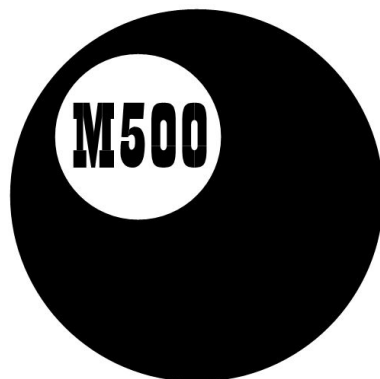


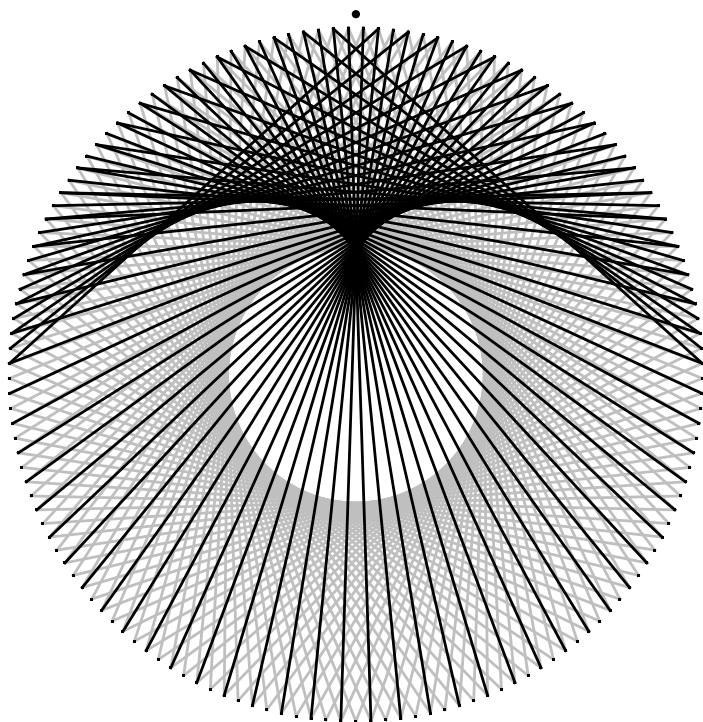
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ISSN 1350-8539



**M500 216**

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## The M500 Society and Officers

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**The M500 Society** is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: [www.m500.org.uk](http://www.m500.org.uk).

**The magazine M500** is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

**MOUTHS** is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

**The September Weekend** is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

**The Winter Weekend** is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

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**Editor** – *Tony Forbes*

**Editorial Board** – *Eddie Kent*

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**Advice to authors.** We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail.

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## Solution 213.1 – Pascal triangle sums

Show that the sums of the reciprocals of the columns of Pascal's triangle, if they converge, are given by the simple formula

$$\sum_{n=1}^{\infty} \frac{k!(n-1)!}{(n+k-1)!} = \frac{k}{k-1}.$$

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

**Nick Hobson**     <http://www.qbyte.org/puzzles/>

Here is a quick solution to this problem, from which convergence (for  $k > 1$ ) naturally drops out.

Let

$$S(k, r) = \sum_{n=1}^r \frac{k!}{n(n+1)\dots(n+k-1)}$$

be the partial sum to  $r$  terms. Also let  $S(k) = \lim_{r \rightarrow \infty} S(k, r)$ . We aim to prove that for  $k > 1$  the limit exists, and that  $S(k) = k/(k-1)$ .

Observe that, for  $k > 1$ ,

$$\begin{aligned} \frac{1}{n(n+1)\dots(n+k-2)} - \frac{1}{(n+1)(n+2)\dots(n+k-1)} \\ = \frac{k-1}{n(n+1)\dots(n+k-1)}. \end{aligned}$$

Hence, when  $k = 2$ ,

$$\frac{S(2, r)}{2} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{r} - \frac{1}{r+1}\right).$$

This is a telescoping sum in which all but the first and last terms cancel, leaving  $S(2, r)/2 = 1 - 1/(r + 1)$ . As  $r \rightarrow \infty$ ,  $1/(r + 1) \rightarrow 0$ . Thus  $S(2)/2 = 1$ , and  $S(2) = 2$ .

Next,

$$\frac{S(3, r)}{3} = \sum_{n=1}^r \frac{1}{n(n+1)} - \sum_{n=1}^r \frac{1}{(n+1)(n+2)}.$$

Again, this is a telescoping sum, yielding

$$\frac{S(3, r)}{3} = \frac{1}{2} - \frac{1}{(r+1)(r+2)}.$$

Thus  $S(3)/3 = 1/2$ , and  $S(3) = 3/2$ .

The general case is similar. We have

$$\begin{aligned} \frac{S(k, r)}{k} &= (k-2)! \left( \sum_{n=1}^r \frac{1}{n(n+1)\dots(n+k-2)} \right. \\ &\quad \left. - \sum_{n=1}^r \frac{1}{(n+1)(n+2)\dots(n+k-1)} \right). \end{aligned}$$

Hence

$$\frac{S(k, r)}{k} = (k-2)! \left( \frac{1}{(k-1)!} - \frac{1}{(r+1)(r+2)\dots(r+k-1)} \right).$$

Thus  $S(k)/k = 1/(k-1)$ , and  $S(k) = k/(k-1)$ .

## Problem 216.1 – Rotations

Let  $\phi$  be a rotation of the sphere about a given axis. Let  $\psi$  be another rotation of the sphere about a different axis. Then the composition  $\phi \circ \psi$  is a rotation of the sphere through  $\alpha$  degrees, say, about a certain axis, in general distinct from the rotation axes of  $\phi$  and  $\psi$ . Also,  $\psi \circ \phi$  is a rotation of the sphere through  $\beta$  degrees, say, about an axis not necessarily the same as the other three rotation axes.

Prove that  $\alpha = \beta$ .

