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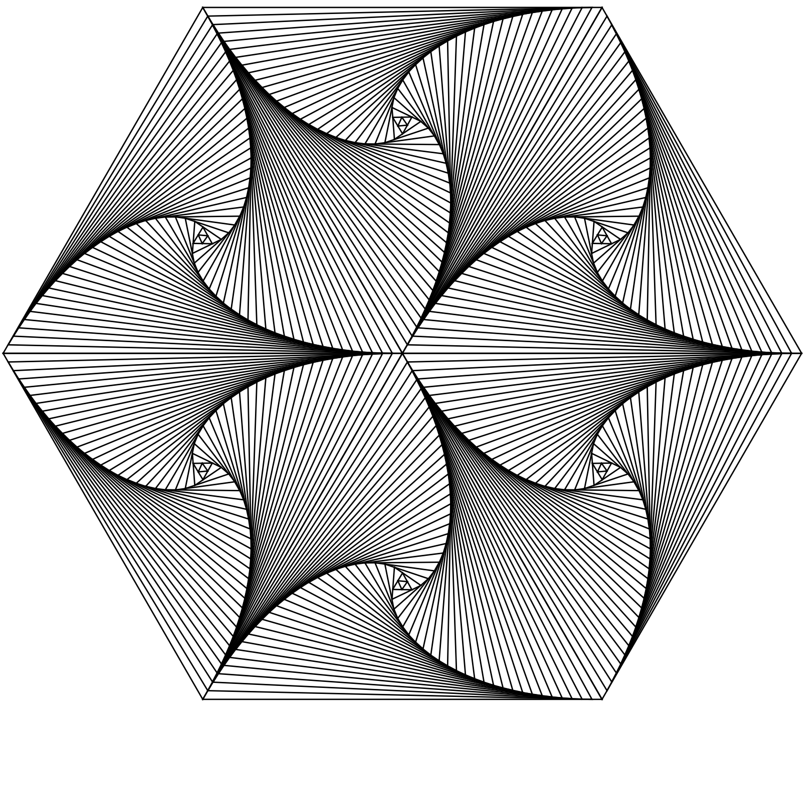
M500 211



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail.

What mathematics will be taught in 20 years time?

Dick Boardman

My schooldays were BC (before calculators) and I was taught to compute using logarithm tables. Books of tables included not only sine, cosine and tangent but also log sine, log cosine and log tan. A book of four-figure tables was about the size of the M500 journal but five-figure tables were half the size of a typical paperback book, and a set of seven-figure tables was a substantial volume. Geometry was taught using Euclid's methods but the rigorous derivation of theorems from Euclid's postulates and 'common notions' had been replaced by a less formal approach. The algorithms for 'long multiplication' and 'long division' were still taught but the equivalent algorithm for finding square roots had been dropped. In those days, money was in pounds, shillings, pence and farthings, and I vividly remember being taught a method for calculating compound interest to the nearest 0.25 d. Determinants were taught as a way of solving linear equations and were evaluated by hand. Quadratic equations were solved but cubics and quartics were only solved graphically. In the sixth form we were taught elementary calculus, and many homework hours were spent differentiating awkward expressions and evaluating integrals using various substitutions. Before logarithms were invented (by a Scotsman) all calculation had to be done by hand. Kepler, who used logarithms towards the end of his life, is said to have remarked that logarithms added years to the life of an astronomer by reducing the drudgery of calculation. My college course was very 'pure'. Nothing practical and nothing less than a hundred years old was included, although, in the world outside, there was a fad for so-called 'modern mathematics', and Boolean algebra was taught to primary school children. (Is it still?) The kids seemed to manage but their parents were terrified!

When I started work as an engineer, things changed dramatically. For rough numerical work, people used a 'guessing stick', more properly known as a slide-rule, which was an analog form of logarithms and gave an answer to three decimal places. More lengthy calculations were given to a young lady called a 'computor' (note the 'o') who worked on a noisy mechanical calculating machine. This was an electric powered device, about the size of a large typewriter, which could do twenty-digit addition, subtraction, multiplication and division. A long-division sum could take up to ten seconds. All results had to be read by eye and the computor gave you long columns of copper plate figures which you then plotted on to a graph. An awkward, nonlinear, differential equation could occupy her for a couple of days and she would solve it using numerical methods. I had to learn a whole range of new methods, some algebraic, like Laplace transforms, some numeric, such as Newton's method. Graphical methods were very popular as were things called nomograms. I also had to learn how much accuracy I really needed. Anything more than four decimal places took much longer and was therefore expensive.

The first revolution in calculation was the so-called analog computer (they were originally called analogue computers but the 'ue' got lost somehow). In these devices, a set of electric voltages were made to obey the same rules (equations) as physical quantities such as distance, velocity, acceleration, pressure and concentration. Operations like addition, subtraction, differentiation, integration and multiplication by constants were easy. Operations like multiplication and division of variables were much more difficult. Most of the time, they modelled linear differential equations, but they could do so on a quite heroic scale, say tenth order in nine or ten variables. The accuracy was quite low, only two or three decimal places, but the calculations were very rapid, with the analog system running at the same rate as the real system, sometimes much faster, whereas calculating the results by hand would have taken months. Physically, they were great racks of electronic components, full of hot glowing valves. They occupied large rooms and consumed kilowatts of power, which was turned into heat and had to be removed by equally large cooling systems.

Next came early digital computers. Physically, these were similar to the analog computers, with the same problems of power supply and heat. They gave less computing power than today's pocket calculators with storage capacities equivalent to tens or hundreds of bytes and cycle times in milliseconds. For historical reasons, they shared peripheral devices with the punched card systems used for censuses and the like, and a computer room would have been full of card readers, card punches, line printers and electric typewriters. They were hot, noisy and dusty. As I am sure you all know, internally, almost all computers use binary arithmetic and, as a programmer, I had to learn to count in binary (base 2), octal (base 8) hexadecimal (base 16) as well as the usual decimal and to change numbers from one base to another. The transfer of binary numbers in and out of memory, through adders, multipliers and dividers is controlled by binary numbers called instructions. Nowadays, computer instructions are generated by software in large groups, but in those days, the number controlling each individual transfer had to be created by hand, in binary. Programming was incredibly slow and incredibly error-prone. Furthermore, computers were very expensive to run, so that a single, misplaced binary digit could cost someone half a day's wages. Under these circumstances, one had to check one's work very very carefully!

