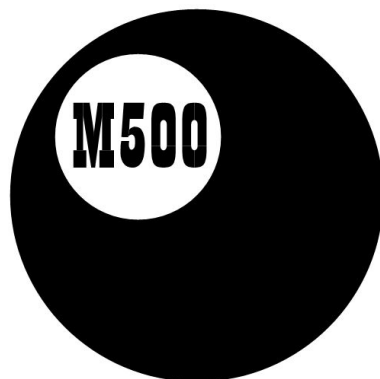
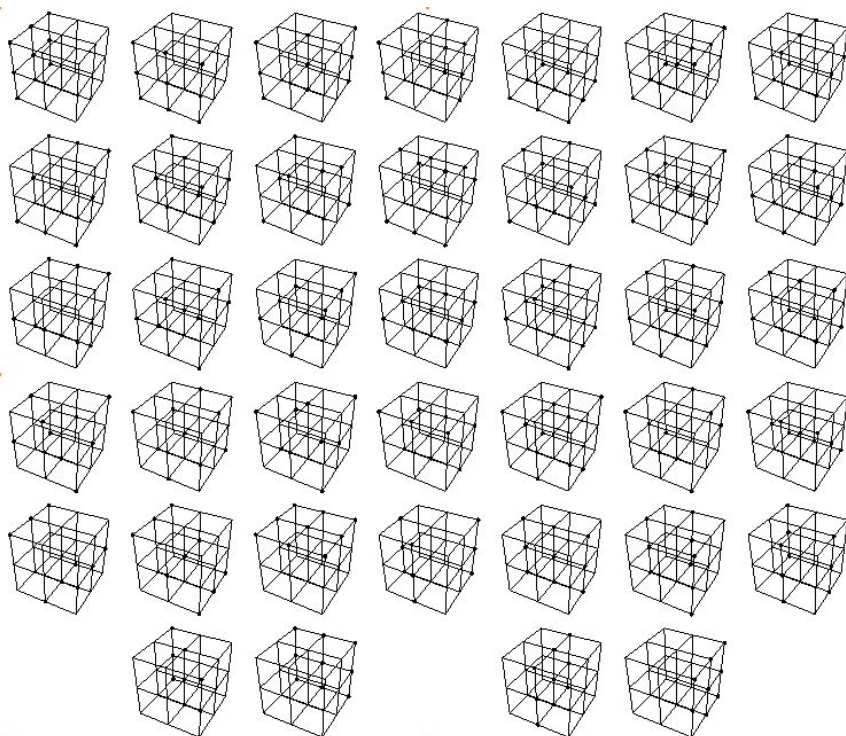


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M500 219



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The Geochron world clock

Rob Evans

This article shall mainly be concerned with solving the following problem.

Let P denote an arbitrary point on the Earth's surface. Then, on the assumption that the Sun appears directly overhead at P , what is the equation (in terms of P) of the boundary between day and night on a map of the Earth's surface that is based on Mercator's projection?

Readers of this magazine who have had a subscription for the last ten years or longer can confirm that the above problem (slightly re-worded) is Problem 151.1. It appears after a description of the so-called 'Geochron World Clock'. In addition to giving the time in each of the world's time zones, this clock shows the boundary between day and night on a map of the world. The author of Problem 151.1 made the assumption that this map is based on Mercator's projection.

The first thing to note about the problem is that the applicability of Mercator's projection to the mapping of the Earth's surface depends on the assumption that that surface is (perfectly) spherical. Thus, we begin this article by making that assumption. In the remainder of this article, we shall adopt the following notation and conventions.

The symbol \mathbf{S} denotes the (perfectly) spherical surface of the Earth; N and S denote the north and south poles respectively. An arbitrary semi-circle whose endpoints are N and S is designated as being 'the central semi-meridian'. Points in \mathbf{S} shall be located by means of a spherical coordinate system. With regards to this coordinate system, $\theta_{\mathbf{S}}$ and ϕ indicate longitude and co-latitude respectively. Longitude is measured eastward from the central semi-meridian. Co-latitude is measured southward from N . Furthermore, we stipulate that $\theta_{\mathbf{S}} \in (-180^\circ, 180^\circ]$ and $\phi \in [0, 180^\circ]$.

(N.B. Firstly, the term 'co-latitude' is a contracted form of the term 'complement of latitude'. In other words, the co-latitude of a given point in \mathbf{S} is 90° minus the latitude of that point. Secondly, throughout this article, despite appearances to the contrary, angles are always measured in radians. The 'degree' symbol is used merely as a function symbol that is written in superscript after the independent variable. Concretely, we have that $x^\circ = 2\pi x/360$ for each $x \in \mathbb{R}$.)

The symbol Π denotes an arbitrary plane. In Π , an arbitrary point is designated as being 'the origin', and is denoted by O . In Π , an arbitrary pair of mutually orthogonal directions are designated as being 'east' and

‘north’. Points in Π shall be located by means of both a rectangular coordinate system and a polar coordinate system. With regards to the rectangular coordinate system, x and y indicate displacement east and north of O respectively. With regards to the polar coordinate system, θ_{Π} and r indicate bearing and distance from O respectively. Furthermore, we stipulate that $\theta_{\Pi} \in (-180^{\circ}, 180^{\circ}]$ and $r \geq 0$.

Here, by definition, bearings are ‘angles of location’ that are measured clockwise from north about O . With regards to the measurement of these angles of location, the determination of which rotary sense is ‘clockwise’ is made by the following requirement. Points east of O have a bearing equal to 90° .

With the preceding notation and conventions understood, the definition of Mercator’s projection that we shall work with is as follows. Mercator’s projection is the function $m : \mathbf{S} \setminus \{N, S\} \rightarrow \Pi$ defined by the rule

$$(\theta_{\mathbf{S}}, \phi) \mapsto (x, y) = (\theta_{\mathbf{S}}, \log \cot \phi/2). \quad (\text{See Figure MP1.})$$

Mercator’s projection has the following characteristic property. Every curve in $\mathbf{S} \setminus \{N, S\}$ that can be traversed by maintaining a given bearing is mapped to a straight line in Π which can be traversed by maintaining the same bearing. (See Figure MP2.) We shall not prove that our definition of Mercator’s projection gives rise to this characteristic property. However, (as readers can confirm) this is a relatively straightforward thing to do.