## Solution 213.8 – Definite integral

Compute

$$\mathcal{I} = \int_0^1 \frac{5x^{114} + 5x^{112} - \frac{4}{71}}{x^2 + 1} dx.$$

A popular problem. Nick Hobson, Simon Geard, Basil Thompson, Martin Orman and Steve Moon sent in solutions. To 8 decimal places the answer is approximately zero, as can be seen by splitting the integrand into two parts and integrating them separately:

$$\begin{aligned} \mathcal{I} &= 5 \int_0^1 x^{112} dx - \frac{4}{71} \int_0^1 \frac{dx}{x^2 + 1}. \\ &= \frac{1}{71} \left( \frac{355}{113} - \pi \right) \approx 0. \end{aligned}$$

---

### Nick Hobson

This somewhat surprising result occurs because  $355/113$  is an unusually good approximation for  $\pi$ . It is ‘unusual’ in the sense that with a denominator of only three decimal digits, the approximation is correct to six decimal places. It is one of the continued fraction convergents for  $\pi$ :  $3$ ,  $22/7$ ,  $333/106$ ,  $355/113$ ,  $103993/33102$ ,  $104348/33215$ ,  $208341/66317$ ,  $312689/99532$ ,  $833719/265381$ ,  $\dots$ . The jump in size of the denominator from  $113$  to  $33102$  arises because of the relatively large term  $292$  in the continued fraction expansion of  $\pi$ :  $[3; 7, 15, 1, 292, 1, 1, 1, 2, \dots]$ .

The approximation  $355/113$  is good enough for almost all practical purposes. For instance, the circumference of a circle of diameter  $113$  miles differs from  $355$  miles by less than  $2$  inches!

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See Problem 202.1 – Squaring the circle, and its solution in **M500 204**, for an interesting ruler-and-compasses construction (due to Srinivasa Ramanujan) of a line segment having length  $\sqrt{355/113}$ —**ADF**.

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### Free book

*The Natural Algebras: The higher dimensional complex number algebras*  
by **Dennis Morris**

This book is available freely as an internet download to readers of M500. For details contact Dennis at [superden4@aol.com](mailto:superden4@aol.com).

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## Solution 208.3 – Concentric circles

Explain the prominent circles that appear in the circular representation of the complete graph  $K_n$  when  $n$  is reasonably large.

### Ken Greatrix

I noticed that the circle in question is the  $K$ -number divided by 8 (then rounded to the nearest whole number). This led me to investigate the possibility of large  $K$ s.

As  $K_n$  tends to infinity, the chord in question subtends an angle of  $135^\circ$  at the centre, but I can't see why this should be significant. From the physics classes at school, I remembered that the focal point of a spherical mirror is halfway along the radius; thus an angle of  $120^\circ$  would be subtended. Using a simple graphics routine, I proceeded as follows.

Define  $n$  points on a circle, where  $n \equiv 1 \pmod{8}$ —this makes the graphics easier since the `int()` function rounds downwards. Select an arbitrary point at the top of the circle as a reference (this makes the final picture look nice!) between two of the above defined points.

Starting from the reference point, take the first pair outwards and join them to the appropriate points so that they are tangent to the first circle, and they cross over above the centre. Take the second pair of points, joining them to their partners so that they are tangent to the second circle. Repeat this process until sufficient points have been joined by chords.

You should see a picture which vaguely looks like a bird.

The tail of the bird is concentrated on the required circle (the fourth in your example on page 20 of M500 **212**). The next few cross outside this circle and the process continues to make the bird's wings.

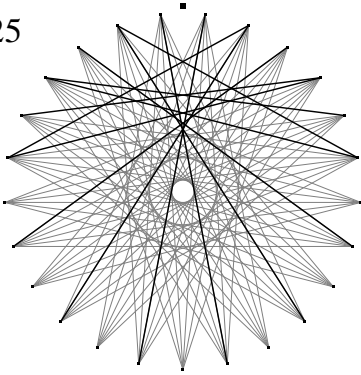
As for an explanation of this, I suspect some sort of focussing that would occur with a conic section (e.g. in an ellipse, from one focus to the other; in a parabola, parallel lines reflect to a single focal point).

An example is shown on the front cover of this issue— $K_{145}$ ; any higher  $K$ -value becomes cluttered in the graphics. The light grey lines are tangent to the 18th circle. The black lines are those described above. The black blob at the top is my chosen reference point.

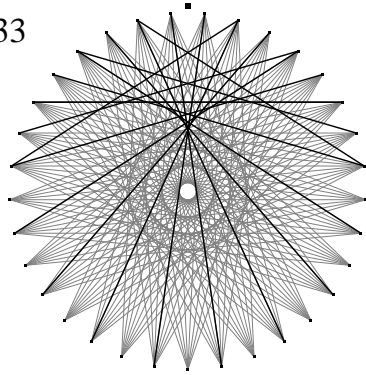
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**ADF** writes—This is very interesting. On the next few pages I have provided some examples with various values of  $n$ . The thin lines in the background are some of the edges of  $K_n$  and they indeed generate the familiar pattern of concentric circles. Notice how the focus of the thick lines seems to be coincident with the  $\frac{1}{8}(n-1)$ th circle, thus making it the most prominent one when the remaining chords are added.

