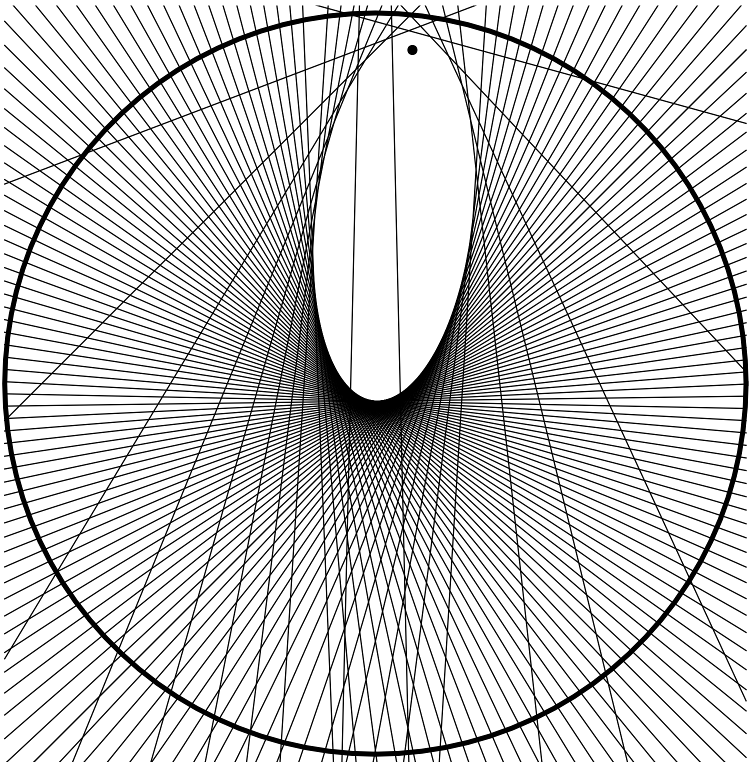
*

ISSN 1350-8539

M500 208



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail.

Drawing the harmonic series

Sebastian Hayes

A clear geometrical drawing should not be seen as a mere illustration: it demonstrates that a proposed theorem is true in at least this particular case. The challenge is now to generalize it, to show that the truth of the theorem does not depend on features specific to this one case, such as a triangle being right-angled for example. This approach is inductive rather than deductive and comes far more naturally to the vast majority of people, which is one of the main reasons why it does not find favour with the mathematical establishment. It is a completely different process from what passes as mathematical (i.e. algebraic) proof today. A modern theorem may well have been 'proved' and yet not one instance of it be presentable; this is the case for 'existence proofs', which mathematicians at one time viewed with distrust—Euclid's proofs are usually constructive. Worse still, the proved theorem may appear to be nonsense or plain wrong like the Banach-Tarski two-sphere theorem, which demonstrates how a sphere can be dissected in a particular way so as to eventually furnish two spheres each the size of the original one.

There are many elegant proofs of the famous inequality

$$\text{H.M.} \;\leq\; \text{G.M.} \;\leq\; \text{A.M.},$$

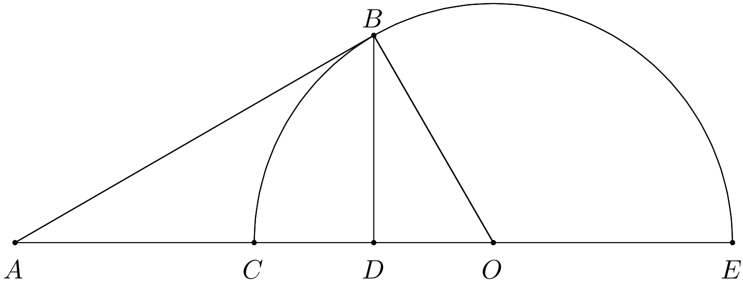
relating the harmonic, geometric and arithmetic means, but as far as I am concerned the most convincing one is an elementary geometric construction.

Ruler and compasses construction of the harmonic mean

We start as the Greeks did with the lengths of line segments; that is, with strictly positive quantities. Numerically, the harmonic mean, H.M., of two quantities a and b is defined as $2ab/(a+b)$.

Lengths $AC = a$ and $AE > AC = b$ are given. Draw AB tangent to circle centre O. Then

$$\begin{aligned} AO \;=\; AC + CO \;&=\; (2AC + 2CO)/2 \\ &=\; (AC + (AC + CE))/2 \\ &=\; (AC + AE)/2 \\ &=\; \text{A.M. of } AC \text{ and } AE. \end{aligned}$$



Since $AB^2 = AC \cdot AE$ by the secant theorem, AB is the G.M. of AC and AE. Also, since $AD/AB = AB/AO$ (similar triangles),

$$\begin{aligned} AD = AB^2/AO &= \frac{(\text{G.M.})^2}{\text{A.M.}} = \frac{AC \cdot AE}{(AC + AE)/2} \\ &= \text{H.M. of } AC \text{ and } AE. \end{aligned}$$

This diagram is perfectly general. Also, if there is a circle at all,

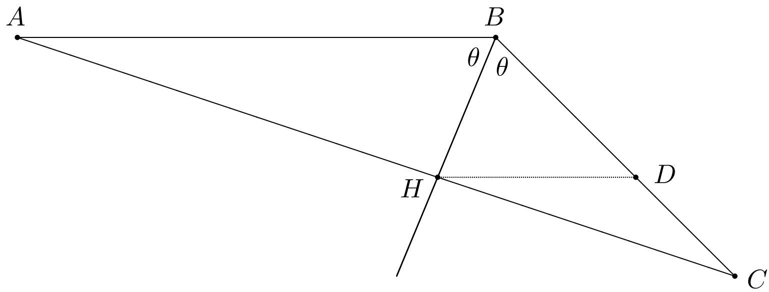
$$\text{H.M.} < \text{G.M.} < \text{A.M.}$$

As the constructed circle diminishes, AC/AE approaches unity and we may consider the degenerate case when $AC = AE$ as the mathematical limit, i.e. a circle of zero radius. In such a case, clearly H.M. = G.M. = A.M. No further argument is needed.

But this method does not readily generalize to the construction of a harmonic series with more than three quantities.

Construction by bisection

Lengths AB and BC are given. Draw BC at any angle to AB and bisect the angle ABC. Join AC meeting the angle bisector at H. Then HD drawn parallel to AB is $(AB \cdot BC)/(AB + BC) =$ H.M./2.

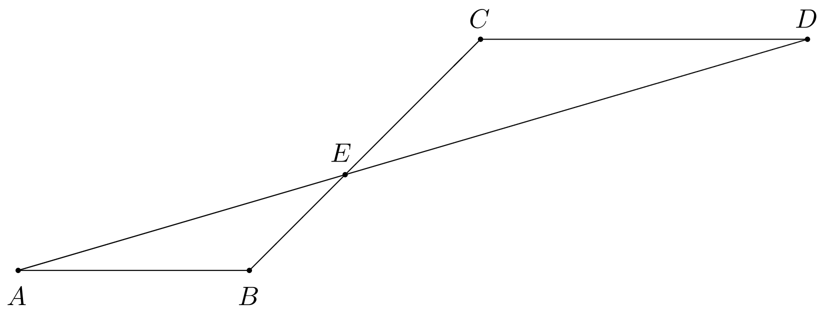


Angle $ABH = \angle HBD$ by construction; $\triangle ABC$ is similar to $\triangle HDC$. Therefore $HD/AB = DC/BC = (BC - BD)/BC$. But $\angle HDC = \angle ABC$ because AD is parallel to HD. Also $\angle HDC = \angle HBD + \angle BHD$ (exterior angle). Therefore $\triangle HBD$ is isosceles and $BD = HD$. Therefore $HD/AB = (BC - HD)/BC$. Hence

$$HD \;=\; \frac{AB \cdot BC}{AB + BC} \;=\; \frac{\text{H.M.}}{2} \text{ of } AB \text{ and } BC.$$

Construction of the H.M. by straight-edge and set square

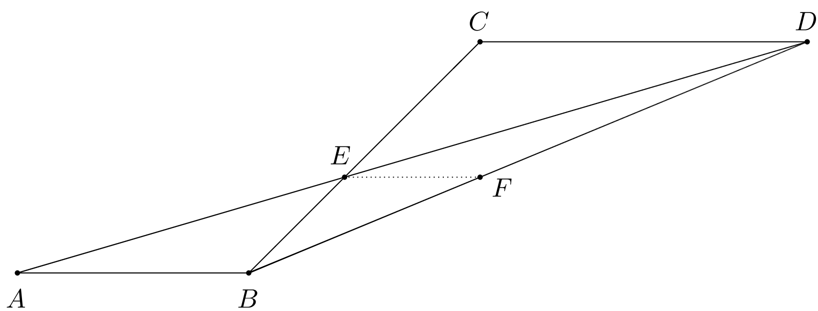
The following method only requires the use of a straight edge and some means of drawing parallel lines.