Here, by definition, bearings are ‘angles of travel’ that are measured clockwise from north. With regards to the measurement of angles of travel in $\mathbf{S} \setminus \{N, S\}$ and Π , the determination of which rotary sense is ‘clockwise’ is made by the following requirement. Easterly travel corresponds to a bearing equal to 90° .

In our solution to the above problem, we shall also need a definition of stereographic projection. The definition of that projection that we shall work with is as follows. Stereographic projection is the function $s : \mathbf{S} \setminus \{N, S\} \rightarrow \Pi$ defined by the rule

$$(\theta_{\mathbf{S}}, \phi) \mapsto (\theta_{\Pi}, r) = (\theta_{\mathbf{S}}, \cot \phi/2). \quad (\text{See Figure SP1.})$$

From this definition, we can immediately see that stereographic projection has the following particularly simple geometrical interpretation.

Firstly, imagine that Π has been rigidly moved so that we end up with Π tangent to \mathbf{S} , O coincident with \mathbf{S} and the polar coordinate system of Π ‘aligned with’ the spherical coordinate system of \mathbf{S} . Put formally, this

condition of alignment is as follows. For each $\theta \in (-180^\circ, 0]$, we have that

$$\Pi_\theta \cap \Pi = \{(\theta_\Pi, r) \in \Pi : \theta_\Pi \in \{\theta, \theta + 180^\circ\}\},$$

where Π_θ denotes the plane in space which is defined implicitly by

$$\Pi_\theta \cap \mathbf{S} = \{(\theta_\mathbf{S}, \phi) \in \mathbf{S} : \theta_\mathbf{S} \in \{\theta, \theta + 180^\circ\}\}.$$

Secondly, with regards to the polar coordinate system of Π , define unit length to be the length of the line segment NS . Then stereographic projection maps $\mathbf{S} \setminus \{N, S\}$ into Π according to the rule

$$X \mapsto NX \cap \Pi,$$

where NX is the line which passes through N and X . (See Figure SP(GI).)

Note that stereographic projection has the following characteristic property. Every circle in $\mathbf{S} \setminus \{N\}$ is mapped onto a circle in Π . (See Figure SP2.) We shall not prove that our definition of stereographic projection gives rise to this characteristic property. However, as readers can confirm, with the aid of the above geometrical interpretation of that projection this is a relatively straightforward thing to do.

Let f denote the function that maps $\Pi \setminus \{O\}$ into Π according to the rule

$$(\theta_\Pi, r) \mapsto (x, y) = (\theta_\Pi, \log r).$$

Then, on inspection of the rules which define m , s and f respectively we have

$$m(\theta_\mathbf{S}, \phi) = f(s(\theta_\mathbf{S}, \phi)) \quad \text{for each } (\theta_\mathbf{S}, \phi) \in \mathbf{S} \setminus \{N, S\}.$$

The fact that this is so suggests the possibility that we can solve the above problem by exploiting the aforementioned characteristic property of s .

A solution to the above problem now follows.

Let P denote an arbitrary point on \mathbf{S} . Moreover, let $O_\mathbf{S}$ denote the centre of \mathbf{S} . In turn, let $\partial(P)$ denote the boundary between day and night on \mathbf{S} when the Sun appears directly overhead at P . Moreover, let $\perp(P)$ denote the plane in space which contains $O_\mathbf{S}$ and which is orthogonal to the line $O_\mathbf{S}P$.

Then, from considerations of symmetry which pertain to the geometry of \mathbf{S} we know that $\partial(P)$ is the great circle $\perp(P) \cap \mathbf{S}$. (Result $\partial(P)$)

Here, we assume that light from the Sun reaches the Earth as a parallel beam of light.

Consequently, from considerations of symmetry which pertain to the definition of m we know that if P moves along a circle of constant latitude in \mathbf{S} (in the easterly direction) then $m(\partial(P))$ moves rigidly in $m(\mathbf{S} \setminus \{N, S\})$ (in the easterly direction).

In reality, P only ever ‘moves’ along a circle of constant latitude in S at the instants of the summer and winter solstices. And, of course, in reality, P never ‘moves’ in the easterly direction! However, this is beside the point. The above paragraph requires that we imagine that the Sun, and thus P , can be moved at will. Moreover, with regards to the corresponding rigid movement of $m(\partial(P))$ in $m(\mathbf{S} \setminus \{N, S\})$ (in the easterly direction) we need to imagine that the set $m(\mathbf{S} \setminus \{N, S\})$ has been rolled up into an ‘infinitely long’ cylinder whereby for each $y_0 \in \mathbb{R}$ we have that the point $(x, y) = (-\pi, y_0)$ now coincides with the point $(x, y) = (\pi, y_0)$.

Moreover, from other considerations of symmetry which pertain to the definition of m we know that if P' is the image of P under a reflection in the equatorial plane then $m(\partial(P'))$ is the image of $m(\partial(P))$ under a reflection in the line $y = 0$. Consequently, without a real loss of generality we shall continue this solution on the assumption that $(\theta_{\mathbf{S}}(P), \phi(P)) = (0, \alpha)$, where $\alpha \in [0, 90^\circ]$.

Firstly, we shall consider the special case whereby

$$(\theta_{\mathbf{S}}(P), \phi(P)) = (0, 0); \quad \text{i.e. } P = N.$$

From Result $\partial(P)$ we can immediately see that $\partial(P)$ (i.e. $\partial(N)$) is the great circle $\phi = 90^\circ$ (the equator). Consequently, from the definition of m we have that $m(\partial(P))$ is the line $y = 0$.

Secondly, we shall consider the special case whereby

$$(\theta_{\mathbf{S}}(P), \phi(P)) = (0, 90^\circ).$$

From Result $\partial(P)$ we can immediately see that $\partial(P)$ is the great circle $\theta_{\mathbf{S}} = \pm 90^\circ$. Consequently, from the definition of m we have that $m(\partial(P) \setminus \{N, S\})$ is the line-pair $x = \pm\pi/2$. For this special case we have that $\{N, S\} \subseteq \partial(P)$. However, $m(N)$ and $m(S)$ are undefined. So, in turn, $m(\partial(P))$ is undefined. Hence, we have to be content with finding $m(\partial(P) \setminus \{N, S\})$.

Having dealt with the above two special cases, we continue this solution on the assumption that $(\theta_{\mathbf{S}}(P), \phi(P)) = (0, \alpha)$ where $\alpha \in (0, 90^\circ)$.