Programming techniques and languages improved steadily over the years, but, for a long time, computing was about numbers. Computers were not much use to mathematicians since they could not prove anything, only examine lots of individual cases. However, they can disprove things. If someone had found a single counter-example to Fermat's Last Theorem, that would have killed it completely, and several similar conjectures were settled in just this way. In the 1980s I started to hear about programs which did algebra as opposed to arithmetic. Programs like MAXIMA and REDUCE could manipulate algebraic expressions, solve equations symbolically, carry out factorization, integration, differentiation and matrix operations. Over the years, these programs have been improved and expanded, and are now very widely used by engineers who have to do a lot of algebra. However, they are expensive and not much used in schools.

So, what will budding engineers be taught in the future. Calculators are now extremely common and may be taken into some examinations. I expect that long division, long multiplication, and the use of log tables will soon leave the curriculum if they haven't gone already. Metrication has greatly simplified the calculations required for many practical problems and also for school physics. Simple algebra, and the methods of solving quadratic equations will still need to be taught and so will techniques for solving simultaneous equations. Our honoured editor suggested in M500 issue **202** that the methods for solving cubic equations should also be included, but I don't think engineers have ever used these formulae. Graphical, followed by simple numerical methods are more reliable and less work. Decreases in the cost of computing power may start to bridge the gap between calculators and lap-tops. Maybe the use of spreadsheets will start to be taught in schools. Word processors are used by many children already. Simple, cheap symbolic calculation is now available. Much of the drudgery involved in learning calculus could be eliminated and students could concentrate on ideas rather than drill techniques.

What about proofs? Mathematicians have traditionally been very sniffy about proofs of theorems which involve computers. They consider that proofs like that of the four-colour map problem are decidedly inferior. They maintain that a humanly produced chain of reasoning, refereed and published in a suitable journal is the nearest thing to perfect truth that mankind has yet achieved. I wonder!

Velocity and time contraction in H-space

Dennis Morris

Study numbers

The Study numbers are

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} : |a| > |b|.$$

We *assume* that the real number within the study number matrix, the a, manifests itself to our senses as time and that the other number, the b, manifests itself as a spatial distance. Velocity thus becomes

$$v = \frac{b}{a}.$$

Since $|a| > |b|$, there is an upper limit of 1 upon velocity.

We *assume* that the distance within this space–time is invariant under velocity. That is, that all observers see the same space–time distance regardless of their velocity. We have, for two people, Jane and Fred,

$$d^2_{\text{Jane}} = a^2_{\text{Jane}} - b^2_{\text{Jane}} = d^2_{\text{Fred}} = a^2_{\text{Fred}} - b^2_{\text{Fred}}.$$

From Jane's point of view, she is standing still, and so the distance between two points in this space time is purely time. Thus we have

$$a^2_{\text{Jane}} = a^2_{\text{Fred}} - b^2_{\text{Fred}} = a^2_{\text{Fred}} - v^2 a^2_{\text{Fred}} = a^2_{\text{Fred}}(1 - v^2),$$

leading to

$$a_{\text{Fred}} = \frac{a_{\text{Jane}}}{\sqrt{1 - v^2}}.$$

So time slows down when you are moving through hyperbolic space.

Complex numbers

If we had done the above with the complex numbers, we would have arrived at

$$a_{\text{Fred}} = \frac{a_{\text{Jane}}}{\sqrt{1 + v^2}}.$$

So time speeds up when you are moving through euclidean space.

$L^1 H^2_1$

The $L^1 H^2_1$ algebra is

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}.$$

We again *assume* that the real number, the a, manifests itself as time. If we assume $v_b = b/a$ and take the change in the c direction to be zero, we get

$$a_{\text{Fred}} = \frac{a_{\text{Jane}}}{\sqrt[3]{1+v^3}},$$

and similarly for $v_c = c/a$. So, in three-dimensional H-space time speeds up for moving observers. If we had done this in the $L^1E_1^2$ algebra, there would have been a minus sign under the cube root instead of a plus.

$L^1H_3^3$

The $L^1H_3^3$ algebra is

$$L^1H_3^3 = \begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}.$$

This algebra has a two-dimensional subalgebra:

$$\text{SUB}_{[L^1H_3^3]}2_1 = \begin{bmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ c & 0 & a & 0 \\ 0 & c & 0 & a \end{bmatrix}.$$

The determinant of this subalgebra is

$$\det\left(\text{SUB}_{[L^1H_3^3]}2_1\right) = a^4 - 2a^2c^2 + c^4 = (a^2-c^2)^2.$$

Taking a to represent time and $v_c = c/a$, we get

$$a_{\text{Fred}} = \frac{a_{\text{Jane}}}{\sqrt[4]{(1-v_c^2)^2}} = \frac{a_{\text{Jane}}}{\sqrt{1-v_c^2}},$$

as is the case for the Study numbers. With $v_b = b/a$, taking $c = d = 0$, we get

$$a_{\text{Fred}} = \frac{a_{\text{Jane}}}{\sqrt[4]{1-v_b^4}} = \frac{a_{\text{Jane}}}{\sqrt[4]{(1-v_b^2)(1+v_b^2)}},$$

with similar result for $v_d = d/a$, taking $c = b = 0$. Time slows down for moving observers in four-dimensional H-space.

$L^1H_3^4$

The $L^1H_3^4$ algebra is

$$L^1H_3^4 \;=\; \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}.$$

This algebra has three two-dimensional subalgebras:

$$\mathrm{SUB}_{[L^1H_3^4]}2_1 \;=\; \begin{bmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & a \end{bmatrix}.$$

The determinant of the b subalgebra is

$$\det\left(\mathrm{SUB}_{[L^1H_3^4]}2_1\right) \;=\; a^4 - 2a^2b^2 + c^4 \;=\; (a^2-b^2)^2.$$

Taking a to represent time and $v_b = b/a$, we get

$$a_{\mathrm{Fred}} \;=\; \frac{a_{\mathrm{Jane}}}{\sqrt[4]{(1-v_b^2)^2}} \;=\; \frac{a_{\mathrm{Jane}}}{\sqrt[2]{(1-v_b^2)}},$$

as is the case for the Study numbers. The same happens for the other two subalgebras.