The given lengths are AB and BC. Draw CD parallel to AB and equal to BC. Then BE is the H.M./2 of AB and BC.

Proof. $\triangle ABE$ is similar to $\triangle CDE$ ($\angle ABE = \angle ECD$ and $\angle AEB = \angle CED$). Therefore $EB/EC = AB/CD$; $EB = EC \cdot AB/CD = (BC - EB) \cdot AB/BC$ (since $BC = CD$); $EB(BC + AB) = BC \cdot AB$; $EB = BC \cdot AB/(BC + AB) = \text{H.M.}/2$.

Also, if we join up BD and draw EF parallel to AB, then $EF = BE = \text{H.M.}/2$.

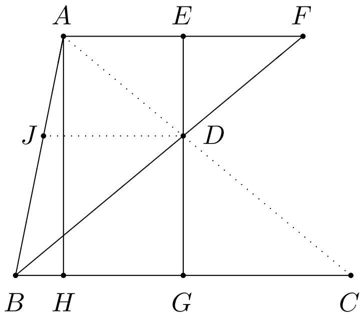


Since EF is parallel to CD, $\angle EFB = \angle CDB$. But $BC = CD$ by construction, therefore $\triangle BCD$ is isosceles. Also $\angle CDB = \angle DBC$. But $\triangle BEF$ is similar to $\triangle BCD$. Therefore $\triangle BEF$ is also isosceles. Therefore $BE = EF =$ H.M./2.

To inscribe a square in a triangle

This suggests a method for constructing an inscribed square in a triangle whose base angles are acute.

If the triangle is BAC with height AH, we draw $AF = AH$ parallel to BC.



We have $AH = EG = AF$ (construction); $ED/DG = AF/BC$ (heights of similar triangles); $ED/DG + 1 = AF/BC + 1$; $EG/DG = (AF + BC)/BC$; $DG = BC \cdot EG/(AF + BC)$ $DG = BC \cdot AH/(AH + BC)$ (since $AH = EG = AF =$ height).

Thus $DG =$ (height $\cdot$ base)/(height + base).

Also, if we draw a line through D meeting BA at J, $JD = DG$; $JD/BC = ED/AH$ (similar triangles); $JD = (BC \cdot ED)/AF = BC \cdot (ED/AF)$ (since $AH = AF$). But $ED/AF = DG/BC$ (similar triangles). Therefore $JD = DG =$ (height $\cdot$ base)/(height + base).

We recall that, for two quantities a and b,

$$\text{A.M.} = \frac{a+b}{2}, \quad \text{G.M.} = \sqrt{ab}, \quad \text{H.M.} = \frac{2ab}{a+b} = \frac{(\text{G.M.})^2}{\text{A.M.}}$$

Now H.M. $\leq$ G.M. $\leq$ A.M. with equality only occurring when $a = b$. Thus

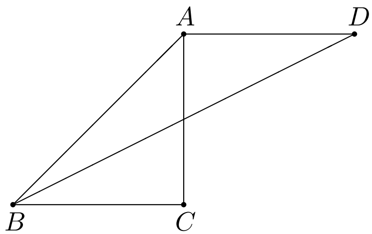
$$\begin{aligned} DG \cdot JD &= \text{(inscribed square)} \\ &= \frac{\text{height} \cdot \text{base}}{\text{height} + \text{base}} \leq \frac{\text{height} \cdot \text{base}}{4} = \frac{\text{area } \triangle}{2}. \end{aligned}$$

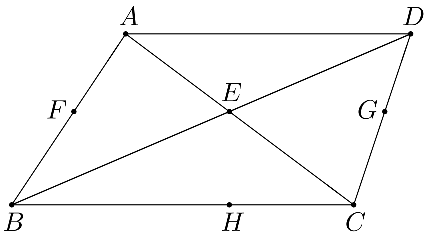
The area of the inscribed square is thus always less than half the area of the triangle except for a triangle whose height equals its base, when the area of the inscribed square is exactly half that of the triangle and a quarter that of the larger square.

Note that the two preceding constructions are not the same (except in the case of a right-angled triangle)—in the first we laid off a length equal to a side and in the second we laid off a length equal to the height. Here $CA = AD =$ height (left-hand diagram, below).

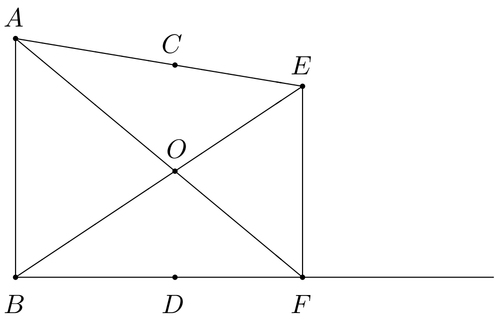
If we make the parallel line AD equal to AC—and not equal to the height—we can construct an inscribed rectangle but it will not be a square.

Thus $AC = AD$ and $\triangle ACD$ is isosceles; but since FEG is drawn parallel to BC, this means that angle $EGC =$ angle $ADC =$ angle DCA, which makes $\triangle EGC$ isosceles also. Therefore, $EG = EC = BC \cdot AC/(BC + AC) =$ H.M./2. But FE is also H.M./2 of BC and AD, or BC and AC. Therefore $FE = EG = EC$. However, $EH \neq FE$ (right-hand diagram).





Standard straight-edge and parallel line construction



If AB and EF are the given lengths, we simply draw them parallel and join the extremities. Then a line through the join parallel to AB is the H.M. Why? Since AB, CD and EF are parallel, triangles BOD and BEF are similar; likewise, triangles FOD and FAB.

With $OD = d$, $AB = a$, $EF = c$, $DF = n$, we have

$$\frac{m+n}{a} = \frac{n}{d}, \qquad \frac{m+n}{c} = \frac{m}{d}.$$

Adding,

$$\frac{m+n}{d} = \frac{m+n}{a} + \frac{m+n}{c}.$$

Dividing by $m + n$ gives

$$\frac{1}{d} = \frac{1}{a} + \frac{1}{c}, \qquad \text{or} \qquad d = \frac{ac}{a+c}.$$

Also, triangles EOC and EBA are similar; likewise triangles AOC and AFE. With $OC = e$,

$$\frac{e}{a} = \frac{EO}{EB} = \frac{n}{m+n} \qquad \text{and} \qquad \frac{e}{c} = \frac{BO}{BE} = \frac{m}{m+n}.$$

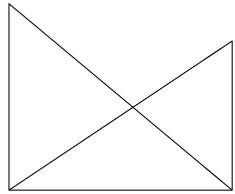
Adding, we have $e/a+e/c = 1$, or $1/e = 1/a+1/c$. Therefore $d = e = CD/2$.

Setting $CD = b$, $b = 2ac/(a + c)$, which is the definition of H.M., the reciprocals of the three lengths are in A.P. since $2/b = 1/a + 1/c$.