From Result $\partial(P)$ we can immediately see that $\partial(P)$ is a great circle which passes through the two points $(\theta_{\mathbf{S}} = (0, 90^\circ + \phi(P)))$ and $(\theta_{\mathbf{S}} = (180^\circ, 90^\circ - \phi(P)))$. Consequently, from the definition of s and the fact that

s maps circles in $\mathbf{S} \setminus \{N\}$ onto circles in Π we have that $s(\partial(P))$ is a circle which passes through the two points $(\theta_{\Pi}, r) = (0, \cot((90^\circ + \phi(P))/2))$ and $(\theta_{\Pi}, r) = (180^\circ, \cot((90^\circ - \phi(P))/2))$.

However, from considerations of symmetry that pertain to $\perp(P)$ in relation to the geometry of \mathbf{S} we also know that $\partial(P)$ is invariant under a reflection in Π_0 , where Π_0 denotes the plane in space which is defined implicitly by

$$\Pi_0 \cap \mathbf{S} = \{(\theta_{\mathbf{S}}, \phi) \in \mathbf{S} : \theta_{\mathbf{S}} \in \{0, 180^\circ\}\}.$$

Consequently, from considerations of symmetry which pertain to the definition of s we also have that $s(\partial(P))$ is invariant under a reflection in the line $\{(\theta_{\Pi}, r) \in \Pi : \theta_{\Pi} \in \{0, 180^\circ\}\}$.

Finally, with respect to the ordering of points on the line $\{(\theta_{\Pi}, r) \in \Pi : \theta_{\Pi} \in \{0, 180^\circ\}\}$ we can make the following two deductions. Firstly, since the cotangent function is positive on the interval $(0, \pi/2)$ we deduce that O lies strictly between the two points $(\theta_{\Pi}, r) = (0, \cot((90^\circ + \phi(P))/2))$ and $(\theta_{\Pi}, r) = (180^\circ, \cot((90^\circ - \phi(P))/2))$. Secondly, since the cotangent function is strictly decreasing on the interval $(0, \pi/2)$ we deduce that the centre of $s(\partial(P))$ lies strictly between the two points O and $(\theta_{\Pi}, r) = (180^\circ, \cot((90^\circ - \phi(P))/2))$.

From the facts which are laid out in the last three paragraphs we know that $s(\partial(P))$ is the circle which passes through the two points $(\theta_{\Pi}, r) = (0, \cot((90^\circ + \phi(P))/2))$ and $(\theta_{\Pi}, r) = (180^\circ, \cot((90^\circ - \phi(P))/2))$ and whose centre lies on the *open half-line* $\{(\theta_{\Pi}, r) \in \Pi : \theta_{\Pi} = 180^\circ, r \neq 0\}$. Moreover, O lies inside $s(\partial(P))$. (See Figure $s(\partial(P)).1$.)

We continue our solution as follows.

Let O' denote the centre of $s(\partial(P))$. Moreover, let X denote a variable point on $s(\partial(P))$. Then, from our description of $s(\partial(P))$ we know that

$$|OO'| = \frac{1}{2} \left(\cot \frac{90^\circ - \phi(P)}{2} - \cot \frac{90^\circ + \phi(P)}{2} \right), \quad (1)$$

$$|O'X| = \frac{1}{2} \left(\cot \frac{90^\circ - \phi(P)}{2} - \cot \frac{90^\circ + \phi(P)}{2} \right) \text{ for all } X \in s(\partial(P)). \quad (2)$$

However, from standard trigonometric identities we have

$$\cot \frac{90^\circ \pm \phi(P)}{2} = (1 \mp t)(1 \pm t), \text{ where } t = \tan \frac{\phi(P)}{2}.$$

Consequently, from equations (1) and (2) together with some straightforward algebraic manipulation we obtain

$$\begin{aligned}
 |O'X| &= \frac{1+t^2}{1-t^2} && \text{for all } X \in s(\partial(P)), \\
 |OO'| &= \frac{2t}{1-t^2}, && \text{where } t = \tan \frac{\phi(P)}{2}.
 \end{aligned}$$

Hence, using the relevant half-angle formulae we have

$$|O'X| = \sec \phi(P) \text{ for all } X \in s(\partial(P)), \text{ and } |OO'| = \tan s(\partial(P)).$$

(See Figure $s(\partial(P)).2$.)

From an application of the law of cosines to triangle $OO'X$ we obtain

$$\begin{aligned}
 s(\partial(P)) &= \{(\theta_\Pi, r) \in \Pi \setminus \{O\} : \\
 &\quad \sec^2 \phi(P) = r^2 + \tan^2 \phi(P) - 2r \tan(\phi(P)) \cos(\pi - \theta_\Pi)\} \\
 &= \{(\theta_\Pi, r) \in \Pi \setminus \{O\} : \\
 &\quad \sec^2 \phi(P) = r^2 + \tan^2 \phi(P) + 2r \tan(\phi(P)) \cos(\theta_\Pi)\} \\
 &= \{(\theta_\Pi, r) \in \Pi \setminus \{O\} : \\
 &\quad r^2 + 2r \tan(\phi(P)) \cos(\theta_\Pi) + \tan^2 \phi(P) - \sec^2 \phi(P) = 0\} \\
 &= \{(\theta_\Pi, r) \in \Pi \setminus \{O\} : \\
 &\quad r^2 + 2r \tan(\phi(P)) \cos(\theta_\Pi) - 1 = 0\} \\
 &= \left\{ (\theta_\Pi, r) \in \Pi \setminus \{O\} : \right. \\
 &\quad \left. r = -\tan(\phi(P)) \cos(\theta_\Pi) + \sqrt{\tan^2(\phi(P)) \cos^2(\theta_\Pi) + 1} \right\}. \quad (3)
 \end{aligned}$$

However, as was observed immediately prior to this solution we have that $m(\theta_{\mathbf{S}}, \phi) = f(s(\theta_{\mathbf{S}}, \phi))$ for each $(\theta_{\mathbf{S}}, \phi) \in \mathbf{S} \setminus \{N, S\}$, where f is as defined on page 3. Note that we can rewrite the rule for f as

$$f : (\theta_\Pi, r) \mapsto (x, y) \text{ such that } (\theta_\Pi, r) = (x, \exp(y)).$$