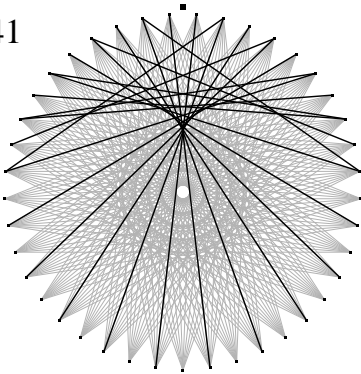
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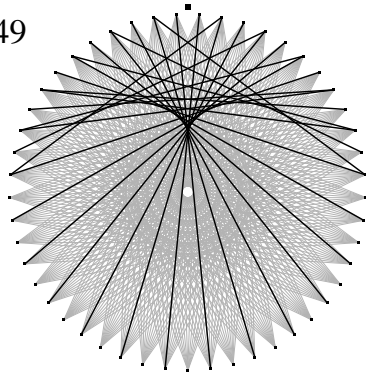
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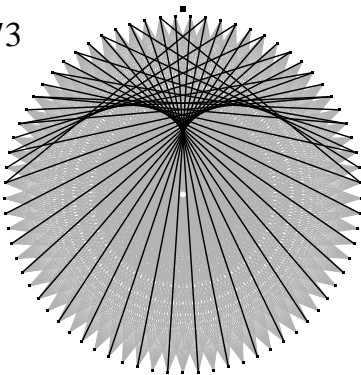
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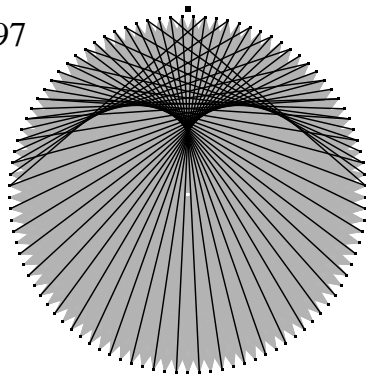
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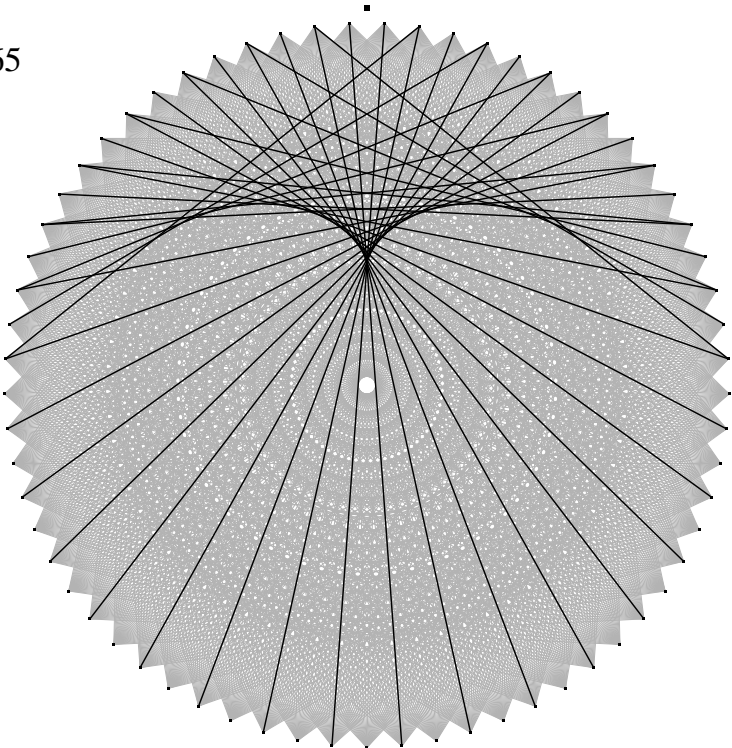
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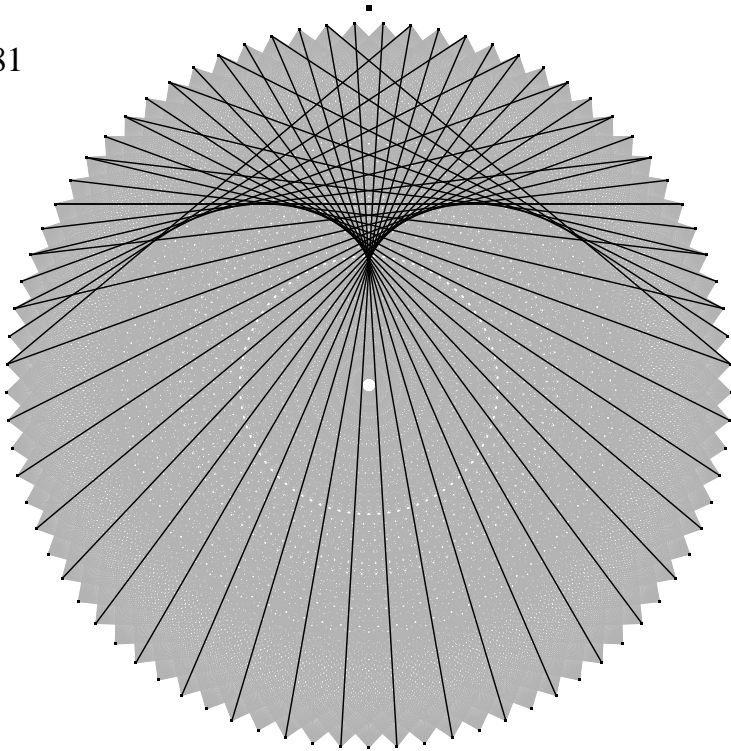


65





81



## Solution 213.6 – What is the number?

There is a certain positive integer. When divided by three it has remainder two; when divided by five it has remainder three; when divided by seven it has remainder two. What is the number? In other words, solve

$$x \equiv 2 \pmod{3}, \quad (1)$$

$$x \equiv 3 \pmod{5}, \quad (2)$$

$$x \equiv 2 \pmod{7}. \quad (3)$$

### Nick Hobson

Here are a couple of approaches to this problem. By Fermat's little theorem,  $2^7 \equiv 2 \pmod{7}$ ; hence  $x = 2^7$  satisfies (3). Further, by the same theorem,  $2^7 = (2^2)^3 \cdot 2^1 \equiv 1^3 \cdot 2 \equiv 2 \pmod{3}$ , and similarly  $2^7 = (2^4) \cdot (2^3) \equiv 1 \cdot 8 \equiv 3 \pmod{5}$ . Hence  $x = 2^7$  is a solution and, by the Chinese remainder theorem, it is unique modulo  $3 \cdot 5 \cdot 7 = 105$ . So the general solution is  $x \equiv 23 \pmod{105}$ .

Alternatively, we could note that (1) and (3) have solution  $x \equiv 2 \pmod{21}$ , and by inspection  $x = 23$  also satisfies (2), again giving the general solution  $x \equiv 23 \pmod{105}$ .

---

### Tony Forbes

Steve Moon and Paul Richards also sent in the same solution arrived at by similar methods. However, the real difficulty with the problem is in the interpretation of the mysterious hint that came with it.

*Three people walking together, 'tis rare that one be seventy,  
Five cherry blossom trees, twenty-one branches bearing flowers,  
Seven disciples reunite for the half-moon,  
Take away one hundred and fives and you shall know.*

After a little research I think I can offer an explanation.

Making use of a general method of solving systems of linear congruences we can write down the solution of (1)–(3):

$$x \equiv 2a + 3b + 2c \pmod{105}, \quad (4)$$

where  $a$ ,  $b$  and  $c$  are given (modulo 105) by

$$\begin{aligned} a &= 5 \cdot 7 \left( (5 \cdot 7)^{-1} \pmod{3} \right), \\ b &= 3 \cdot 7 \left( (3 \cdot 7)^{-1} \pmod{5} \right), \\ c &= 3 \cdot 5 \left( (3 \cdot 5)^{-1} \pmod{7} \right). \end{aligned}$$

Since 3, 5 and 7 are prime, computing the multiplicative inverse  $z^{-1}$  of any non-zero  $z$  is a legitimate operation. Hence quantities like  $((5 \cdot 7)^{-1} \pmod{3})$  are well defined.