Note that the distance between a and c, BF in the above diagram, is irrelevant to the construction. This is very remarkable and means that a series of uprights forming a harmonic progression can be 'squashed together' or spread out at will. The three lines, representing the quantities a, b and c, do not need to be at right angles to the baseline but they must be parallel.

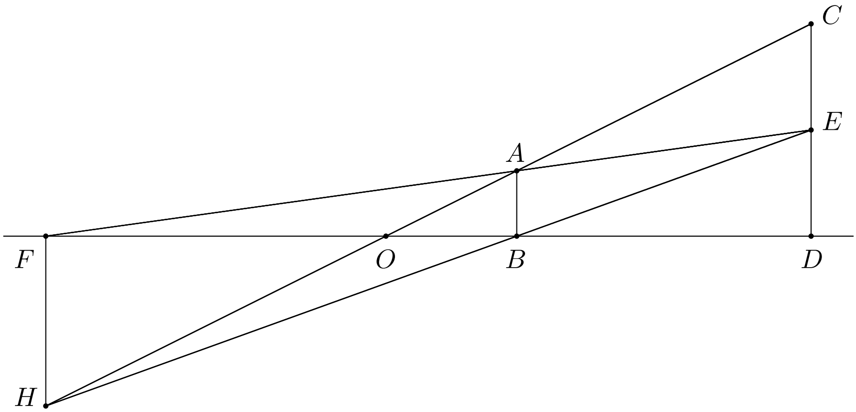
The 'Construction by Bisection' and the second 'Straight Edge and Set Square Construction' are special cases of the standard construction. In the first, bisecting the angle ABD means that the width between the uprights is equal to the second upright—in the standard construction diagram $EF = c = BF = m + n$. In the second we make one of the linking diagonals, BE in the standard construction diagram, equal to c.

Whenever we see two parallel lines with their extremities joined and the baseline drawn in, we have a harmonic progression with a and b and the half harmonic mean $ab/(a+b)$ in between. The half harmonic mean crops up in physics much more frequently than the harmonic mean itself: we come across the relation $1/f = 1/u + 1/v$ in optics for example.



'Negative' harmonic series

However, if, in a case where $a > b/2$, we produce the lines through the mid-point of b in the opposite direction we find they do meet the baseline and the diagonal produced so that we can define a joining line c.



Here $AB = a$ and $CD = b$ are the given lengths and are drawn parallel; $AB < CD/2$. Make $CE = ED = CD/2$. Join EB and produce to meet the line CAO produced at H. Join EA and produce to meet the baseline at F. Join FH.

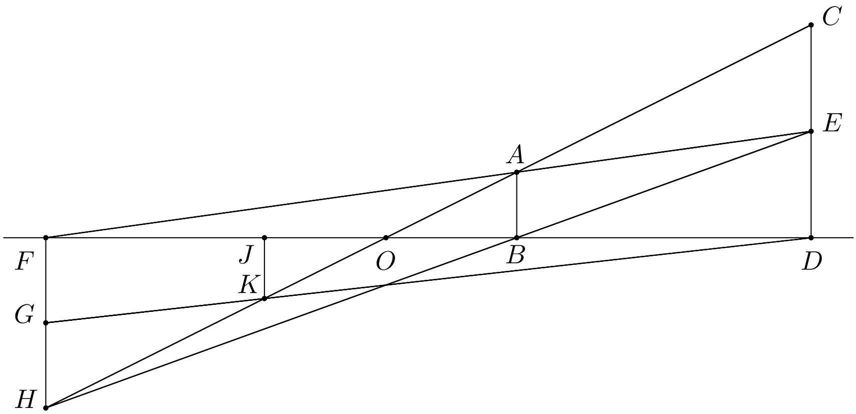
If FH is parallel to AB, we have a case of the standard construction and the lengths CE, $2AB$ and FH form an ascending harmonic progression. Therefore, AB is H.M./2 of CE and FH; i.e. $AB = FH \cdot CE/(FH + CE)$.

Since $CE = CD/2 = b/2$ and $AB = a$, $FH = 2a(b/2)/(2(b/2) - 2a) = ab/(b - 2a)$.

However, it must be shown first that FH is parallel to AB and CD. If $\triangle EBD$ is similar to $\triangle FBH$, $\angle HFB = \angle EDB = 90°$ and FH is parallel to AB and CD. Also $\angle FBH = \angle EBD$ since they are opposite angles. It is thus sufficient to show that two sides of the respective triangles are proportional; e.g. $FB/BD = HB/BE$.

Since $\triangle HCE$ is similar to $\triangle HAB$, $HE/HB = CE/AB$. But $CE = ED = CD/2$; therefore $HE/HB = ED/AB$. Also, since $\triangle AFB$ is similar to $\triangle FED$, $ED/AB = FD/FB$. Therefore $HE/HB = FD/FB$, or $(HB + BE)/HB = (FB + BD)/FB$, whence $BE/HB + 1 = BD/FB + 1$; $BE/HB = BD/FB$, as required.

If now we continue using the same construction method, i.e. drawing lines from the extremities of one upright through the mid-point of the next, we obtain an upright JK that is closer to O.



It can be shown in a similar way that JK is parallel to FH and CD.

This is a case of the standard construction on the baseline GKD, which means that FG, $2JK$ and CD form an increasing harmonic progression; i.e. $JK = FG \cdot CD/(FG + CD)$. Since, in terms of the original lengths a and b, FH is $ab/(b-2a)$ and CD is b, this means

$$JK \;=\; \frac{b \cdot \frac{1}{2}ab/(b-2a)}{b + \frac{1}{2}ab/(b-2a)} \;=\; \frac{ab}{2b-3a}.$$

This is the 'absolute' value—line segments are always positive—but if we consider lines below the baseline to be negative we obtain (after the first two terms a and b) a diminishing negative harmonic series:

$$\frac{ab}{2a-b},\quad \frac{ab}{3a-2b},\quad \frac{ab}{4a-3b},\quad \ldots,\quad \frac{ab}{(n+1)a-nb},\qquad n = 1, 2, 3, \ldots.$$

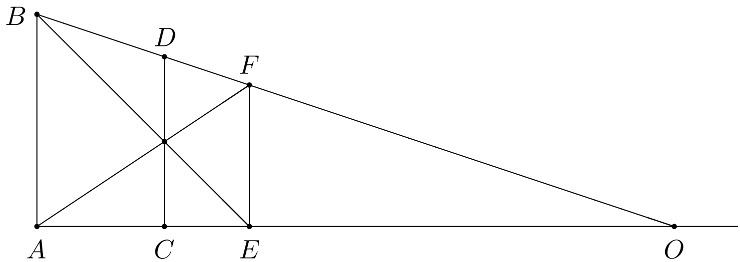
A numerical example will make this clear. If we set $a = 1$, $b = 3$, we have in effect the condition $0 < a < b/2$ and the next term, c, is $1 \cdot 3/(2-3) = -3$. The negative series proceeds -3, $3/(3-2\cdot 3) = -1$, $-3/5$, $-3/7$, $-3/9$, ..., or

$$\frac{-3}{1},\quad \frac{-3}{3},\quad \frac{-3}{5},\quad \frac{-3}{7},\quad \ldots,\quad \frac{-3}{2n-(n-1)},\qquad n = 1, 2, 3, \ldots.$$

This series diminishes to zero obviously like $ab/((n+1)a - nb)$, $a > b$.

Drawing an indefinitely extendable harmonic progression

Wherever we can spot the standard diagram, made up of three parallel lines and two linking diagonals, we know we have before us three lengths forming a harmonic progression. Given any two lengths, a and c, we can always construct an intermediate length, b, to form a harmonic progression with three terms. Also, if we have any two lengths, a and b, with $a > b$, we can very easily draw a third term, c, with $b > c$, to form a diminishing harmonic progression. All that is necessary is to define the so-called 'point at infinity' and to bisect the angle. Now join the top (or bottom) of the first upright to the mid-point of the second and produce it to meet the baseline (or slanting top line). An upright parallel to the first and second lines is the desired line.