Consequently, from the above expression for $s(\partial(X))$ we obtain

$$\begin{aligned}
m(\partial(P)) &= f\left(\left\{(\theta_{\Pi}, r) \in \Pi \setminus \{O\} : \right. \right. \\
&\quad \left. \left. r = -\tan(\phi(P)) \cos(\theta_{\Pi}) + \sqrt{\tan^2(\phi(P)) \cos^2(\theta_{\Pi}) + 1}\right\}\right) \\
&= \left\{(x, y) \in f(\Pi \setminus \{O\}) : \right. \\
&\quad \left. \exp(y) = -\tan(\phi(P)) \cos(\theta_{\Pi}) + \sqrt{\tan^2(\phi(P)) \cos^2(\theta_{\Pi}) + 1}\right\} \\
&= \left\{(x, y) \in (-\pi, \pi] \times \mathbb{R} : \right. \\
&\quad \left. y = \log\left(-\tan(\phi(P)) \cos(\theta_{\Pi}) + \sqrt{\tan^2(\phi(P)) \cos^2(\theta_{\Pi}) + 1}\right)\right\}.
\end{aligned}$$

In other words, $m(\partial(P))$ is the graph of the function which maps $(-\pi, \pi]$ into \mathbb{R} according to the rule

$$x \mapsto y = \log\left(-\tan(\phi(P)) \cos(\theta_{\Pi}) + \sqrt{\tan^2(\phi(P)) \cos^2(\theta_{\Pi}) + 1}\right).$$

With regards to the description of $m(\partial(P))$ obtained by the foregoing solution we note the following. On the assumption that $(\theta_{\mathbb{S}}(P), \phi(P)) = (0, \alpha)$, where $\alpha \in (0, 90^\circ)$, it is (as readers can confirm) a straightforward matter to show that $m(\partial(P))$ has all the properties of symmetry, tangency and intersection that one would expect. (See Figure $m(\partial(P))$.)

Figure S

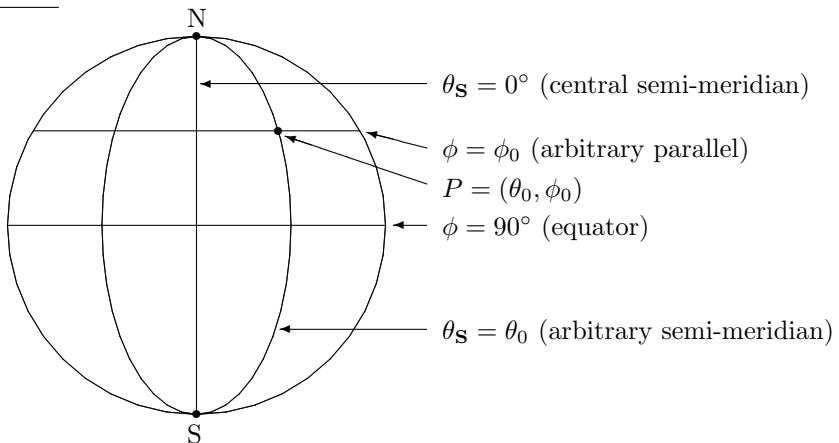
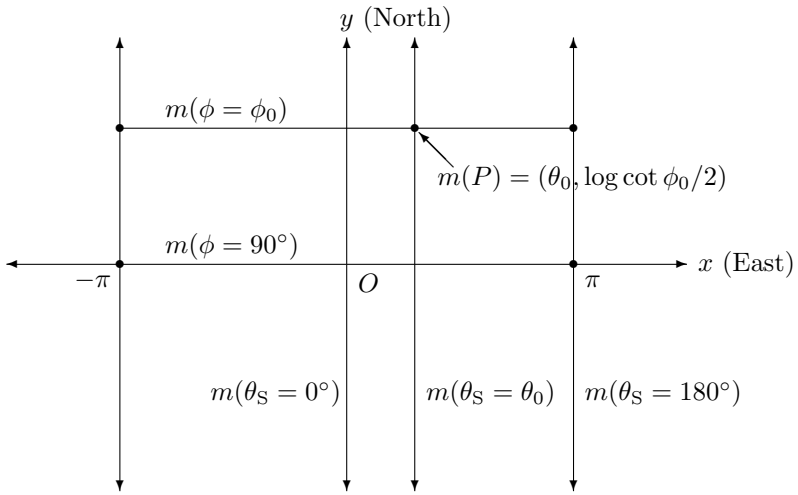
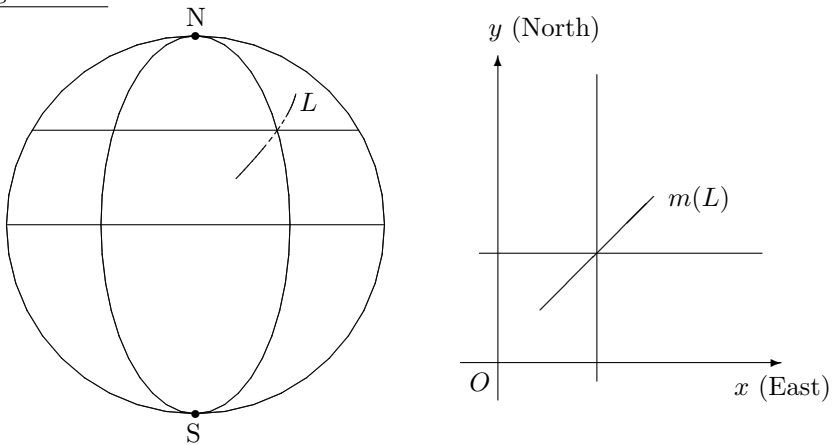


Figure MP1



(See Figure S for interpretation of ϕ_0 and θ_0 .)

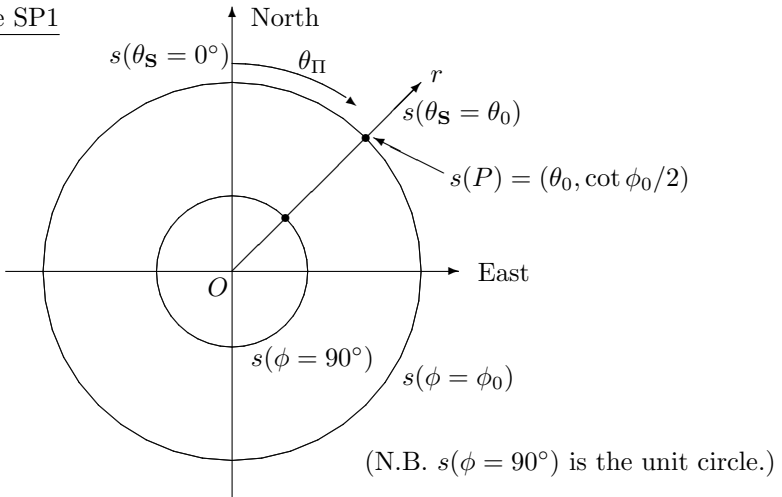
Figure MP2



L is a path on a bearing of k° (in \mathbf{S})

$\Rightarrow m(L)$ is a path is on a bearing of k° (in $\mathbf{\Pi}$).