To see how this solution works, substitute  $a$ ,  $b$  and  $c$  into (4) and reduce modulo 3, 5 and 7 in turn. For instance, working modulo 3 we have  $a = (5 \cdot 7)^{-1} \cdot (5 \cdot 7) = 1$  since the  $(5 \cdot 7)^{-1}$  and  $(5 \cdot 7)$  cancel. Also the expressions for  $b$  and  $c$  each have a factor 3 in them. Hence  $b = c = 0$ . Thus (4) reduces to  $x \equiv 2 \pmod{3}$ .

Now we compute  $a$ ,  $b$  and  $c$ :  $a = 70$ ,  $b = 21$  and  $c = 15$ , giving

$$x \equiv 2 \cdot 70 + 3 \cdot 21 + 2 \cdot 15 \equiv 23 \pmod{105}.$$

Lo and behold! The coefficients of 2 and 3, namely 70 and 21, are the same as the 70 and 21 that occur in the rhyme. Moreover, 70 is associated with 3 (people walking together) and 21 with 5 (cherry blossom trees). But where is the 15? The reference to the half-moon and 7 (disciples) suggests half the rotational period of the moon in days—in approximate agreement with 14.765295, the figure given in one of Patrick Moore's astronomy books.

There's an interesting application of the CRT on page 18.

## Problem 216.2 – Ramanujan's continued fraction

### Sebastian Hayes

As it is recounted by Kanigel, *The Man who Knew Infinity* (Abacus, 1991), a Hindu friend of Ramanujan's, Mahalanobis, when he and Ramanujan were both at Cambridge, read out to him a puzzle from *Strand* magazine about an inhabitant of Louvain (which had just been burned by the Germans). This Belgian lived in a house on a long street which was numbered 1, 2, 3, . . . consecutively along his side of the street. The number of his house had a curious property: the sum of all the house numbers before it was the same as the sum of all the house numbers that came after it. The magazine stated that there were more than fifty houses and less than five hundred houses on that side of the street. So what was the Belgian's house number? Ramanujan thought for a moment and then dictated the first few convergents of a continued fraction which included all the solutions to the problem (not just the one falling within the 50–500 range).

What was the continued fraction?

### Solution 213.3 – Triangles

Consider a graph  $T$  consisting of  $2n$  separate triangles,  $n \geq 1$ . Now add  $3n$  more edges such that (i) each new edge joins vertices of  $T$  belonging to two distinct triangles, and (ii) each vertex of  $T$  is adjacent to precisely one new edge. The result is a cubic graph,  $G_n$ , say. For which values of  $n$  is  $G_n$  3-edge-colourable? For which values of  $n$  is  $G_n$  planar?

#### Ken Greatrix

Looking at the re-drawing of this puzzle, I have ‘stretched’ it into an aligned format and coloured the lines appropriately (see Figure 1)—then I realized that M500 is in B/W; so I put letters  $x, y, z$  on the lines!

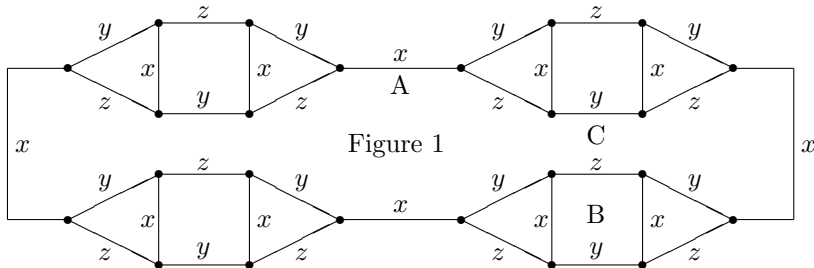


Figure 1

A single line with two extra triangles can be ‘inserted’ into a line of the same colour (e.g. at point A in Figure 1, using Figure 2). A double line with two extra triangles can be inserted into a pair of lines of the same colour orientation (e.g. Figure 3 placed into Figure 1 at point B).

The simplest figure is two triangles (which can be seen if the ends of Figure 2 are joined), thus for all  $n > 0$  the puzzle is solved.

Figure 2

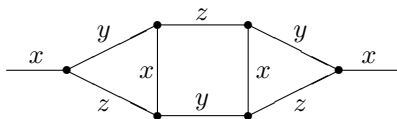
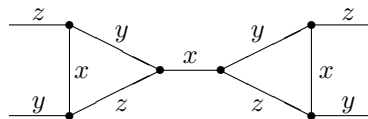


Figure 3



You will note that the orientation of the colours could be any of 6 possible combinations. This implies that any single line of any of the three colours could be interrupted with Figure 2 in the appropriate colour orientation. Also, Figure 3 could be copied to any double-line position of the appropriate colours, provided that those lines are adjacent in the plane (for example, if Figure 3 is rotated through  $180^\circ$ , it could be inserted at point

C). So, the graph need not be a single loop.

Since I like to answer a question with a question (which also generates feed-back for further issues for M500) . . . . If you are given a multiple-triangle graph, you are able to reduce it with the reverse of the above additions to its simplest form. How many simplest forms are there? Does this always reduce to a simple ‘loop’ of triangles as in Figure 1?

Now I’d like to take the graph out of the Euclidean plane.

On the surface of three of the Platonic solids (tetrahedron, cube and dodecahedron) three edges meet at each vertex. So, each vertex could be replaced by a triangle. Does the graph of each of these solids have a solution? Can the graph be projected onto the plane without lines crossing?

I suspect that only the tetrahedron has such a solution but can this be proven? (Hint: consider the ‘opposite pairs’ of the solids).

What if you decide on a Riemann surface, or other non-Euclidean surfaces. What forms do these graphs have?

## Solution 213.5 – Cubic

Show that the roots of  $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$  are  $\tan 20^\circ$ ,  $\tan 80^\circ$  and  $\tan 140^\circ$ .

### Nick Hobson

Another very nice puzzle! Let  $x = \tan a$ . Using the tangent addition formula, it is easily shown that

$$\tan 3a = \frac{3x - x^3}{1 - 3x^2}.$$

If  $a = 20^\circ$ ,  $80^\circ$ , or  $140^\circ$ , then

$$\tan 3a = \sqrt{3} = \frac{3x - x^3}{1 - 3x^2}.$$

Rearranging, we obtain

$$x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0.$$

That is,  $\tan 20^\circ$ ,  $\tan 80^\circ$ , and  $\tan 140^\circ$  are the (only) roots of the cubic.

Solved in a similar manner by **Steve Moon** and **Basil Thompson**.