$L^1H_4^1$

The $L^1H_4^1$ algebra is

$$L^1H_4^1 \;=\; \begin{bmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ c & d & e & a & b \\ b & c & d & e & a \end{bmatrix}.$$

This algebra has no subalgebras. With $v_b = b/a$, taking $c = d = e = 0$, we get

$$a_{\mathrm{Fred}} \;=\; \frac{a_{\mathrm{Jane}}}{\sqrt[5]{1+v_b^5}},$$

with a similar result for each of the other directions. So time speeds up for moving observers in five-dimensional H-space.

Note: This work has not been peer reviewed, and the reader should bear this in mind.

Problem 211.1 – A to B

ADF

I walk from A to B, a distance of d, at constant speed w. However, there are bus-stops at A and B; so while I am walking I constantly scan the horizon at distance h behind me. If a bus appears and if it reduces my journey time to do so, I return to A, catch the bus and continue my journey at speed b. The mean time between buses is t. *What is my expected journey time?*

In my typical application $d = 1$ mile, $h = 0.5$ miles, $w = 4$ mph, $b = 30$ mph and $t = 10$ mins. (In rural areas larger values would be more realistic, say $d = 100$ miles, $t = 24$ hours.) On one occasion the bus was full and didn't stop, thus causing me to walk part of the route three times. I imagine you can reasonably ignore this additional complication.

Problem 211.2 – Pond

A fence of height f surrounds a flat disc of radius R in which there is a central pond of radius r. You look through a hole in the fence at height h. What proportion of the top of the fence can you see reflected in the pond?

Problem 211.3 – Trigonometric limit

Find n given that

$$0 < \lim_{\theta \to 0} \frac{\theta^n \sin^n \theta}{\theta^n - \sin^n \theta} < \infty.$$

Problem 211.4 – Trigonometric product

Compute

$$\left(\sin\theta \, \cos\tfrac{1}{2}\theta\right)^{1/2} \left(\sin\tfrac{1}{2}\theta \, \cos\tfrac{1}{4}\theta\right)^{1/4} \left(\sin\tfrac{1}{4}\theta \, \cos\tfrac{1}{8}\theta\right)^{1/8} \ldots .$$

Problem 211.5 – Product

Prove that

$$\frac{\left(\dfrac{\pi^2}{4} + 1\right)\left(\dfrac{\pi^2}{4} + \dfrac{1}{9}\right)\left(\dfrac{\pi^2}{4} + \dfrac{1}{25}\right)\cdots}{\left(\dfrac{\pi^2}{4} + \dfrac{1}{4}\right)\left(\dfrac{\pi^2}{4} + \dfrac{1}{16}\right)\left(\dfrac{\pi^2}{4} + \dfrac{1}{36}\right)\cdots} = \frac{e^2 + 1}{e^2 - 1}.$$

An interesting relation between π^2 and e^2, don't you think?

Deal or no deal

Dick Boardman

Re: M500 **210**, *Deal or no deal*. I am very sceptical about this game. This is a real situation. Someone, somewhere, has devised this programme and persuaded a sceptical television producer to stage it. The producer will wish to have a maximum of control over the game. He will wish to ensure that it is exciting television and that it runs to time. How will the creator have tried to convince the producer of the game's value?

The perfect game, from the producer's point of view is as follows: (i) Lasts 7 rounds. (ii) Has C (the contestant) choosing between £0.01 and £250000.00 with the other high value boxes being eliminated late on. (iii) Has C apparently having difficulty making his choices and with the audience excited. (iv) Allows Noel Edmonds to finish the program at precisely the right moment with a popular contestant winning.

The worst possible game: (i) Lasts 7 rounds. (ii) Has C choosing between £0.01 and £0.10 with all the high value boxes eliminated early on. (iii) Has C either choosing very quickly or very very slowly and looking like a computer. (iv) Finishes just too late to start another game so that Noel Edmonds has to flannel his way through several minutes without excitement.

Within these limits, the game's design gives the producer a number of ways of keeping the situation under control. (i) If I were in charge, I would choose the contestants. I would avoid people who look wealthy and also probably tramps. I would avoid dull looking people who behave like computers. I would avoid smart alecs and people who look too clever. (ii) I would pre-record several sets of 45 minute programs in front of one audience, possibly switching the order to produce programmes of the right lengths. (iii) I would record one or two extra games so as to eliminate dull ones. (iv) I would like to control the banker's offers. This opens up further possibilities for controlling the game. If a session looks dull or threatens the time schedule, I would advise the banker to make C an offer he can't refuse, possibly an offer substantially greater than the average of the remaining boxes. Conversely, if the game is entertaining, nicely on schedule and with a popular contestant, I would make low offers that C will refuse.

How therefore should a potential contestant behave? To get selected, you should look like an average viewer, slightly exhibitionist and with a passion for gambling. Once playing, you should endeavour to have the banker make you an offer you can't refuse, either by behaving unpredictably or by making the game dull.

Cynics for ever!

ADF writes — Let's dispose of another strategic decision which might affect your play. Recall that if the game lasts the distance, the banker may allow you to exchange your box for the one remaining unopened box. Well, do you swap? Probabilistically speaking, it makes no difference. You can argue that the boxes are always opened at random. Or you can, as I did, play 1,000,000 games to the end, swapping in half of them but not in the other half, and notice that in each case your winnings amount to about £12,800 M.

It's not the same as that other well-known game, where there are three closed boxes, one containing a car and two containing goats. The host knows what is in each box. You choose a box at random, the host opens one of the others to reveal a goat and invites you to exchange your box for the other unopened box. Here it does make a difference. You increase your chance of winning the car from 1/3 to 2/3 by accepting the offer to swap.

A feasible time machine

Eddie Kent

Time travel would be easy if one could develop a material with negative density, or construct an infinitely long tube. These facts are well known. It would even work if the time machine could resist quantum-mechanical fluctuations, perhaps using the flux capacitor of *Back to the Future*. But in reality all of these devices must await their moment.