Figure SP1



(See Figure S for interpretation of ϕ_0 and θ_0 .)

Figure SP(GI)

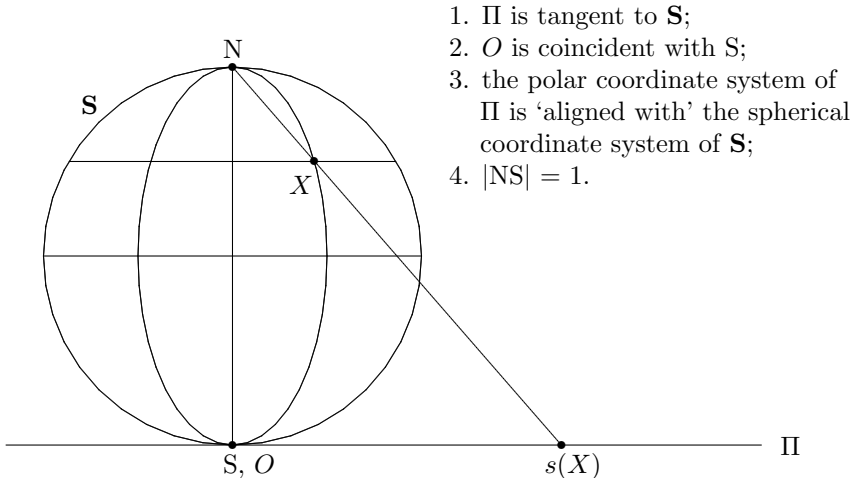
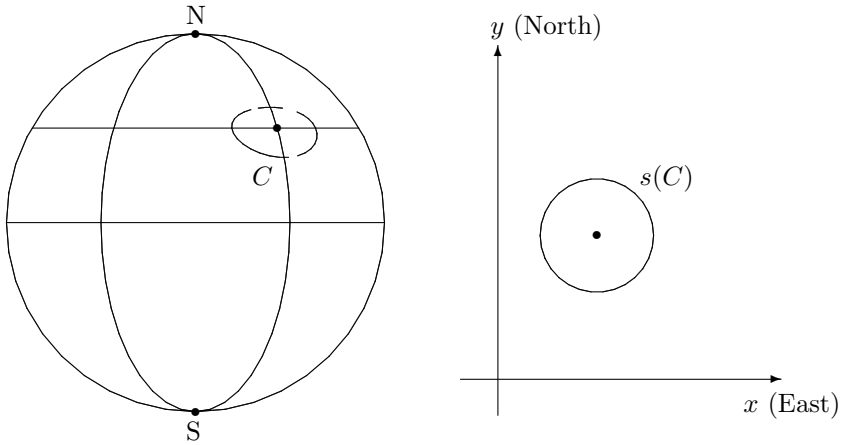
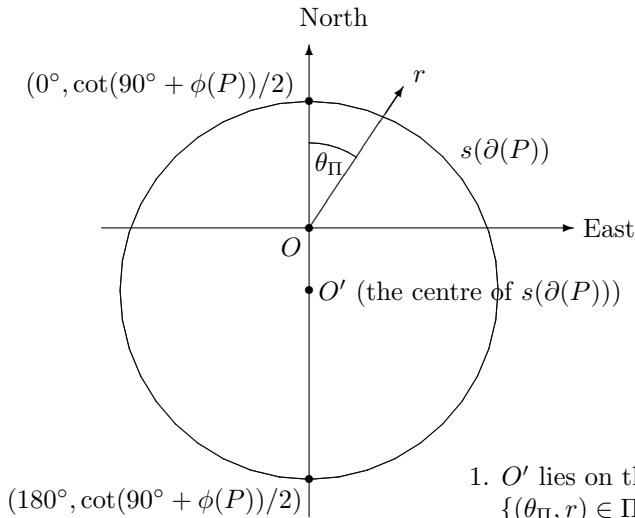


Figure SP2



C is a circle (in \mathbf{S}) $\Rightarrow s(C)$ is a circle (in Π).

Figure $s(\partial(P)).1$



1. O' lies on the open half-line $\{(\theta_{\Pi}, r) \in \Pi : \theta_{\Pi} = 180^\circ, r \neq 0\}$;
2. O lies inside $s(\partial(P))$.