## Solution 212.2 – Area of a triangle

Draw a triangle with side lengths  $a$ ,  $b$  and  $c$ . Extend the sides to infinity in both directions. Draw the four circles each of which touches the three (extended) sides. One of these is inside the triangle (the in-circle); let this have radius  $r$ . The other three circles lie outside the triangles; join their centres to make a big triangle. Prove that the new triangle has area  $abc/(2r)$ .

### A. J. Moulder

In co-operation with the leader of our U3A Mathematics group I offer the following solution to the above problem. Let

- (i)  $ABC$  be the original triangle with, as convention,  $BC = a$ ,  $AC = b$ , and  $AB = c$ ;
- (ii)  $I$  be the centre of the in-circle with  $K$ ,  $L$ ,  $M$  the points of contact of the in-circle with the sides  $b$ ,  $c$ ,  $a$  respectively;
- (iii)  $I_1$ ,  $I_2$ ,  $I_3$  be the centres of the three circles outside the triangle which touch the sides, extended where necessary, and  $S$ ,  $T$  the points of contact of circle centre  $I_3$  with  $c$  and  $a$  extended,  $U$ ,  $V$  the points of contact of the circle centre  $I_1$  with  $c$  extended and  $a$ , and  $W$ ,  $X$  the points of contact of the circle centre  $I_2$  with  $a$  extended and  $c$  extended.

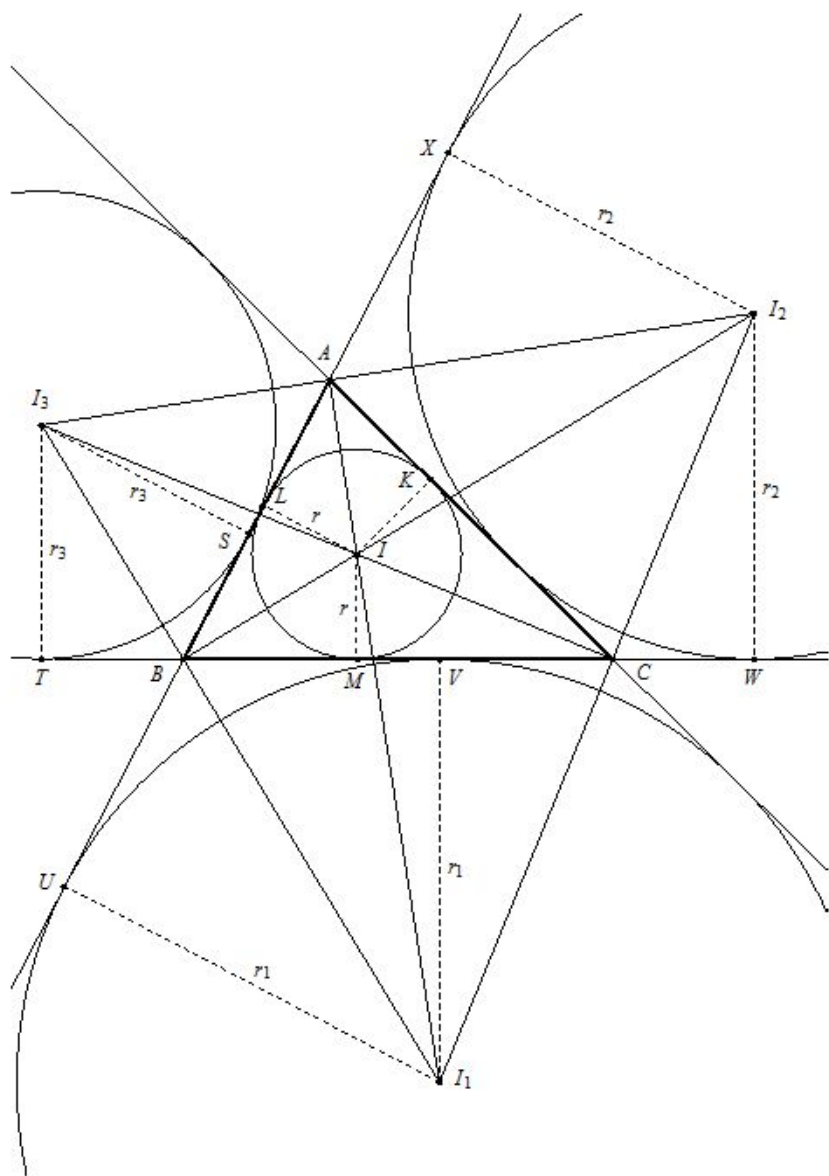
All as shown in the diagram, opposite.

Consider the triangles  $I_3BT$ ,  $I_3BS$ . Then  $I_3T = I_3S = r_3$ , the radius of the circle. Angle  $I_3TB = \text{angle } I_3SB = \frac{1}{2}\pi$  since  $BT$ ,  $BS$  are tangents to the circle. Also  $I_3B$  is common and forms the hypotenuse in both triangles.

Therefore triangles  $I_3BT$ ,  $I_3BS$  are congruent. In particular, angle  $I_3BT = \text{angle } I_3BS$ ; i.e.  $I_3B$  bisects angle  $TBS$ . But angle  $TBS = \pi - B$ ; therefore angle  $I_3BT = I_3BS = \frac{1}{2}\pi - \frac{1}{2}B$ .

Similarly it may be proved that  $I_1B$  bisects angle  $UBV$  and, since  $UBV = \pi - B$ , angle  $UBI_1 = \text{angle } VBI_1 = \frac{1}{2}\pi - \frac{1}{2}B$ . Also angle  $TBS$  and  $UBV$  are vertically opposite, formed by the sides  $a$  extended and  $c$  extended, and therefore equal. So  $I_3B$  and  $I_1B$  is a straight line; i.e. the line  $I_1I_3$  passes through  $B$ . Similarly, the line  $I_1I_2$  passes through point  $C$  and the line  $I_2I_3$  passes through  $A$ .

By considering triangles  $ILB$  and  $IMB$  it may be proved that  $IB$  bisects angle  $B$  and from triangles  $I_2XB$ ,  $I_2WB$  it is seen that  $I_2B$  also bisects angle  $B$  and hence  $I$  lies on the line  $I_2B$ .



We have

$$\angle I_1 B I_2 = \pi - \angle I B S - \angle A B I_3 = \pi - \frac{B}{2} - \left( \frac{\pi}{2} - \frac{B}{2} \right) = \frac{\pi}{2}$$

and

$$\angle I_2 I_3 I_1 = \pi - \angle S B I_3 - \angle S A I_3 = \pi - \frac{\pi - B}{2} - \frac{\pi - A}{2} = \frac{\pi}{2} - \frac{C}{2}$$

since  $A + B + C = \pi$ .