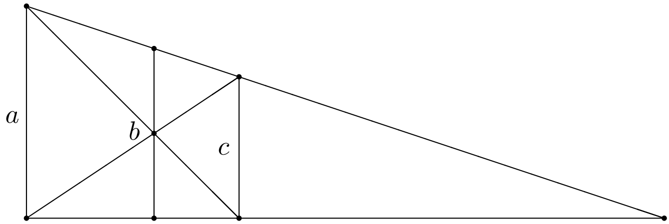
One can thus continue a diminishing harmonic progression indefinitely—within the limits of technical ability. Numerically, supposing $a > b > c > \ldots$, we have $c = ab/(2a - b)$ and using this relation repeatedly we derive the following series

$$a,\ b,\ \frac{ab}{2a-b},\ \frac{ab}{3a-2b},\ \frac{ab}{4a-3b},\ \ldots, \frac{ab}{na-(n-1)b}.$$

We may define it generally as $ab/(na - (n - 1)b)$, where $0 < b < a$ and $n = 0, 1, 2, \ldots$. This can be proved by induction:

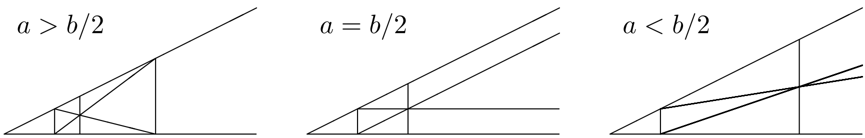
$$\begin{aligned} t(n+1) &= \frac{t(n-1)t(n)}{2t(n-1)-t(n)} \\ &= \frac{\dfrac{ab}{(n-1)a-(n-2)b}\cdot\dfrac{ab}{na-(n-1)b}}{\dfrac{2ab}{(n-1)a-(n-2)b}-\dfrac{ab}{na-(n-1)b}} \\ &= \frac{ab}{(n+1)a-nb} \;=\; t(n+1). \end{aligned}$$

To construct a third length c when a and b are given and $a > b > c$ it is only necessary to draw diagonals through the mid-point of b.



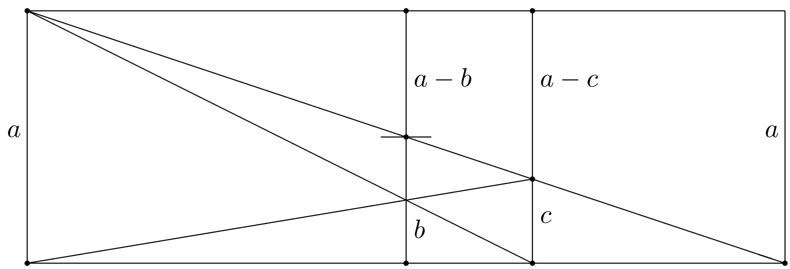
Then $c = ab/(2a-b)$ and the process can be continued indefinitely. The simplest example is what is often known as the Harmonic Series, namely $1, 1/2, 1/3, 1/4, 1/5, \ldots$.

It seems reasonable to suppose that the same method of construction would apply to an increasing harmonic progression but this is not always the case. By trial we find that only if $a > b/2$ can the third upright be drawn.



Clearly, in the rightmost picture lines through the mid-point of b will never meet the baseline or the diagonal line so c cannot be drawn.

Finally, from the next diagram, we can see that if a, b and c are in descending harmonic progression, then so are a, $a-c$ and $a-b$. The details are left to the interested reader.



Clock-watchers

Rob Evans

Recently, whilst looking through back copies of this magazine for interesting maths problems to solve, I came across a problem entitled 'Clock'. That problem (slightly reworded and generalized) is as follows.

> The hour-hand of a clock is h units long. Its minute-hand is m units long, where $m > h$. Determine the angle between the clock's hands when their outermost points are travelling apart from each other at the greatest speed.

If s is the distance between the outermost points of the clock's hands and t is the amount of time which has elapsed since an arbitrary fixed instant of time, the above problem can be succinctly restated as follows.

> Determine the angle between the clock's hands when ds/dt is maximized.

A solution to this problem appeared in M500 **170**. Essentially, it is as follows.

Let $t = 0$ at a given instant of time when the clock's hands are pointing in the same direction. In turn, let $\theta = \theta(t)$ be the clockwise angle through which the minute-hand has moved relative to the hour-hand since $t = 0$. Then the cosine rule yields

$$s \;=\; \sqrt{h^2 + m^2 - 2hm\cos\theta} \qquad (1);$$

see Figure 1. (Various well-known identities involving the cosine function guarantee that (1) holds for all possible values of θ.) Consequently, we have

$$\frac{ds}{dt} \;=\; \frac{ds}{dq}\frac{dq}{dt} \;=\; \frac{hm\sin\theta}{s}\frac{d\theta}{dt}. \qquad (2)$$

Next, we find an expression for d^2s/dt^2. On the reasonable assumption that the clock runs uniformly (i.e. $d\theta/dt$ is constant) we have

$$\frac{d^2s}{dt^2} \;=\; -\frac{hm}{s^3}\left(hm\cos^2\theta - (h^2+m^2)\cos\theta + hm\right)\left(\frac{d\theta}{dt}\right)^2.$$

From this expression, we deduce

$$\frac{d^2s}{dt^2} = 0 \;\Rightarrow\; hm\cos^2\theta - (h^2+m^2)\cos\theta + hm = 0$$

$$\Rightarrow \ (m\cos\theta - h)(h\cos\theta - m) = 0 \ \Rightarrow \ \cos\theta = \frac{h}{m} \text{ or } \frac{m}{h}.$$

In other words, since $m > h$ and $\cos\theta \le 1$, we must have

$$\frac{d^2 s}{dt^2} = 0 \ \text{ if and only if } \ \cos\theta = \frac{h}{m}. \tag{3}$$

To find unique value of θ at which ds/dt is maximized we shall now restrict θ to the interval $[0, 2\pi)$. From (2) and (3) and the reasonable assumption that the clock runs forwards (i.e. $d\theta/dt$ is always positive) we have

$$\frac{ds}{dt} \text{ is maximized if and only if } \theta = \arccos\frac{h}{m}. \ \text{ Q.E.D.}$$

(Figure 2).

This solution in M500 **170** was immediately followed by a contribution from the editor to the effect that the simplicity of the final result suggests that there ought to exist a correspondingly simple way of looking at the problem. I will now demonstrate that this is indeed so!

Let t and $\theta = \theta(t)$ be defined as before. Denoting the centre of the clock by O and the outermost points of the hour- and minute-hands by H and M respectively and then applying the sine rule to triangle OHM we have

$$\frac{\sin\theta}{s} = \frac{\sin(\pi - \phi)}{m} = \frac{\sin\phi}{m}, \tag{1'}$$

where $\phi = \phi(t)$ is the clockwise angle through which the line HM has moved relative to the line OH since $t = 0$. (See Figure 1′.)

However, recalling the earlier expression for ds/dt we have

$$\frac{ds}{dt} = \frac{hm\sin\theta}{s}\frac{d\theta}{dt}.$$

Consequently, on combining the last two equations we have

$$\frac{ds}{dt} = h\sin\phi\frac{d\theta}{dt}.$$

So, on the assumption that the clock runs uniformly and forwards we have

$$\frac{ds}{dt} = h(\sin\phi)\omega, \ \text{ where } \omega = \frac{d\theta}{dt} \text{ is a positive constant.} \tag{2'}$$

To find a unique values of θ and ϕ at which ds/dt is maximized we shall now restrict θ and ϕ to the interval $[0, 2\pi)$. From (2′) it is clear that ds/dt

attains its maximum value of ωh if, and only if, $\phi = \pi/2$. In other words, ds/dt attains its maximum value of ωh if, and only if, $\theta = \arccos(h/m)$. Q.E.D. (See Figure 2′.)

So, as well as having found the value of θ for which ds/dt attains its maximum value we have found what that value is. Moreover, in doing so we have found that this value is independent of m.