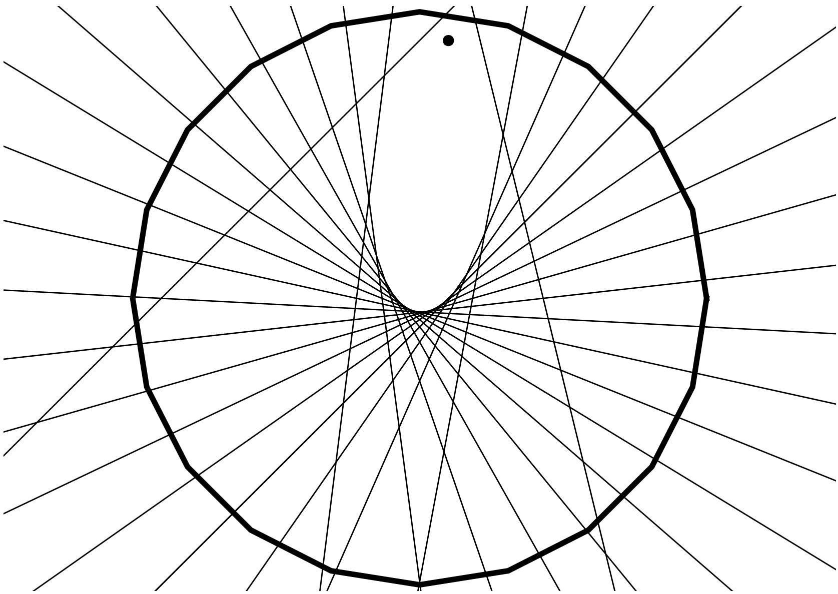
Closer to fruition is recent work by Amos Ori of the Israel Institute of Technology, Haifa. He has shown that a torus-shaped vacuum containing a certain doughnut-shaped gravitational field in otherwise ordinary space could constitute a time machine. Fly your spaceship into the toroid and emerge whenever you like, as far back as the creation of the machine.

He now has to tackle the question of how to build one. He speculates that setting it up might require some method of focusing gravitational waves, or perhaps whirling some really massive object around. Finally he would have to show that it is stable. For full (well, fuller) details, see *Physical Review Letters* for 7 July 2005.

In the days of Lord Reith, a learned professor, on being told that his talk had been accepted for broadcasting on the Third Programme for a fee of £25, was said to have replied thanking the BBC and with words to the effect: 'I enclose a cheque for £25.'

Solution 208.4 – Folding a polygon

> Choose a point, X, inside a regular n-gon. For each vertex V, draw the line that perpendicularly bisects XV. These lines define a polygon $\mathcal{P}$ with X in its interior. What is the maximum number of sides that $\mathcal{P}$ can have? The case $n = 20$ with $X = (1/10, 9/10)$ is illustrated below. Letting n tend to infinity, the lines appear to fill space except for an elliptical hole. What is the equation of the ellipse?



Chris Pile

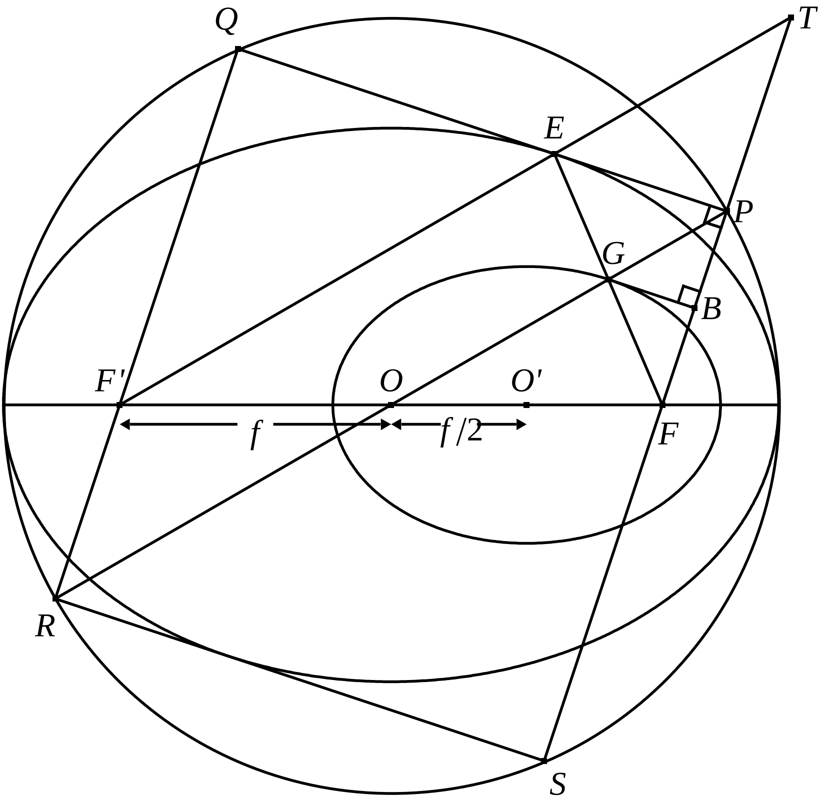
An ellipse can be generated from chords of a circle that are perpendicular to lines through a fixed internal point. Let O be the centre of a circle, radius r, and let F be a fixed point inside, distance f from O. Draw FP to any point P on the circumference and extend FP to T so that $PT = FP$. Draw PQ perpendicular to FP and complete the rectangle $PQRS$. Let QR cut the diameter through F at the corresponding point F'. Join TF' and let it cut PQ at E. Join EF. Triangles PET and PEF are congruent. Therefore $EF = ET$. Also $TF'RP$ is a parallelogram. Therefore $TF' = PR = 2r$. Therefore $EF' + EF = 2r$ which is the requirement for an ellipse with foci F and F'. The major axis is $2r$ and the minor axis is $2\sqrt{r^2 - f^2}$. The equation is therefore

$$\frac{x^2}{r^2} + \frac{y^2}{r^2 - f^2} \;=\; 1.$$

If PF is perpendicularly bisected at B, it will meet PO and EF at G. Triangle OGF is similar to triangle $F'EF$ and half scale. Therefore G traces out a half-scale ellipse, foci O, F and major axis r centred on O', where $OO' = f/2$. The equation is therefore

$$\frac{(x - f/2)^2}{(r/2)^2} + \frac{y^2}{(r/2)^2 - (f/2)^2} \;=\; 1.$$

For this problem, $r = 1$ and $f = \sqrt{(0.1)^2 + (0.9)^2} = \sqrt{0.82}$. The equation simplifies to $y^2 = 0.045 - 0.18(x - \sqrt{0.205})^2$.