Now  $a = BV + VC$  and

$$\frac{r_1}{BV} = \tan \angle V B I_1 = \tan \left( \frac{1}{2}\pi - \frac{1}{2}B \right) = \cot \frac{1}{2}B;$$

i.e.  $BV = r_1 \tan \frac{1}{2}B$ . Similarly,  $CV = r_1 \tan \frac{1}{2}C$ . Therefore

$$\begin{aligned} a &= r_1 \left( \tan \frac{1}{2}B + \tan \frac{1}{2}C \right) \\ &= r_1 \frac{\sin(\frac{1}{2}B) \cos(\frac{1}{2}C) + \cos(\frac{1}{2}B) \sin(\frac{1}{2}C)}{\cos(\frac{1}{2}B) \cos(\frac{1}{2}C)} \\ &= r_1 \frac{\sin(\frac{1}{2}B + \frac{1}{2}C)}{\cos(\frac{1}{2}B) \cos(\frac{1}{2}C)} = r_1 \frac{\sin(\frac{1}{2}\pi - \frac{1}{2}A)}{\cos(\frac{1}{2}B) \cos(\frac{1}{2}C)} \\ &= r_1 \frac{\cos \frac{1}{2}A}{\cos(\frac{1}{2}B) \cos(\frac{1}{2}C)}. \end{aligned}$$

Therefore  $r_1 = a \cos(\frac{1}{2}B) \cos(\frac{1}{2}C) / \cos(\frac{1}{2}A)$ .

By the sine rule for triangle  $ABC$ ,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

where  $R$  is the radius of the circumscribed circle. Therefore

$$\begin{aligned} a &= 2R \sin A = 4R \sin(\frac{1}{2}A) \cos(\frac{1}{2}A), \\ r_1 &= \frac{4R \sin(\frac{1}{2}A) \cos(\frac{1}{2}A) \cos(\frac{1}{2}B) \cos(\frac{1}{2}C)}{\cos(\frac{1}{2}A)} \\ &= 4R \sin(\frac{1}{2}A) \cos(\frac{1}{2}B) \cos(\frac{1}{2}C). \end{aligned}$$

Similarly, using  $c = BS + AS$ ,

$$r_3 = 4R \cos(\frac{1}{2}A) \cos(\frac{1}{2}B) \sin(\frac{1}{2}C).$$



Now

$$\frac{r_3}{I_3 B} = \sin \angle I_3 B S = \sin \frac{1}{2}(\pi - B) = \cos \frac{1}{2} B.$$

Therefore

$$I_3 B = r_3 \sec \frac{1}{2} B = \frac{4R \cos(\frac{1}{2} A) \cos(\frac{1}{2} B) \sin(\frac{1}{2} C)}{\cos \frac{1}{2} B} = 4R \cos(\frac{1}{2} A) \sin(\frac{1}{2} C).$$

Similarly,  $I_1 B = 4R \sin(\frac{1}{2} A) \cos(\frac{1}{2} C)$ . But

$$\begin{aligned} I_1 I_3 &= I_1 B + I_3 B = 4R \left( \sin(\frac{1}{2} A) \cos(\frac{1}{2} C) + \cos(\frac{1}{2} A) \sin(\frac{1}{2} C) \right) \\ &= 4R \sin(\frac{1}{2} A + \frac{1}{2} C) = 4R \cos \frac{1}{2} B \end{aligned}$$

since  $A + B + C = \pi$ . Similarly  $I_2 I_3 = 4R \cos \frac{1}{2} A$ . Now

$$\begin{aligned} \text{area of } \triangle I_1 I_2 I_3 &= \frac{1}{2} I_1 I_3 \cdot I_2 I_3 \cdot \sin \angle I_2 I_3 I_1 \\ &= \frac{1}{2} \cdot 4R \cos(\frac{1}{2} B) \cdot 4R \cos(\frac{1}{2} A) \cdot \sin(\frac{1}{2} \pi - \frac{1}{2} C) \\ &= 8R^2 \cos(\frac{1}{2} A) \cos(\frac{1}{2} B) \cos(\frac{1}{2} C). \end{aligned}$$

But the area of triangle  $ABC$  is  $rs$ , where  $s$  is the semi-perimeter; i.e.  $s = \frac{1}{2}(a + b + c)$ . Hence, by use of the sine rule,

$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2} r a + \frac{1}{2} r b + \frac{1}{2} r c \\ &= \frac{1}{2} r (2R \sin A + 2R \sin B + 2R \sin C) \\ &= R r (\sin A + \sin B + \sin C) \\ &= R r \left( 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B) + 2 \sin(\frac{1}{2} C) \cos(\frac{1}{2} C) \right) \\ &= 2R r \cos(\frac{1}{2} C) \left( \cos \frac{1}{2}(A - B) + \sin \frac{1}{2} C \right) \\ &= 2R r \cos(\frac{1}{2} C) \left( \cos \frac{1}{2}(A - B) + \cos \frac{1}{2}(A + B) \right) \\ &= 4R r \cos(\frac{1}{2} A) \cos(\frac{1}{2} B) \cos(\frac{1}{2} C). \end{aligned}$$

Therefore

$$\frac{\text{area of } \triangle I_1 I_2 I_3}{\text{area of } \triangle ABC} = \frac{8R^2 \cos(\frac{1}{2} A) \cos(\frac{1}{2} B) \cos(\frac{1}{2} C)}{4R \cos(\frac{1}{2} A) \cos(\frac{1}{2} B) \cos(\frac{1}{2} C)} = \frac{2R}{r}.$$

Therefore (area of  $\triangle I_1 I_2 I_3$ ) =  $2R$ (area of  $\triangle ABC$ )/ $r$ . But the area of triangle  $ABC$  is  $\frac{1}{2} b c \sin A$ . From the sine rule,