Figure $s(\partial(P)).2$

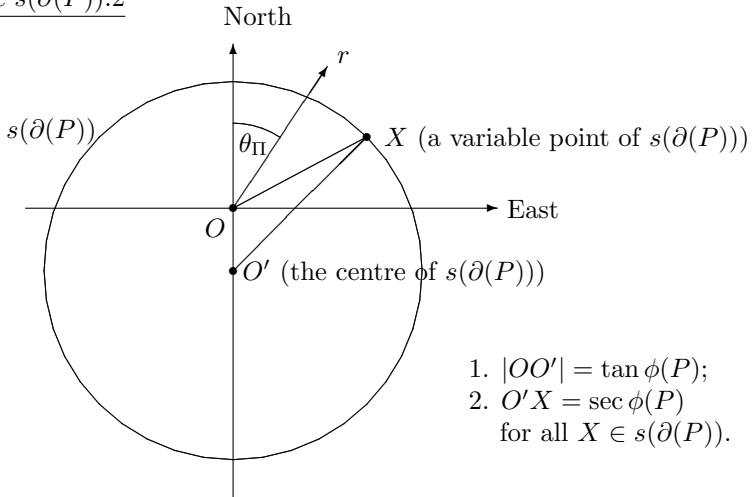
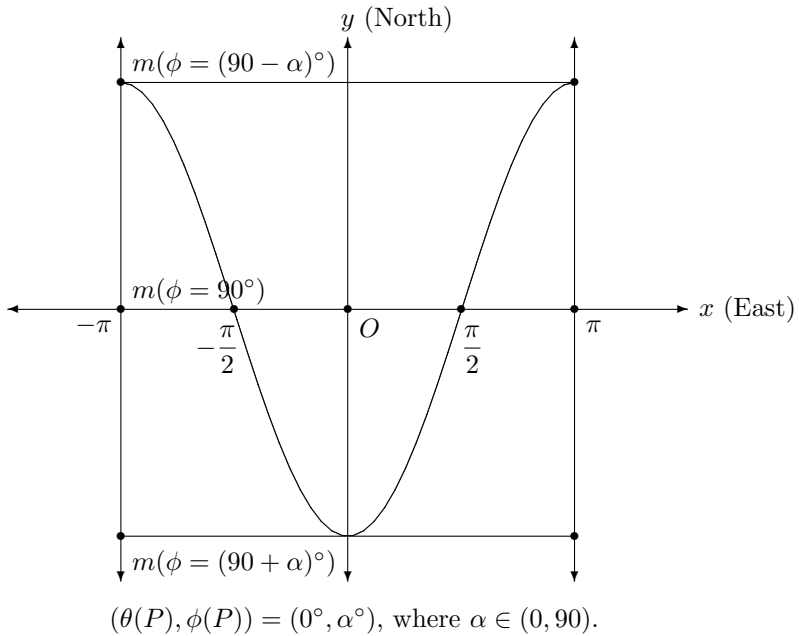


Figure $m(\partial(P))$



Solution 216.2 – Ramanujan’s continued fraction

As it is recounted by Kanigel, *The Man who Knew Infinity* (Abacus, 1991), a Hindu friend of Ramanujan’s, Mahalanobis, when he and Ramanujan were both at Cambridge, read out to him a puzzle from *Strand* magazine about an inhabitant of Louvain (which had just been burned by the Germans). This Belgian lived in a house on a long street which was numbered $1, 2, 3, \dots$ consecutively along his side of the street. The number of his house had a curious property: the sum of all the house numbers before it was the same as the sum of all the house numbers that came after it. The magazine stated that there were more than fifty houses and less than five hundred houses on that side of the street. So what was the Belgian’s house number? Ramanujan thought for a moment and then dictated the first few convergents of a continued fraction which included all the solutions to the problem (not just the one falling within the 50–500 range).

Tommy Moorhouse

The continued fraction most readily associated with this problem is the infinite continued fraction expansion of $\sqrt{2}$.

The reasoning is as follows. Let N be the number of the last house in the street. Then the sum of all the house numbers minus the number of the Belgian’s house, n , is twice the sum of the house numbers up to (not including) that of the Belgian’s house. That is

$$\frac{1}{2}N(N+1) - n = 2 \times \frac{1}{2}n(n-1)$$

by the usual summation formula. This gives

$$N(N+1) = 2n^2$$

which tells us that $N = 2\alpha^2$ and $N+1 = \beta^2$ for some integers α and β . We have chosen to associate the factor of 2 with N and we will see that this makes essentially no difference to the conclusion.

Since $(N+1) - N = 1$ we have

$$\beta^2 - 2\alpha^2 = 1$$

which is an example of Pell’s equation (see Burton [1]) which can be solved in terms of the convergents of the continued fraction expansion of $\sqrt{2}$. We

write down a list of the convergents $p_i/q_i = C_i$ and select those such that $p_i^2 - 2q_i^2 = 1$ (the other convergents satisfy $p_i^2 - 2q_i^2 = -1$ and are related to the other choice for N and $N + 1$). From above we see that

$$n = \alpha\beta, N = 2\alpha^2.$$

For example $C_1 = 3/2$ giving $n = 6$, $N = 2 \times 2^2 = 8$ with the sum 15. $C_3 = 17/12$ leading to $n = 204$, $N = 288$ with a sum of 20706. All the relevant solutions show up in the list of convergents. The maths program MAPLE will find some very large solutions!

[1] D. M. Burton, *Elementary Number Theory*, McGraw–Hill, 1997.

Solution 212.2 – Area of a triangle

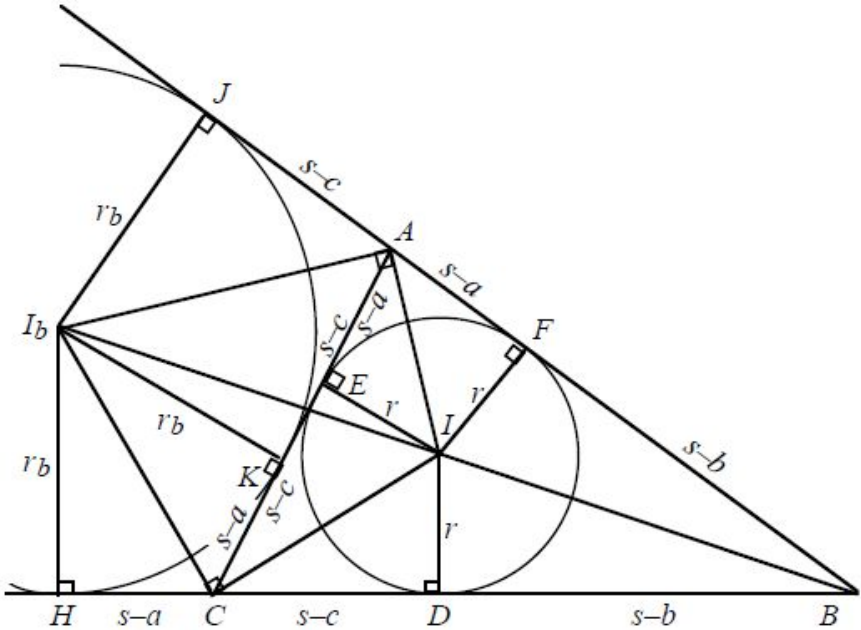
Draw a triangle with side lengths a , b and c . Extend the sides to infinity in both directions. Draw the circles, each of which touches the three (extended) sides. One of these is inside the circle (the in-circle); let this have radius r . The other three circles lie outside the triangle; join their centres to make a big triangle. Prove that the new triangle has area $abc/(2r)$.