Solution 208.1 – 3 ratios

Show that $a/b = c/d = e/f = \alpha$ implies that

$$\alpha = \sqrt[3]{\mathcal{A}}, \quad \text{where } \mathcal{A} = \frac{2a^2c + 3c^3e + 4e^2c}{2b^2d + 3d^3e + 4f^2d}.$$

Steve Moon

Substitute $a = b\alpha$, $c = d\alpha$, $e = f\alpha$ in $\mathcal{A}$ to get

$$\mathcal{A} = \frac{2b^2\alpha^2 d\alpha + 3d^3\alpha^3 f\alpha + 4f^2\alpha^2 d\alpha}{2b^2d + 3d^3 f\alpha + 4f^2d} = \alpha^3.$$

ADF

Peter Fletcher sent a similar answer. But **Nick Hobson** asserts:

> This result isn't true! For example, if you take $a = b = 1$, $c = d = \sqrt{2}$, $e = f = -1$, or, if we're allowing complex numbers (with some proviso for the cube root), $a = c = e = 0$, $b = \sqrt{2}$, $d = 1$, $f = i$, then the denominator and numerator of $\mathcal{A}$ vanish.
>
> I find the result holds if $a = c = e = 0$ and $b^2 + 2f^2 \neq 0$, or a, c, e non-zero and $2a^3 + 3ec^2 + 4ae^2 \neq 0$. So the puzzle could be rescued by specifying that a through f be positive reals.

However, one can argue that it is perfectly legitimate to divide by zero provided there is some kind of limiting process involving continuous variables. Taking the first example above, let $a = b = 1 + \epsilon$, $c = d = \sqrt{2} + \epsilon$ and $e = f = \epsilon - 1$. Then

$$\mathcal{A} = \frac{\epsilon\left(\epsilon + \sqrt{2}\right)\left(3\,\epsilon^2 + \left(3 + 6\sqrt{2}\right)\epsilon - 6\sqrt{2} + 2\right)}{\epsilon\left(\epsilon + \sqrt{2}\right)\left(3\,\epsilon^2 + \left(3 + 6\sqrt{2}\right)\epsilon - 6\sqrt{2} + 2\right)} \to 1 \qquad \text{as } \epsilon \to 0.$$

Proverbs

It is inadvisable to rely on a census of one's ornithological livestock taken whilst the animals remain *in ovo*.

Dust-jacket appraisal is not by itself necessarily a reliable basis for credible literary criticism.

An unmarried female human is no worse than 1609.3440 m.

£0.0041666667 wise, 453.59237 g foolish.

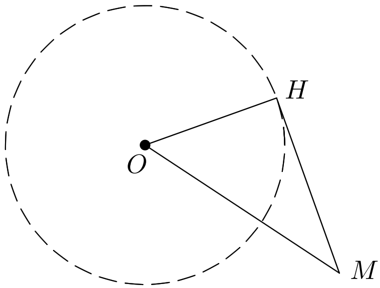
Clocks

Recall that in M500 **168** we asked at what times are the tips of the hands of a clock travelling apart at the fastest speed.

Nick Hobson

Here is yet another approach to this puzzle, which doesn't use calculus at all.

If we regard the minute hand (OM) as fixed and the hour hand (OH) as moving, then it is clear that when MH increases (decreases) at the greatest rate MH is tangent to the circle whose centre is O and radius is OH, because at this time H is moving exactly away from (towards) M. So: MH is tangent to the circle implies MH is perpendicular to OH, and the result follows.



Solution 208.5 – Rain

It is raining. Do you get wetter running or walking?

Steve Moon

Model the person as a horizontal surface of area α joined to a vertical surface of area β. Assume that rain falls vertically at a rate of k uniformly at terminal velocity, that rain is homogeneous in all space, density D, that the distance travelled is d, and that the velocity of the person is v. Rain hits the body in two ways.

(a) Impact on the vertical part: rain intercepted is $D\beta v\, d/v = D\beta d$. (If the person is stationary, $d = 0$ and he doesn't get wet in this way.)

(b) Impact on the horizontal part: rain intercepted is $\alpha k\, d/v$.

Hence the 'wetness' is $D\beta d + \alpha kd/v = K + K'/v$, say, where K and K' are constants. Hence wetness increases as $v \to 0$.

If there is a wind driving rain on to β, wetness is greater than for vertical rain but you still get wetter if you walk.

What do you get if you cross a mountain goat with a rock climber?

Don't be silly; you can't cross two scalers. [scalars]

Letter

Stitched curves

Dear Tony,

Thank you for yet another interesting magazine. The designs for Problems 208.3 and 208.4 can be considered as examples of *Curve Stitching* comprehensively described in the book by Jon Millington (Tarquin Publications, ISBN 0906212650). The designs can all be drawn with straight lines or 'stitched' using coloured threads on card. I have a number of these stitched many years ago, including 'curves of pursuit' [such as the one on the cover of this issue].