Figure 1

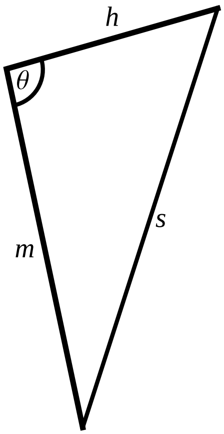


Figure 1'

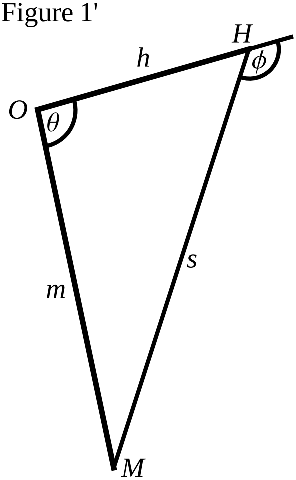
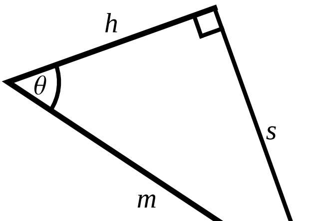


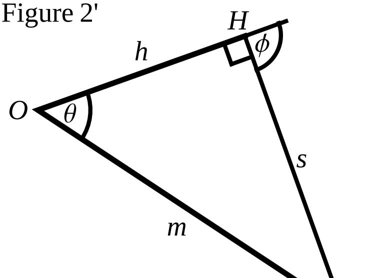
Figure 2



ds/*dt* is maximized

$\theta = \arccos(h/m) \in [0, 2\pi)$

Figure 2'



ds/*dt* is maximized

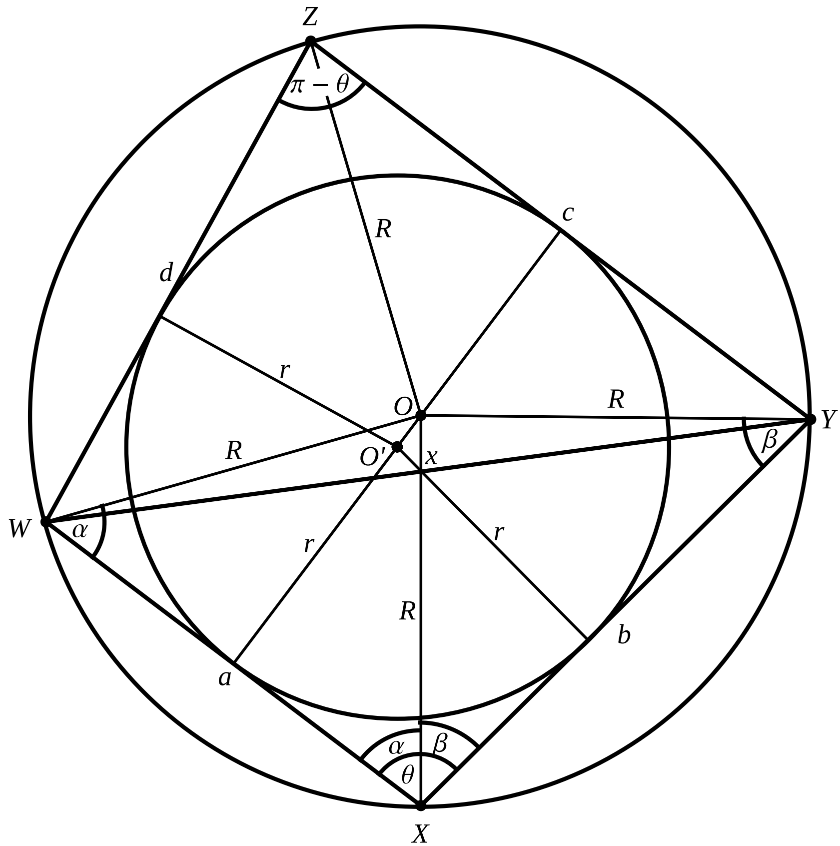
$\phi = \pi/2 \in [0, 2\pi)$

$\theta = \arccos(h/m) \in [0, 2\pi)$

Solution 203.4 – Cyclic quadrilateral

> A cyclic quadrilateral has sides a, b, c and d. Show that R, the radius of the circumcircle, is a function of a, b, c and d. Another cyclic quadrilateral, again with side lengths a, b, c and d, has a circle inside it which is tangent to all four sides. Show that r, the radius of the in-circle, is also a function of just a, b, c and d.

Note that the diagram below shows the special case pertaining to the second part of the problem, where there actually is an in-circle. This suggests a further question. What relation must hold between a, b, c and d for the in-circle to exist?



Ted Gore

Let $WXYZ$ be the quadrilateral. By the cosine rule,

$$x^2 \;=\; a^2+b^2-2ab\cos\theta \;=\; c^2+d^2+2cd\cos\theta;$$

hence

$$\cos\theta \;=\; \frac{a^2+b^2-c^2-d^2}{2ab+2cd}. \tag{1}$$

From a different perspective we have

$$\cos\theta \;=\; \cos(\alpha+\beta) \;=\; \sqrt{\left(1-\frac{a^2}{4R^2}\right)\left(1-\frac{b^2}{4R^2}\right)} \tag{2}$$

since $\cos\alpha = a/(2R)$ and $\cos\beta = b/(2R)$. Putting (1) and (2) together we get

$$R \;=\; \frac{\sqrt{a^2+b^2-2ab\cos\theta}}{2\sin\theta}.$$

Now to find r. Because the quadrilateral is cyclic, we already have formula (1) for $\cos\theta$. Let $\mathcal{A}$ denote the area of the quadrilateral. Then

$$\begin{aligned}\mathcal{A} \;&=\; (\text{area of } XYW) + (\text{area of } YZW)\\ &=\; \frac{ab\sin\theta}{2}+\frac{cd\sin(\pi-\theta)}{2} \;=\; \frac{ab+cd}{2}\sin\theta. \end{aligned}\tag{3}$$

Now the radii of the in-circle meet a, b, c and d at right-angles. So $\mathcal{A}$ is the sum of the areas of triangles XYO', YZO', ZWO' and WXO'. Thus

$$\mathcal{A} \;=\; \frac{r}{2}(a+b+c+d). \tag{4}$$

From (3) and (4) we get

$$r \;=\; \frac{(ab+cd)\sin\theta}{a+b+c+d}.$$

Problem 208.1 – 3 ratios

Show that $a/b = c/d = e/f = \alpha$ implies that

$$\alpha \;=\; \sqrt[3]{\frac{2a^2c+3c^3e+4e^2c}{2b^2d+3d^3e+4f^2d}}.$$

A brief introduction to Study numbers

Dennis Morris

Study numbers are a hyperbolic form of complex numbers. It seems, and I am far from certain about this, that Study numbers are named after Eduard Study, a German mathematician working around the year 1900. I have scanned through some of his works, but I have found nothing at all concerning these hyperbolic complex numbers and I wonder if they are erroneously attributed to him. I cannot claim to have scanned all of his works or any of them thoroughly because, although the maths is written in maths, the English is in German, and I do not speak German. Perhaps someone from the history of mathematics department can enlighten me.