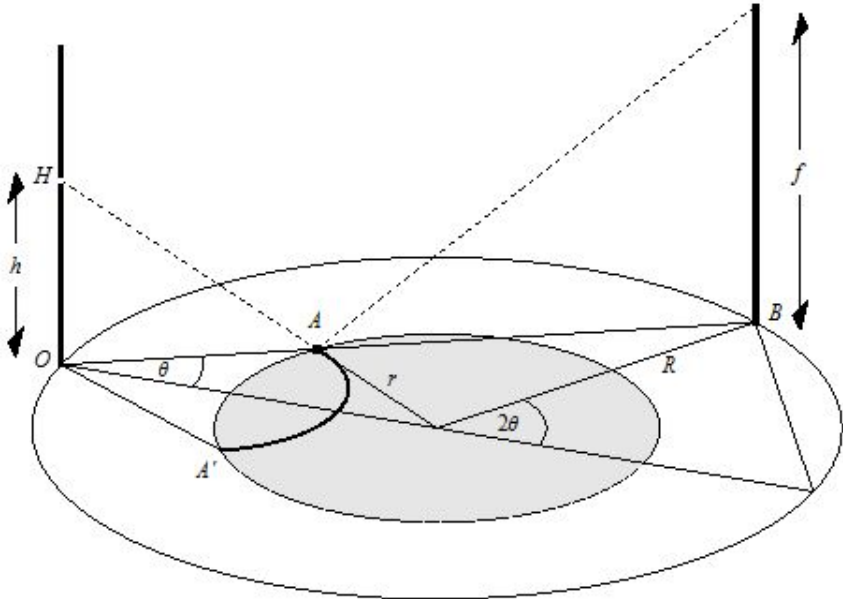
$$R = \frac{a}{2 \sin A} = \frac{abc}{2 b c \sin A} = \frac{abc}{4 (\text{area of } \triangle ABC)}.$$

Hence  $2R$ (area of  $\triangle ABC$ ) =  $abc/2$  and (area of  $\triangle I_1 I_2 I_3$ ) =  $abc/2r$ , as required.

## Solution 211.2 – Pond

A fence of height  $f$  surrounds a flat disc of radius  $R$  in which there is a central pond of radius  $r$ . You look through a hole in the fence at height  $h$ . What proportion of the top of the fence can you see reflected in the pond?

**Steve Moon**



In the general case the visible part of the top of the fence is seen by an observer at  $H$  by reflection in a curve  $AA'$  as shown. Define  $\theta$ ,  $O$ ,  $A$  and  $B$ , and  $H$  as in the diagram;  $H$  is the hole, and we take the origin,  $O$ , to be the point on the ground beneath  $H$ . Then  $\theta$  is the half-angle subtended by the projection on the ground of the visible part of the fence. Let  $a = OA$  and  $b = AB$ .

For reflection, we have  $h/a = f/b$ ,  $a + b = 2R \cos \theta$ , and by the cosine rule,  $r^2 = a^2 + R^2 - 2aR \cos \theta$ . Hence

$$a = \frac{2Rh \cos \theta}{f + h}. \quad (1)$$

Now  $a^2 - 2aR \cos \theta + R^2 - r^2 = 0$ . Hence, using (1),

$$\cos^2 \theta = \frac{(R^2 - r^2)(f + h)^2}{4R^2hf}. \quad (2)$$

To find the proportion of the fence seen, compute

$$\cos 2\theta = 2 \cos^2 \theta - 1 = \frac{(R^2 - r^2)(f + h)^2}{2r^2hf} - 1.$$

By symmetry the length of the top of the fence seen is  $2 \cdot r \cdot \theta = 4R\theta$ . Hence the proportion seen is

$$\frac{4R\theta}{2\pi R} = \frac{2\theta}{\pi} = \frac{1}{\pi} \cos^{-1} \left( \frac{(R^2 - r^2)(f + h)^2}{2R^2hf} - 1 \right).$$

As a check, if  $R = r$ , then  $\theta = \pi/2$  and all the fence top is visible. If  $f = h$  then  $\sin \theta = r/R$  and  $\cos^2 \theta = 1 - r^2/R^2$  to which (2) reduces.

We can find the equation of  $AA'$ , the top of the fence as seen in the pond. Its cartesian coordinates are

$$x = a \cos \theta = \frac{2Rh}{f + h} \cos^2 \theta, \quad y = a \sin \theta = \frac{2Rh}{f + h} \cos \theta \sin \theta.$$

Therefore the reflection of the top of the fence is defined by

$$y^2 = x \left( \frac{2Rh}{f + h} - x \right)$$

with the restriction given by the equation of the pond,  $(x - R)^2 + y^2 \leq r^2$ .

## Problem 216.3 – Reflection

**Tony Forbes**

What is the function you get when you reflect graph of  $y = e^x$  in the line  $y = ax$ , where  $a$  is a constant?

Note that as  $a$  varies from 0 to  $\infty$  via 1, the function goes from  $-e^x$  to  $e^{-x}$  via  $x$ . We would really like to know what happens in between.

## Where I can put my new Theorem of the Day

**Robin Whitty**

The web site <http://myweb.lsbu.ac.uk/~whitty/MathSci/TheoremOfTheDay> maintains a list of, say,  $L$  theorems. These are picked in turn to be displayed as ‘Theorem of the Day’. To do this, a javascript function measures the number of milliseconds since a fixed base date and divides this by 86400000, the number of milliseconds in twenty four hours. The result, rounded down, is a number of days,  $D$ , which increases by one every midnight. The theorem displayed on any given day is chosen as theorem number  $D \pmod{L}$  in the list (indexed from 0 to  $L - 1$ ).

As often as possible I choose another classic theorem and add a description of it to the web site; with respect to the above, this increases  $L$  to  $L + 1$ , while  $D$  remains fixed. My problem is that I want to do this without the order of presentation of the theorems (and today’s already chosen theorem in particular) being disturbed. So while adding the new theorem I have also to cycle the list of theorems round to bring position  $D \pmod{L}$  into position  $D \pmod{L + 1}$ .

Now the question is: how do I discover the current value of  $D$ ? The answer I would like to give is that I use a scrupulously coded algorithm based on Dershowitz and Reingold’s timeless *Calendrical Calculations* (Cambridge University Press, 2001). The reality is that I cheat: I increase the theorem array size by 1, upload the java code, observe what theorem is now (illicitly) occupying today’s slot and make haste to re-upload with the correct theorem cycled into its rightful position.

However, in the process of pulling this fast one, I do get to find out the value of  $D$ ; and it seems worth mentioning how because it is a nice illustration of one of the Theorems of the Day: the *Chinese Remainder Theorem*.