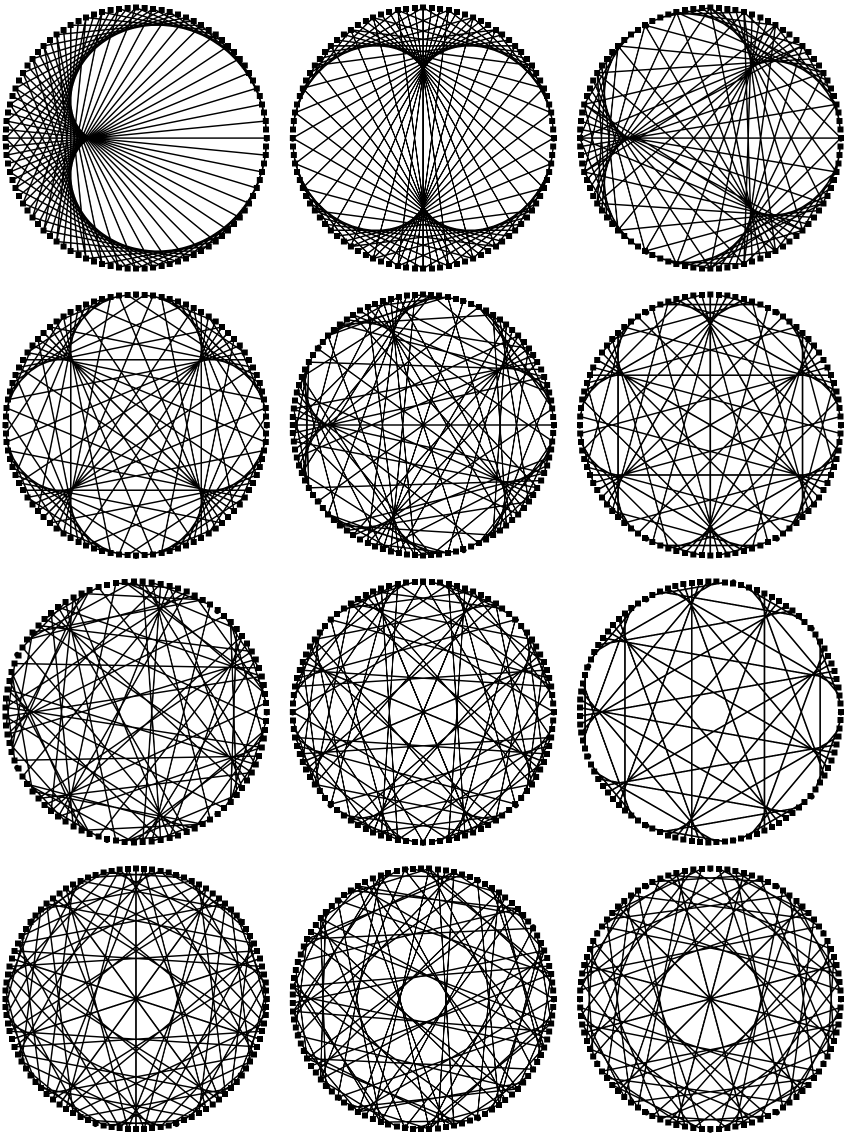
The method is not limited to two dimensions, so you can knit a hyperbolic paraboloid if you have a spare spool of thread and enough motivation. You will note that I have not answered the question in Problem 208.4. The equation should be amended to take account of the orientation of the point specified $(0.1, 0.9)$ or the axes should rotated through 83.7° or so. With my background of years of engineering, I found it easier to turn the page around (p. 11).

I have been going through old copies of M500. My local discount store was selling some inexpensive A5 box files so I thought that it would be a good idea to keep the M500s tidy. Now, four weeks later, I have a dozen empty box files and the living room is decorated with scores of magazines that are too interesting to put away. I think the first A5 issue was No. **29**. I also have some A4 issues (along with Marion's 'Dice Star Trek') but since moving house (four years ago) I have still not unpacked everything. Time is the problem—it is no longer a linear function. I am working on a corollary to a well known principle, i.e. time contracts to limit the amount of work that can be done.

Yours sincerely,

Chris Pile

ADF writes — Some more 'stitched curves' appear on the page opposite, which shows a selection of diagrams formed by arranging N points in a circle and joining point i to point ni, $i = 0, 1, ..., N - 1$. As you can see, there are $n - 1$ petals, $n = 2, 3, ..., 13$. The number of points (N) is one of $\{91, 96, 99, 100\}$ and has been chosen slightly carefully in order to create a pleasing effect. So you can probably determine N for the larger n without doing any counting.



Proof (2005)

Eddie Kent

I finally got to see *Proof*, which was reviewed recently in the LMS *Newsletter*. If not for that review I would never have heard of it, just like you, which is a pity because it is an enjoyable piece of hokum. And it contains Anthony Hopkins and Gwyneth Paltrow, both worth crossing the road for.

Gwyneth, as Catherine, plays her *Royal Tenenbaums* role rather than *Sliding Doors*: silent and moody for the most part but amazingly sharp and perceptive when she is driven to it. Hopkins, dead a week so you only ever see him in flashback, is a mathematician who made breakthroughs on three fronts before he was 23 but produced only gibberish afterwards. He is shown to be happy because he thinks what he is writing down is significant. We know his brain is still working because he is able to tell us the smallest number which is the sum of two cubes in two different ways.

Catherine gave up her studies to look after him and is now being pursued by Jake Gyllenhaal (Hal), a PhD student who also thinks there might be something, hidden in the 103 notebooks. Therein will lie his fame if he can find it. Meanwhile he needs to keep Catherine happy. Catherine and Hal are talking. She: "Can you name any women mathematicians?" He: "No." She: "How about Sophie Germain?" He: "The name is familiar. I might have seen her at conferences but I don't think I've ever met her." She: "She was born in Paris in 1776." He: "Then I definitely never met her." After a pause he thinks of Germain primes, and they both explain (to us), as I am now about to tell you. A *Sophie Germain prime* is a prime p, where $2p+1$ is prime. Sophie generated these numbers in an attempt to prove Fermat's Last Theorem. She succeeded in showing that the number 5, for example, severely restricts FLT: $2 \cdot 5 + 1 = 11$, and if $x^5 + y^5 = z^5$, then x, y or z must be divisible by 5.

This is all relevant for two reasons. In the first place Sophie was self taught, and had no degree and no standing in the academic world, just like Catherine. Secondly, when Sophie eventually won a prize she wouldn't appear in person to collect it since she was certain the judges believed it was really someone else's work, because she was a woman. The basis of the film Proof is that Catherine claims to have written a proof which all around her are convinced was really her father's work. He, of course, would have been capable of doing it, once.

Since Dad ended up barking the assumption is that Catherine, who has followed his route and has convinced herself that the work is her own because she so passionately wants it to be, needs to be a good and worthy daughter. Her sister wants to pack her away and sell the house. Hal makes love to her but is he only using her?