Study numbers are similar to complex numbers, but instead of having $\hat{i} = \sqrt{-1}$ they have $\hat{r} = \sqrt{+1} \neq \pm 1$, where we have put a hat upon the 'imaginary' parts. They are written as $x + \hat{r}y$. The permissible values are restricted. The absolute value of the real part must be greater than the absolute value of the 'imaginary' part: $|x| > |y|$. Addition and multiplication are parallels of the complex numbers:

$$(a + \hat{r}b) + (x + \hat{r}y) \;=\; (a + x) + \hat{r}(b + y),$$

$$(a + \hat{r}b)(x + \hat{r}y) \;=\; (ax + \hat{r}^2 by) + \hat{r}(ay + bx) \;=\; (ax + by) + \hat{r}(ay + bx).$$

The conjugate of the Study number $x + \hat{r}y$ is $x - \hat{r}y$, and division is done with the conjugate as with complex numbers.

$$\frac{a + \hat{r}b}{x + \hat{r}y} \;=\; \frac{(a + \hat{r}b)(x - \hat{r}y)}{(x + \hat{r}y)(x - \hat{r}y)} \;=\; \frac{(a + \hat{r}b)(x - \hat{r}y)}{x^2 - y^2}.$$

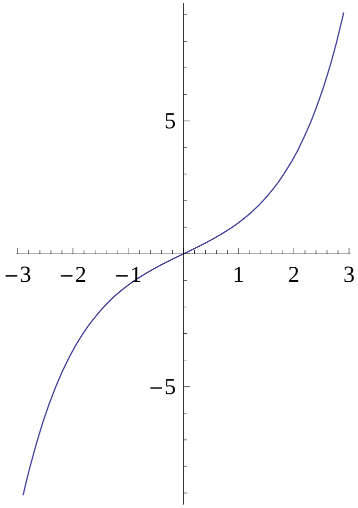
Such division is not defined when $|x| = |y|$, but these values are excluded.

The norm of Study numbers (with $|x| > |y|$) is $x^2 - y^2$. The polar form of Study numbers is $c(\cosh\chi + \hat{r}\sinh\chi)$, and so Study numbers can be thought of as the hyperbolic form of complex numbers. For the reader's convenience, we provide copies of the graphs of these two functions.

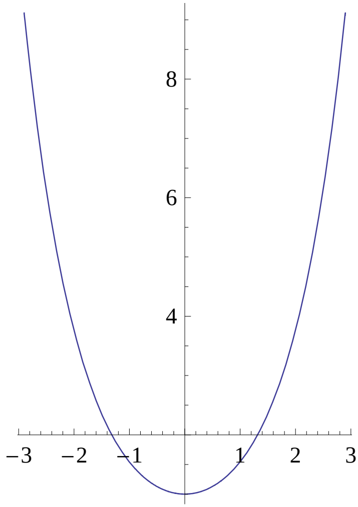
Thus $\cosh\chi$ and $\sinh\chi$ are the parameters of the hyperbola given by $x^2 - y^2 = c^2$, where c is a constant. The reader can compare this to the complex number case, where $\sin\theta$ and $\cos\theta$ are the parameters of the circle given by $x^2 + y^2 = c^2$, where c is a constant. Study numbers sit on a plane in a way similar to the way complex numbers sit on the Argand diagram, but the Study number plane is a hyperbolic space rather than a Euclidean one. The Study number plane is split into two halves by the

45° lines where $|x| = |y|$, and the Study numbers occupy two horizontally opposite quadrants. The real numbers lie on the horizontal x-axis. The lower illustration is an Argand diagram for Study numbers. It is a plot of hyperbolas with $c = 1, 2, 3, 4$. Also plotted are the 45° and minus 45° axes. The value of $\pm x$ where a hyperbola crosses the x-axis corresponds to the value of $\sqrt{c^2}$.

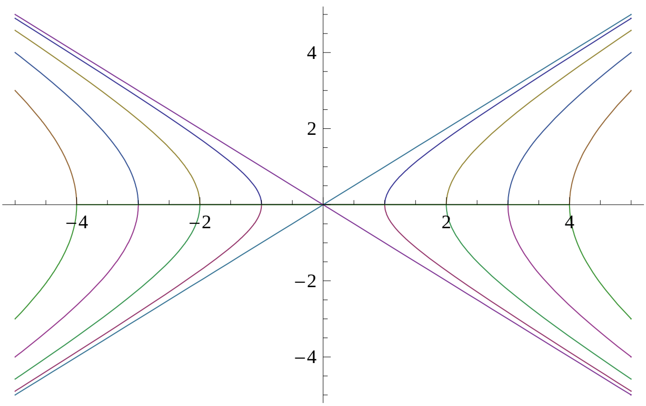
sinh χ



cosh χ



Argand diagram for the Study numbers



The vertically opposite quadrants are not part of the Study number space; they are the realm of numbers of the form $a + \hat{r}b$ with $|a| < |b|$.

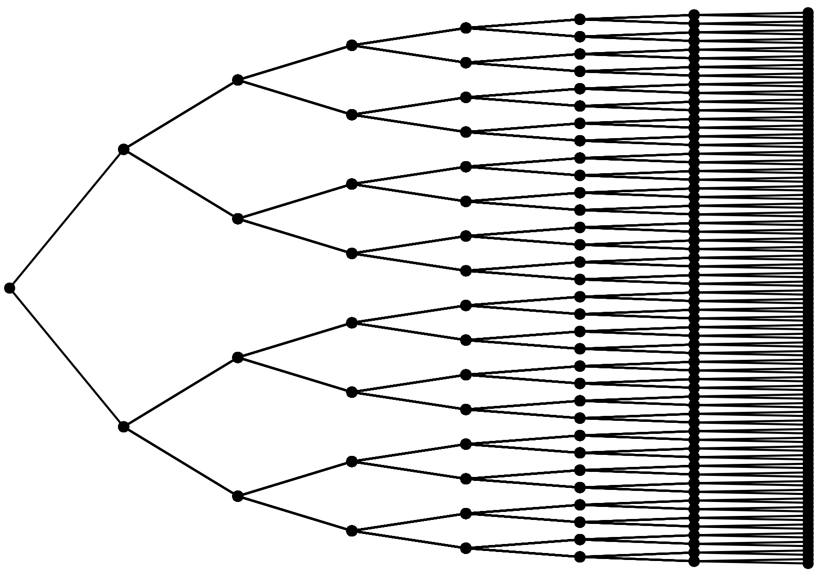
Unlike complex numbers, Study numbers are not algebraically closed; numbers with a negative real part have no square roots, but also unlike complex numbers, functions of them (like $\log_e$) are single valued. Complex numbers can be extended to become quaternions. Similarly, Study numbers can be extended to form a hyperbolic form of quaternion. As with complex numbers, Cauchy–Riemann like equations can be derived for Study numbers.

Study numbers with positive real part have four square roots, but only two of them are Study numbers.

Problem 208.2 – Binary tree

Ian Adamson

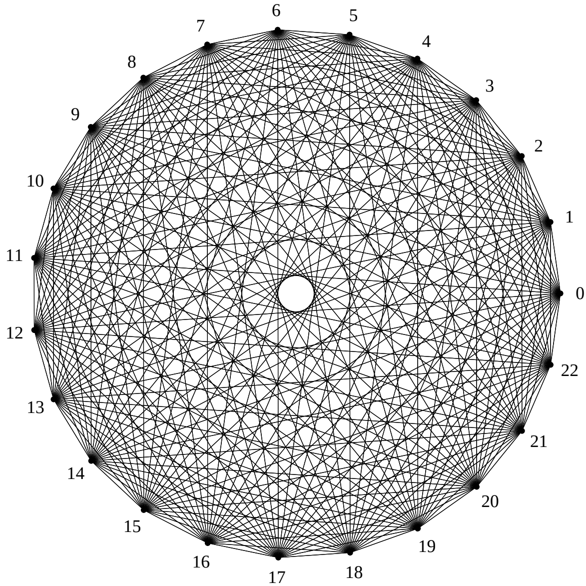
Imagine the picture below extended to infinity left, right, up and down. It is clear that there are infinitely many nodes; but what sort of infinity? Are the nodes countable or uncountable?



Problem 208.3 – Concentric circles

You have n points arranged in a circle with straight lines connecting every possible pair of points. There's an example on the right with $n = 23$. And another on the front of M500 **178**, this time with $n = 19$.

I have been looking these things for as long as I can remember and I have always been puzzled by the optical illusions they generate. So I ask a simple question: Where do all those concentric circles come from?



Problem 208.4 – Folding a polygon

Take a regular n-sided polygon and choose a point, X, in its interior. For each vertex V, draw the line that perpendicularly bisects XV. (You can achieve the same effect by folding the plane so that V coincides with X.) These lines define a polygon $\mathcal{P}$ with X in its interior. What is the maximum number of sides that $\mathcal{P}$ can have?