John Bull

Maybe a slight exaggeration, but in the 1950s problems of this nature arose in GCE syllabus A, and were tackled in the 5th form. About 80 percent of what follows can be found in school text books of the time (see references). The final 20 percent would have been thrown down as a challenge to students who had finished all the exercises, to keep them quiet until the others catch up. Initial results below are ‘well known’.

Focus on the initial triangle and one of the ex-circles as shown in the diagram. Labelling follows a common convention where the side of length a is opposite point A , and so on. Similarly, ex-centre I_b is opposite point B . Radii are r , r_a , r_b , r_c .

Define the semi-perimeter $s = (a + b + c)/2$. Observe that $BDCH$ and $BFAJ$ are tangents to both in- and ex-circles, so that $BD = BF$ and $BH = BJ$. Also observe that $CKEA$ is a tangent to both circles, so that $CH = CK$, $AJ = AK$, $CE = CD$ and $AE = AF$.



Some algebra on the perimeter of triangle ABC gives $BD = BF = s - b$, $CE = CD = s - c$ and $AE = AF = s - a$, as shown. Further algebra using $BH = BJ$ gives $CH = CK = s - a$ and $AJ = AK = s - c$, as shown. Also, note that $BH = BJ = s$.

Triangle ABC is made up of three smaller triangles AIB , BIC and CIA whose areas are easy to compute (using base \times height / 2), so that triangle ABC has area $\Delta ABC = rc/2 + ra/2 + rb/2 = rs$.

Triangle CI_bA has area $\Delta CI_bA = br_b/2$, and there are two other similar triangles contributing to the large triangle $I_aI_bI_c$. We now need r_b in terms of r , and this can be found from the similar triangles BHI_b and BDI , giving $r_b = rs/(s - b)$. Results for other ex-circles and triangles follow similarly.

Putting all the bits together, we have the area of the specified triangle as:

$$\Delta I_aI_bI_c = \frac{rs}{2} \left(2 + \frac{a}{s - a} + \frac{b}{s - b} + \frac{c}{s - c} \right).$$

This is a perfectly good result but not in the required form, so there is

a little more work to do.

Observe that CI bisects the internal angle C , and CI_b bisects the external angle C . As these angles lie on the same straight line HD , we conclude that the internal and external radii CI and CI_b are perpendicular. Hence $\angle HI_bC = \angle DCI$, $\angle HCI_b = \angle DIC$ and triangles HI_bC and DCI are similar. Ratios of sides give $rr_b = (s-a)(s-c)$. Eliminate r_b between this and the earlier result $R - B = rs/(s-b)$ and we derive $sr^2 = (s-a)(s-b)(s-c)$. This bonus is Heron's formula, usually given as area $\triangle ABC = rs = \sqrt{s(s-a)(s-b)(s-c)}$. We might simply have quoted it, but at least this way we prove all our results.

Substituting this in our earlier result we have

$$\begin{aligned} \triangle I_a I_b I_c &= \frac{(s-a)(s-b)(s-c)}{2r} \frac{rs}{2} \left(2 + \frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} \right) \\ &= \frac{1}{2r} (2(s-a)(s-b)(s-c) \\ &\quad + a(s-b)(s-c) + b(s-a)(s-c) + c(s-a)(s-b)). \end{aligned}$$

The rest, as they say, is algebra. Using the definition of s , amazingly this reduces to

$$\triangle I_a I_b I_c = \frac{abc}{2r}$$

as required.

One might imagine we are now finished, but not so. The above method is a traditional route to the solution, but a result so simple and elegant suggests more to discover; something with the tedious algebra inherent. Observations that might offer clues are: (1) that the triangles CI_aB , I_cAB , CAI_b are all similar to each other (angles equal in the orders given), and (2) that the small triangle formed by joining the points where the in-circle of triangle ABC touches the sides is similar to the large triangle $I_aI_bI_c$ whose area we are trying to find. The in-circle of triangle ABC is the circumcircle of this small triangle, and we know that the area of a triangle ABC is $abc/(4R)$, where R is the circumradius. Where next?

References

1. H. S. Hall and F. H. Stevens, *A School Geometry*, Parts I–V, Macmillan 1902, 1930 edition, page 214.
2. B. Russell, *A Sequel to Elementary Geometry*, Oxford 1907, 1913 edition, page 16.

Problem 214.1 revisited – River crossing

There is a river and a rowing boat which can carry at most two people. A number of married couples are on one bank and they want to cross to the other side of the river. For the usual reason a woman must never be in the presence of a man who is not her husband unless her husband is also present.

- (i) Arrange a crossing schedule for one married couple.
- (ii) Arrange a crossing schedule for two couples.
- (iii) Arrange a crossing schedule for three couples.
- (iv) Can four couples cross the river?
- (v) Show that any number of couples can cross if there is an island in the middle of the river.

ADF

Nobody has sent an answer to this. Obviously the difficult part is (iv). I claim that it's impossible to arrange a crossing for four couples. Unfortunately the only proof I have is a hideously complicated case-by-case analysis which I wouldn't want to inflict on you. I suspect there may be a clever, laterally thinking manner in which to view the problem. If you have an elegant solution to (iv), we would certainly like to publish it.

Now, when I said that nobody has sent an answer, what I really meant was that nobody answered the *intended* problem. Admittedly I now see that the wording is slightly ambiguous. So let's clarify. *When the boat docks, it counts as part of the bank.* Otherwise, (iv) has a trivial solution, as several readers pointed out. Well, my excuse is that surely there are enough clues as to the correct interpretation. Apart from the actual wording of part (iv), which suggests a negative answer, one must imagine the problem in a real-life setting. A simple rowing boat could not possibly provide adequate protection for a woman from the men on the river bank while her husband is on the other side.

Problem 219.1 – Walk

John Spencer

You start facing North, you walk a mile then turn through d degrees, walk another mile, then turn through $2d$ degrees, walk another mile then turn through $3d$ degrees, and so on.

If d is a prime greater than 5, how far have you travelled by the time you next face North?

Problem 219.2 – Balanced sudoku puzzles

Tony Forbes

			1				7	
	6	8			5			
4			7			3		6
			3			2	5	8
				5				
8	5	2			6			
6		3			7			4
			5			8	6	
	7				1			

Look at the sudoku puzzle, above. You will notice that it is symmetric; that is, symbols only appear in pairs of diametrically opposite cells. Most published puzzles have this property for no good reason I can think of other than that they look pretty.