I won't give you the details of the detective work that goes into solving this problem, but it's quite as convincing as any plot-line of Agatha Christie's. I'm very glad to have seen the film and I think it's a pity that it is doing so badly. When I left the cinema I asked in the box office how many people have seen it over its five showings, and worked out that if they had all been to one performance they would have half filled the auditorium. It is a tiny cinema. And the music by Stephen Warbeck is very good. It induced me to stay until the lights went fully up.

ADF — It seems that in most popular accounts of the subject Sophie Germain's contribution to Fermat's Last Theorem is grossly understated. What she actually proved goes something like this.

Sophie Germain's theorem. *Let p be a prime and suppose there is another prime q such that*

(i) $x^p + y^p + z^p \equiv 0 \pmod{q}$ *implies one of x, y, z is divisible by q, and*

(ii) *p is not a p-th power modulo q.*

Then case I of Fermat's Last Theorem is true for p.

(Case I of FLT asserts that for prime p, $x^p + y^p = z^p$ has no solution in positive integers x, y, z with xyz not divisible by p, and case II is where p is a odd and divides xyz.)

Let us see how we can use Sophie Germain's theorem to prove case I of FLT for all primes less than 200, say.

Observe that $q = 2p + 1$, q prime, works because a non-zero p-th power modulo q must be ± 1. Hence (i) follows since there is no way of adding three (± 1)s to get zero modulo q; and (ii) follows trivially. Thus we have case I for 2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191; but that still leaves plenty of primes to deal with, 7, 13, 17, 19, ...

Take $p = 7$, for example. We can't use $q = 2p+1$ because it is composite. But $q = 4p + 1 = 29$ is prime. This time the non-zero p-th powers modulo q are ± 1 and ± 12, as you can see by computing $j^7 \bmod 29$ for $j = 1, 2, \ldots, 28$. Again, you can verify that (i) and (ii) hold.

The rest I leave to you. And if you can deal with case II without Wiles's proof of the Shimura–Taniyama conjecture, so much the better!

Solution 204.5 – Circles

Consider a triangle with vertices A, B and C. Let Q be the centre of the in-circle. Let Q_A be the centre of the escribed circle that touches AB extended, AC extended and BC between B and C. Let Q_B and Q_C denote the centres of the other two escribed circles defined similarly. Show that

$$|QQ_A||QQ_B||QQ_C| + d\left(|QQ_A|^2 + |QQ_B|^2 + |QQ_C|^2\right) \;=\; 4d^3,$$

where d is the diameter of the circumcircle.

Norman Graham

Some standard properties of the triangle are required; these are listed as (i)–(v) and proved at the end of this Solution. Let $a = |BC|$, $b = |AC|$, $c = |AB|$, $s = (a+b+c)/2$ and denote the area of the triangle by Δ. The inscribed circle has radius r and touches the sides at A_0, B_0, C_0. The escribed circle with centre Q_A has radius r_A and touches BC at A_1, AC extended at B_1 and AB extended at C_1.

From (i), $|A_0A_1| = |a - 2(s-b)|$. From (ii) and (iii), $r = \Delta/s$ and $r_A = \Delta/(s-a)$, and $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ from (iv). Therefore

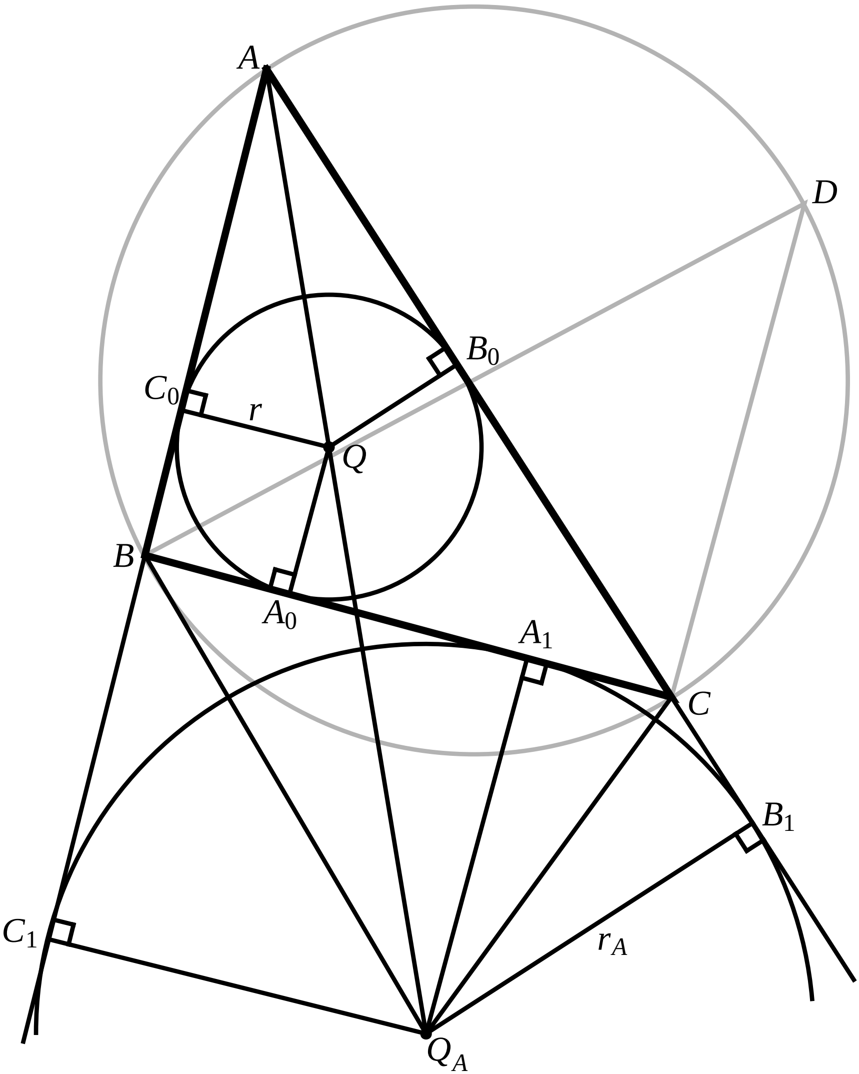
$$\begin{aligned}
|QQ_A|^2 &= (r + r_A)^2 + |A_0A_1|^2 \;=\; \left(\frac{\Delta}{s} + \frac{\Delta}{s-a}\right)^2 + (a - 2s + 2b)^2 \\
&= \left(\frac{2s-a}{s(s-a)}\Delta\right)^2 + (b-c)^2 \\
&= \frac{1}{s(s-a)}\left((b+c)^2(s-b)(s-c) + (b-c)^2 s(s-a)\right) \\
&= \frac{a^2}{4s(s-a)}\left((b+c)^2 - (b-c)^2\right) \;=\; \frac{a^2bc}{s(s-a)}.
\end{aligned}$$