The case $n = 100$ is illustrated on the cover. The thing that looks like a circle is really a 100-sided polygon of 'radius' 1. The point has coordinates $(1/10, 9/10)$. Letting n tend to infinity, the picture becomes a solid black square with an elliptical hole. Can you determine the equation of the ellipse?

Problem 208.5 – Rain

You travel from A to B in the rain. Do you get wetter running or walking?

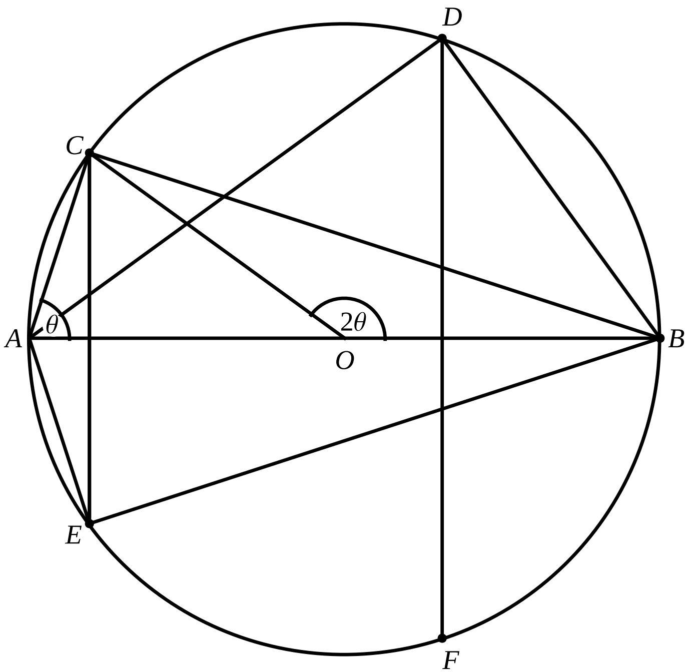
Product of cosines

Sebastian Hayes

In M500 **190**, p. 15, John Smith states the useful result which has cropped up in several problems recently:

$$\prod_{r=1}^{n-1} \cos \frac{r\pi}{n} = \frac{(-1)^{(n-1)/2}}{2^{n-1}} \quad \text{for } n \text{ odd.}$$

(Note the misprint in the magazine: $\pm n/(2n-1)$.) The result, obtainable by way of complex numbers, can be derived by elementary geometry alone.



Since AB is the diameter, $\angle ACB$ is a right angle and thus $AC = AB\cos\theta$ and $CB = AB\sin\theta$, where $\angle COB = 2\theta$.

If $\theta = 2\pi/5$, CB will be the chord joining the vertex of a pentagon placed at $(1,0)$ to the second vertex proceeding anti-clockwise. Then $AC = 2\cos 2\pi/5 = 2\sin \pi/10$ will be the side of a decagon and, more generally, the 'complementary' chords joining $(-1,0)$ to the vertices of a regular n-gon, n

odd, with a vertex placed at $(1, 0)$, are equal in length to the odd-numbered chords of a $2n$-gon, i.e. $2\sin r\pi/2n$, $r = 1, 3, 5, \ldots, n-2$.

This is evident simply by inspection of the figure but, for those who insist on formulae, we have, $\cos(\pi(n-r)/2n) = \sin(r\pi/2n)$. If n and r are both odd, $n - r$ is even and the 2 in $(n-r)\pi/(2n)$ drops out. Thus, for $n = 5$, $\sin \pi/10 = \cos 2\pi/5$, $\sin 3\pi/10 = \cos \pi/5$.

Now $AE = AC$ and $BC = BE$; so, applying Ptolemy's theorem to the cyclic quadrilateral $ACBE$, we have $2AC{\cdot}CB = AB{\cdot}CE = 2CE$ for a circle with unit radius, $= BD =$ side of pentagon $= 2\sin\pi/5$. Also $AD \cdot DB = BE{\cdot}CE = DF = BC$. Multiplying $(AC{\cdot}CB)(AD{\cdot}DB) = BD{\cdot}BC$. Hence $AC \cdot AD = 1$ (for circle with unit radius); i.e. $(2\cos 2\pi/5)(2\cos\pi/5) = 1$ and, by symmetry, $(2\cos 3\pi/5)(2\cos 4\pi/5) = 1$.

The same argument applies to the chords from $(-1, 0)$ to the vertices of every regular n-gon, n odd, with a vertex placed at $(1, 0)$. For example, taking the case of $n = 7$, $2n = 14$, we obtain, applying the same principle and using vertex notation, $v_6 v_1 = v_2$, $v_4 v_3 = v_6$, $v_2 v_5 = v_4$, $v_1 v_3 v_5 = 1$.

Thus we conclude that

$$\left(\cos\frac{\pi}{n}\right)\left(\cos\frac{2\pi}{n}\right)\ldots\left(\cos\frac{\pi(n-1)}{n}\right) \;=\; \frac{(-1)^{(n-1)/2}}{2^{n-1}}$$

for n odd, taking signs of cosines into account.

The sign of the product depends on whether $n-1$ is or is not divisible by 4. For although the cosines will be equally distributed in the two quarters ($(n-1)/2$ of them will be less than $n/2$, and $(n-1)/2$ of them will be greater than $n/2$), if an odd number of cosines are placed in the second quarter (where cos is negative), the resultant sign will be minus and this will occur every time $(n-1)/2$ is odd, i.e. when $n-1$ is not a multiple of 4. Thus, for $n = 7$, we have $(\cos 4\pi/7)(\cos 5\pi/7)(\cos 6\pi/7n) = (-1)^3/2^3 = -1/8$.

We cannot, incidentally, take an equivalent product for n even because $\cos\pi/2 = 0$ will be included and reduce the product to zero.

When $n = 13$, $\prod_{r=1}^{12} \cos r\pi/13 = 1/2^{12}$. Also from the main result of my article 'Product of regular polygon chords' [this M500, page 24], the product of the chords of a 13-gon is given by $\prod_{r=1}^{12} 2\sin r\pi/13 = 13$. Combining, and using the relation $\tan(\pi - \theta) = -\tan\theta$, we have

$$\left(\prod_{r=1}^{6} \tan\frac{r\pi}{13}\right)^2 \;=\; \prod_{r=1}^{12} \tan\frac{r\pi}{13} \;=\; 13,$$

which answers Problem 195.2.

Solution 200.3 – An arithmetic geometric mean

Traditionally the *arithmetic–geometric mean* of a pair of numbers $\{a, b\}$ is the common limit of the process $\{a, b\} \to \{(a + b)/2, \sqrt{ab}\}$. Here we adopt a slightly skewed alternative definition. Let $a_1 = a$, $b_1 = b$, and for $n > 1$ let $a_n = (a_{n-1} + b_{n-1})/2$, $b_n = \sqrt{a_n b_{n-1}}$. Show that

$$a_\infty \;=\; b_\infty \;=\; \frac{\sqrt{b^2 - a^2}}{\arccos a/b}.$$

Norman Graham

Let $c_n = b_n - a_n$, $f_n = \sqrt{b_n^2 - a_n^2}$, $\theta_n = \arccos a_n/b_n$ (assume principal values 0 to π) and $k_n = f_n/\theta_n$. The result is true only for $|a_1| < |b_1|$; otherwise $\arccos a_1/b_1$ does not exist.