So, suppose that the day’s theorem is currently in position  $s$  in the list, the value of  $D \pmod{L}$ , and that, when I add the new theorem, the current theorem changes to  $D \pmod{L + 1}$ , which we notice is position  $t$  in the list. This is written as a pair of congruence equations in the unknown,  $D$ :

$$D \equiv s \pmod{L}, \quad D \equiv t \pmod{L + 1}. \quad (1)$$

Now the Chinese Remainder Theorem, in its simplest form, says that

$$x \equiv a \pmod{m}, \quad x \equiv b \pmod{n},$$

where  $\gcd(m, n) = 1$ , is solved, uniquely mod  $mn$ , by

$$x = am(m^{-1} \bmod n) + bn(n^{-1} \bmod m), \quad (2)$$

where  $m^{-1} \bmod n$  is the least positive multiple of  $m$  which has remainder 1 (mod  $n$ ) (and similarly for  $n^{-1} \bmod m$ ).

Now it is easy to see that  $L^{-1} \bmod (L + 1) = L$ , since  $L^2 = (L + 1)(L - 1) + 1 \equiv 1 \pmod{L + 1}$ , and  $(L + 1)^{-1} \bmod L = L + 1$ , since  $(L + 1)^2 = L^2 + 2L + 1 \equiv 1 \pmod{L}$ . So equation (1) is solved by

$$\begin{aligned} D &\equiv s(L + 1)((L + 1)^{-1} \bmod L) + tL(L^{-1} \bmod (L + 1)) \pmod{L(L + 1)} \\ &\equiv s(L + 1)^2 + tL^2 \pmod{L(L + 1)}. \end{aligned}$$

Example: suppose today's theorem is number 4 in the list of 42. I add a new theorem and now the theorem displayed has changed to number 16 (out of 43). Then

$$D \equiv 4 \times 43^2 + 16 \times 42^2 = 35620 \equiv 1306 \pmod{42 \times 43 = 1806}.$$

A doubt remains in my mind: is there not some clever way to find out the value of  $D$  without making today's theorem temporarily change? Perhaps instead of increasing  $L$  to  $L + 1$  some other increased length  $L'$  would reveal  $D$  without changing from theorem  $s$  to  $t$ . We do not need  $L'$  to be coprime to  $L$ ; provided  $s \equiv t \pmod{\gcd(L, L')}$  an extended version of the Chinese Remainder Theorem still applies. Or perhaps there is some other approach altogether that I have not thought of.

## Problem 216.4 – Four fours

### Tony Forbes

In past years, fours and collections thereof have played a significant role in M500 problems, the classic game being to represent numbers by expressions that involve just four 4s together with the usual mathematical apparatus. So for the positive integers we would have  $1 = 44/44$ ,  $2 = 4!/(4 + 4 + 4)$ ,  $3 = 4/4 + 4/\sqrt{4}$ ,  $4 = (\log \log 4 - \log \log \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{4 \cdot 4}}}}}}})/(\log 4)$ ,  $5 = \dots$  well, you get the idea.

Now try the same thing with the fundamental mathematical constants,  $\pi$ ,  $e$ ,  $i$ , the golden ratio  $\phi = (\sqrt{5} + 1)/2$ , and any others you have a particular fondness for. Don't worry if you can't do it exactly; a reasonable approximation might be perfectly acceptable.

## Finding numbers in a grid

### Tony Forbes

See how many numbers you can find in each of the following arrays.

As in the game Boggle, which is played in a similar manner but with letters and words [M500 213 15], you make a number from the symbols shown by stepping along a non-self-intersecting path travelling N, NW, W, SW, S, SE, E or NE from one square to the next (if any). In the first grid, for example, 1003020004 is valid (1, S, 0, SE, 0, E, 3, N, 0, W, 2, SW, 0, SE, 0, E, 0, E, 4) but 100302004 is not (because I can't see how to do it without illegal multiple use of a zero in one of the central squares); nor are 01003 (because it starts with a zero), 103 and 13.

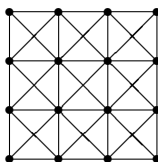
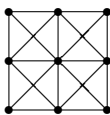
1	0	0	0
0	2	0	0
0	0	3	0
0	0	0	4

1	0	0	2
0	0	0	0
0	0	0	0
3	0	0	4

0	0	0	0
0	1	.	0
0	.	2	0
0	0	0	0

0	0	0	0
0	1	$\frac{1}{2}$	0
0	$\pi$	2	0
0	0	0	0

If you are feeling energetic, perhaps you would like to have a go at a related problem in graph theory. How many paths are there in the planar graph consisting of  $n^2$  vertices arranged in an  $n \times n$  array with edges between nearest neighbours in the eight directions  $0^\circ$ ,  $45^\circ$ ,  $90^\circ$ ,  $\dots$ ,  $315^\circ$ ? For example, the one on the left ( $n = 2$ ) has 4 vertices, 6 edges and contains 30 paths: 6 of length 1, 12 of length 2 and 12 of length 3. The graph on the right corresponds (after multiplication by 2) to the 'Boggle' problem with 16 distinct symbols.



## Problem 216.5 – Equation

### Tony Forbes

Solve

$$x = 3e^{x^2/214}.$$

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*Math Made Visual*  
*Creating Images for Understanding Mathematics*

by **Claudi Alsina and Roger B. Nelson**

Published by The Mathematical Association of America

**Sebastian Hayes**

This is a marvellous book beautifully produced, full of original and interesting ways of proving well-known theorems and also containing many theorems unknown to me. For a long time now algebra has ruled the roost and mathematics has become almost entirely abstract. This puts many people off who otherwise would enjoy mathematics, or parts of it anyway.

‘Is it possible to create mathematical drawings that help students understand mathematical ideas, proofs and arguments? We are convinced the answer is yes’, the authors state in their Introduction. I personally, since I believe that the concrete world actually exists and that mathematics does not transcend it, needed no persuading of the authors’ basic thesis, but even those who are not of this persuasion will probably find the book enlightening. As a pedagogical aid it cannot be praised too highly.

As to price, you will have to go to Amazon and see what is on offer as the book is not available in British bookshops.

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## **Mathematics Revision Weekend 2007**

The **33rd M500 Society Mathematics Revision Weekend** will be held at **Aston University, Birmingham** over **14–16 September 2007**.

The cost, including accommodation (with *en suite* facilities) and all meals from bed and breakfast Friday to lunch Sunday is £195 – £230. The cost for non-residents is £100 (includes Saturday and Sunday lunch). M500 members get a discount of £10. For full details and an application form, see the Society’s web page, [www.m500.org.uk](http://www.m500.org.uk), or send a stamped, addressed envelope to

**Jeremy Humphries, M500 Weekend 2007.**

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. Tutorial sessions start at 19.30 on the Friday and finish at 17.00 on the Sunday. We plan to present most OU mathematics courses.

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