Similarly, $|QQ_B|^2 = ab^2c/(s(s-b))$ and $|QQ_C|^2 = abc^2/(s(s-c))$. Hence

$$|QQ_A||QQ_B||QQ_C| \;=\; \frac{(abc)^2}{s\Delta}.$$

Also

$$\begin{aligned}
&|QQ_A|^2 + |QQ_B|^2 + |QQ_C|^2 \\
&\qquad = \frac{abc}{\Delta^2}\left(a(s-b)(s-c) + b(s-a)(s-c) + c(s-a)(s-b)\right).
\end{aligned}$$


A
D
B_0
C_0
r
Q
B
A_0
A_1
C
B_1
C_1
r_A
Q_A

From (v), $d = abc/(2\Delta)$. Therefore

$$|QQ_A||QQ_B||QQ_C| + d(|QQ_A|^2 + |QQ_B|^2 + |QQ_C|^2)$$
$$= \frac{(abc)^2}{2\Delta^3}\big(2(s-a)(s-b)(s-c) + a(s-b)(s-c) + b(s-c)(s-a) + c(s-a)(s-b)\big),$$
$$= \frac{(abc)^2}{8\Delta^3}\cdot 4abc \;=\; 4\left(\frac{abc}{2\Delta}\right)^3 \;=\; 4d^3.$$

Standard properties of the triangle

(i) Since $|BA_0| = |BC_0|$, etc., we have $|BA_0| + |CB_0| + |AC_0| = (a + b + c)/2 = s$. Therefore

$$|BA_0| \;=\; s - (|CB_0| + |AC_0|) \;=\; s - (|CB_0| + |AB_0|) \;=\; s - b.$$

Also $|AC_1| + |AB_1| = c + b + a = 2s$, and $|AC_1| = |AB_1|$ implies $|AB_1| = s$. Therefore $|CA_1| = |CB_1| = s - b$. Hence

$$|A_0A_1| \;=\; |a - 2(s-b)|.$$

(ii) $\Delta = \text{areas}\,(BQC + CQA + AQB) = \frac{1}{2}\,r\,(a + b + c) = rs$. Hence $r = \Delta/s$.

(iii) $\Delta = \text{areas}\,(AQ_AC + AQ_AB - BQ_AC) = \frac{1}{2}\,r_A\,(b+c-a) = r_A(s-a)$. Hence $r_A = \Delta/(s-a)$.

(iv) By the cosine rule, $\cos A = (b^2 + c^2 - a^2)/(2bc)$. Therefore

$$2\cos^2\frac{A}{2} \;=\; 1 + \cos A \;=\; \frac{(b+c)^2 - a^2}{2bc} \;=\; \frac{2s(s-a)}{bc},$$

$$2\sin^2\frac{A}{2} \;=\; 1 - \cos A \;=\; \frac{a^2 - (b-c)^2}{2bc} \;=\; \frac{2(s-b)(s-c)}{bc}.$$

Hence

$$r_A \;=\; |AB_1|\tan\frac{A}{2} \;=\; \sqrt{\frac{s(s-b)(s-c)}{s-a}}$$

and therefore $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

(v) In the diagram, $|BD|$ is the diameter of the circumcircle. Thus $A = \angle BDC$ and $|BC| = d\sin A$. Hence $\Delta = \frac{1}{2}\,bc\sin A = \frac{1}{2}\,bca/d$. Therefore $d = abc/(2\Delta)$.

Problem 211.6 – Odds again

Ian Adamson

Suppose $1 \in Q$; $m \in Q \Rightarrow 2^n m - 1 \in Q, n \in \mathbb{N}$. Prove that Q is (or is not) the set of positive odd integers.

I agonized over the proof of this, until I read 'Problem 207.3 – Odds' when it became obvious. Naturally I would like to prove that $1 \in R$; $m \in R \Rightarrow (2^n m - 1)/3 \in R, n \in \mathbb{N}$ & n such that $2^n m \equiv 1 \pmod 3) \in Q$!

ADF — Recall that Problem 207.3 asks for a proof of or counterexample to the assertion that the set T, defined by (i) $1 \in T$, (ii) $n \in T \Rightarrow 2n + 1 \in T$, (iii) $3n \in T \Rightarrow n \in T$, is the set of positive odd integers.

Two readers have sent in solutions claiming that 9 is the smallest odd integer which is not in T. I have not had time to study the details but I have to admit that both of these proofs look convincing except for the unfortunate fact that there must be an error somewhere. To show that $9 \in T$ needs a bit of work. From (i) and (ii) we have $1, 3, 7, 15, 31, 63, 127, 255 \in T$. Then from (iii), $255 \in T \Rightarrow 85 \in T$. Then $85 \in T \Rightarrow 171 \in T$ by (ii); $19 \in T$ by two applications of (iii); $39 \in T$ by (ii); $13 \in T$ by (iii); $27 \in T$ by (ii); and finally $9 \in T$ by (iii).

Problem 207.3 was posted by Alf van der Poorten to the Internet mailing list *NMBRTHRY*, which also contains some discussion relating it to the famous Syracuse conjecture alluded to above: that the process $\{n \to n/2$ if n is even, $n \to 3n + 1$ if n is odd$\}$ yields 1 after sufficiently many iterations.

Sentences

Jeremy Humphries

This sentence contains 27 letters.

This sentence contains thirty-six letters.

It's not too difficult to see another number in English which works. Can you find one in Roman numerals which works? Clearly there is no limit to the number of numerical solutions. You just have to insert any expression whose value is 27:

This sentence contains $28 - 1$ **letters.**

How about other solutions in words? Here's one:

This sentence contains thirty plus ten letters.

Can you prove that the number of solutions in words is also unlimited?

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