We have

$$f_{n+1} \;=\; \sqrt{b_{n+1}^2 - a_{n+1}^2} \;=\; \sqrt{a_{n+1}(b_n - a_{n+1})} \;=\; \sqrt{\frac{b_n + a_n}{2} \cdot \frac{b_n - a_n}{2}}.$$

Therefore

$$f_{n+1} \;=\; \tfrac{1}{2}\sqrt{b_n^2 - a_n^2} \;=\; \tfrac{1}{2} f_n. \tag{1}$$

and hence

$$f_n \;=\; \frac{1}{2} f_{n-1} \;=\; \frac{1}{2^2} f_{n-2} \;=\; \ldots \;=\; \frac{1}{2^{n-1}} f_1.$$

As $n \to \infty$,

$$f_n \to 0, \quad b_n^2 - a_n^2 \to 0, \quad c_n = b_n - a_n \to 0. \tag{2}$$

Also

$$\begin{aligned}\cos 2\theta_{n+1} \;&=\; 2\cos^2\theta_{n+1} - 1 \;=\; \frac{2a_{n+1}^2}{b_{n+1}^2} - 1 \\ &=\; \frac{2a_{n+1}}{b_n} - 1 \;=\; \frac{a_n + b_n}{b_n} - 1 \;=\; \frac{a_n}{b_n} \;=\; \cos\theta_n.\end{aligned}$$

Therefore

$$2\theta_{n+1} \;=\; \theta_n, \qquad \theta_{n+1} = \tfrac{1}{2}\,\theta_n. \tag{3}$$

Dividing (1) by (3), $f_{n+1}/\theta_{n+1} = f_n/\theta_n$, or $k_{n+1} = k_n$. Hence

$$k_n \text{ is independent of } n. \tag{4}$$

For large n, $c_n \to 0$ and c_n^2 can be neglected. Therefore

$$f_n = \sqrt{(b_n + a_n)(b_n - a_n)} = \sqrt{(2b_n - c_n)c_n} \approx \sqrt{2b_n c_n}. \qquad (5)$$

Also as $b \to \infty$, $\cos\theta_n = a_n/b_n = 1 - c_n/b_n \to 1$. Therefore $\theta_n \to 0$ and $\cos\theta_n \approx 1 - \frac{1}{2}\theta_n^2$. Hence

$$\frac{\theta_n^2}{2} \approx \frac{c_n}{b_n} \quad \text{and} \quad \theta_n \approx \sqrt{\frac{2c_n}{b_n}}. \qquad (6)$$

Dividing (5) by (6), $k_n = f_n/\theta_n \to b_n$ as $n \to \infty$. Hence, using (2) and (4), a_n and $b_n \to k_1$ as $n \to \infty$.

A numerical example

n	a_n	b_n	f_n	θ_n	k_n
1	1	3	2.82843	1.23096	2.29774
2	2	2.44949	1.41421	0.61548	2.29774
3	2.22474	2.33441	0.707107	0.30774	2.29774
4	2.27958	2.30683	0.353553	0.15387	2.29774
5	2.29321	2.30001	0.176777	0.076935	2.29774
6	2.29661	2.29831	0.0883883	0.0384675	2.29774
7	2.29746	2.29788	0.0441942	0.0192337	2.29774
8	2.29767	2.29778	0.0220971	0.00961687	2.29774
9	2.29772	2.29775	0.0110485	0.00480844	2.29774
10	2.29774	2.29774	0.00552427	0.00240422	2.29774

Amendment for negative b_1

If b_1 is negative, a_2 is negative and θ_2 is in the range $(\pi/2, \pi)$. Hence $\theta_1 = \theta_2$ must be in the range $(\pi, 2\pi)$.

n	a_n	b_n	f_n	θ_n	k_n
1	1	-3	2.82843	1.91063	1.48036
2	-1	1.73205	1.41421	2.18628	0.64686
3	0.366025	0.796225	0.707107	1.09314	0.64686
4	0.581125	0.680225	0.353553	0.546569	0.64686
5	0.630675	0.654982	0.176777	0.273285	0.64686
6	0.642829	0.648877	0.0883883	0.136642	0.64686

There is no problem with negative a_1, since then a_2 is positive.

Product of regular polygon chords

Sebastian Hayes

We consider a regular n-gon, centre $(0,0)$, with one vertex placed at the point $(1,0)$. Then the complex numbers ω_0 $(=1)$, ω_1, $\ldots$, ω_{n-1} represent the positions of the vertices taken in order; i.e. $(\cos 0, \sin 0), (\cos 2\pi/n, \sin 2\pi/n), \ldots, (\cos 2\pi(n-1)/n, \sin 2\pi(n-1)/n)$.

The distances from $(1,0)$ to the other vertices taken in order are given by $|1-\omega_1|$, $|1-\omega_2|$, &c., and these complex numbers represent the lengths of chords from a vertex to each of the others. Thus

$$\prod \text{chords} \;=\; |1-\omega_1|\,|1-\omega_2|\ldots|1-\omega_{n-1}|\,.$$

(There are $n-1$ chords only for an n-gon because we exclude the distance from a vertex to itself.)

The distances $|1-\omega_1|, |1-\omega_2|, \ldots$ should be $(2\sin\pi/n)$, $(2\sin 2\pi/n)$, $\ldots$. Take

$$\begin{aligned} |1-\omega_1| &= \left|\left(1-\cos\frac{2\pi}{n}, \sin\frac{2\pi}{n}\right)\right| \;=\; \sqrt{\left(1-\cos\frac{2\pi}{n}\right)^2+\left(\sin\frac{2\pi}{n}\right)^2} \\ &= \sqrt{2\left(1-\cos\frac{2\pi}{n}\right)} \;=\; 2\sin\frac{\pi}{n}, \end{aligned}$$

since $\cos 2\theta = 1-2\sin^2\theta$.

We now look at the equation $z^n-1=0$, where z is complex. There are n roots and the expression can be written $(z-\omega_0)(z-\omega_1)\ldots(z-\omega_{n-1})$. But $z^n-1=(z-1)(z^{n-1}+z^{n-2}+\cdots+1)$ and moreover $1=(1,0)$ is a root of the complex equation. Therefore

$$(z-1)(z^{n-1}+z^{n-2}+\cdots+1) \;=\; (z-1)(z-\omega_1)(z-\omega_2)\ldots(z-\omega_{n-1}),$$

where we identify 1 with ω_0 and write out the second part in terms of its roots.

Setting $z=1$ in $z^{n-1}+z^{n-2}+\cdots+1=(z-\omega_1)(z-\omega_2)\ldots(z-\omega_{n-1})$, we obtain

$$(1-\omega_1)(1-\omega_2)\ldots(1-\omega_{n-1})=n.$$

Hence $\prod \text{chords} = n$.

Question. What's one-sided and swims in the sea?
Answer. Möbius Dick.

Letters to the Editor

Scepticism in mathematics

Tony,

I had always assumed that mathematics had no connection with physical law. But it had always seemed to me that various mathematical structures (if 'structures' is the right word) could be used as reasonably good, or often very good, mathematical models to describe various physical situations. So I found Sebastian Hayes's point about the apparent impossibility of different factorizations in other universes very intriguing.

I don't see how you can have a counting system in any universe that does not have the same prime numbers as we do. That suggests that the mathematics forces the physical laws, rather than describing them. However, any given number of tennis balls could probably be distributed fairly equally among any given number of bins in a universe that had a very large value for Planck's constant.

On another topic, and of no relevance to the above, I just read in our parish magazine that Neil Kinnock, when describing his experience of working as a European Commissioner, said it was: 'Like drawing a diagram using x and y coordinates, where x is the unknown quantity, and y is the question.'

Colin Davies

Pins

Over the years, as I have had bank cards replaced (often after a fraud) and their PINs changed, I have noticed that the four-digit numbers supplied have always had some childish mnemonic tricks in them. For example: (x, y, y, x) twice, (x, y, y, $x+1$ or $x+2$) three times and (x, y, (2-digit product of x and y)) twice.

These are seven instances out of—I think—ten, and the others have also been instantly memorisable in other ways, such as dates.

There are now 6-digit identification numbers for some actions on the web, and the same tendency is apparent. It seems unsafe: only a tiny subset of available numbers is being doled out.

Is this just my bank, or have you had a similar experience?

I have changed them, as one can, to what seem like bland numbers, but for all I know these may be madly significant. Is there anywhere a database of Really Boring 4- and 6-digit numbers, from which one might make a safe selection?

Ralph Hancock

Contents — M500 208 – February 2006