If you look more closely, however, you will see that the symbols in pairs of diametrically opposite cells are always identical. Well, nearly always. The only exception is the pair $\{3, 6\}$ in the central 3×3 box. This leads to a very interesting problem.

Either (i) Construct a sudoku puzzle where *all* pairs of diametrically opposite cells have identical symbols; or (ii) prove that (i) is impossible.

For the record, I suspect that (ii) might be easier.

Problem 219.3 – Circumcircle

A triangle has sides which are the three roots of the cubic

$$x^3 - ax^2 + bx - c.$$

Show that its circumcircle has radius

$$\frac{c}{\sqrt{4a^2b - 8ac - a^4}}.$$

Symmetry and the Monster

by Mark Ronan

Oxford University Press

Dennis Morris

This book is a very easy read. I thoroughly enjoyed it. It is concerned with the history of the classification of the 26 sporadic groups, of which the Monster is the largest. It is full of interesting biographical information about the mathematicians that contributed to the understanding of the sporadic groups. There is almost no mathematics that is not written in simple prose. Only a general overview of the sporadic groups is given. It culminates with comments on Borcherd's proof of the Monster Moonshine theorem for which he was awarded a Fields Medal recently.

Given that the only likely readership of this book is people who know what a group is, the author weakens the text by using non-technical terms like 'atoms of symmetry' instead of 'finite simple groups'. Further, understanding sporadic groups requires profound technical expertise, and no attempt is made to present this expertise—it would not be an easy read if such an attempt was made. Nonetheless, it is a good look at this area of mathematics. It is light bedtime reading.

Problem 219.4 – Pairs

Let T be the set of triples $\{x, y, z\}$ where x, y and z are distinct integers such that $x + y + z = 0$. Characterize those pairs of integers which are not contained in some member of T .

Solution 215.1 – Pythagoras's theorem

A triangle has sides of length a, b and c opposite angles α, β and γ respectively. Prove that $\operatorname{sgn}(\alpha + \beta - \gamma) = \operatorname{sgn}(a^2 + b^2 - c^2)$.

Use the cosine rule to obtain $a^2 + b^2 - c^2 = 2ab \cos \gamma$, note that $\gamma = \pi - \alpha - \beta$ and observe that ab is positive. Then, as **John Spencer** points out, $\operatorname{sgn}(\alpha + \beta - \gamma) = \operatorname{sgn}(\pi - 2\gamma) = \operatorname{sgn}(\cos \gamma)$.

Now devise similar extensions to other familiar trigonometric formulae.

Rearrange the letters of TRIANGLE and you get INTEGRAL. Similarly, as we have already seen in **M500 196**, TWO PLUS ELEVEN becomes ONE PLUS TWELVE. Are there any other anagrammatically related mathematical concepts?

[Ken Greatrix]

Do you know your left from your right?

Ian Adamson

The axioms of a group G are usually given as follows.

- (1) Closure: $\forall a, b \in G, ab \in G$.
- (2) Associativity: $\forall a, b, c \in G, (ab)c = a(bc)$.
- (3) Existence of identity: $\forall a \in G, \exists e \in G$ satisfying $ea = a = ae$.
- (4) Existence of inverses: $\forall a \in G, \exists a' \in G$ satisfying $a'a = e = aa'$.

Now (3) and (4) could be rewritten so that only right identity and inverses are postulated and we can then prove existence of left identity and inverses. Thus (3) and (4) would become

- (3') $\forall a \in G, \exists e \in G$ satisfying $a = ae$;
- (4') $\forall a \in G, \exists a' \in G$ satisfying $e = aa'$.

But what if *right* identity and *left* inverses are postulated? Do you think we could then prove existence of also *left* identity and *right* inverses?

You can't!

Proof. Consider the set $\{a, b\}$ where $aa = a, ab = a, ba = b, bb = b$.

	a	b
a	a	a
b	b	b

Now here there is no left identity or right inverse. Both a and b are right identities, a is the left inverse of both a and b with respect to a ; b is the left inverse of both a and b with respect to b . Closure is obvious from the multiplication table and you can check associativity.

Supplies are running low again. We really want articles of, say, 2-6 pages on mathematical topics and preferably aimed at undergraduate students. Also I note that there are quite a lot of outstanding problems whose solutions could generate interesting publishable material. So we need you out there to get writing.

'If life is a vector space, then what is its basis?'

[Cheng Ku]

Ladders

ADF

Draw a diagram similar to the one on the right. There are $n \geq 2$ vertical lines (*uprights*) and a positive number of horizontal segments (*rungs*) which can go anywhere you like provided that they link two uprights and are pairwise disjoint. In the picture the rungs link adjacent uprights but this is not necessary.

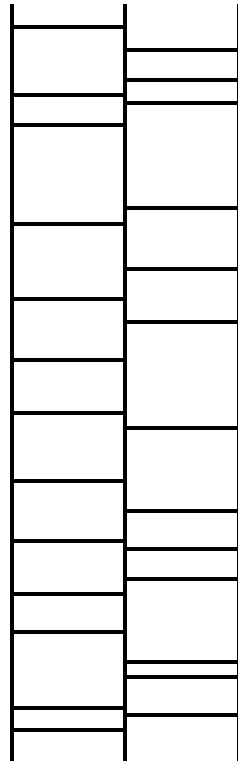
The collection of two uprights and the rungs that link them is called a *ladder*, although the ones that I have seen builders use have their rungs more or less evenly spaced. The whole diagram is then a collection of closely coupled ladders in the sense that adjacent ladders share an upright. Number the uprights $i = 1, 2, \dots, n$.

Now for each i , trace a route from from the base of upright i to the top of upright j , say, according to the following rules.

(a) You must always travel upwards or horizontally, never downwards.

(b) If whilst going upwards you meet a junction between a rung and the upright, you must make a $\pm 90^\circ$ turn onto the rung.

The amazing fact is that two distinct starting points always lead to two distinct end points. See if you can prove it.



Thanks to **Robin Whitty** for showing me this construction. Like the river-crossing problem on page 16, this, too, has practical application.

Suppose three people are to spend the night in a flat. There is a bed, a sofa and a floor. Who gets which? Draw a diagram with three uprights labelled with the names of the persons at the bottom and the words ‘bed’, ‘sofa’, ‘floor’ at the top together with about 30 rungs distributed at random amongst the ladders. Now individuals can trace their routes up the ladders to decide how they are to spend the night.

Mondegreens

ADF

The word *mondegreen* was coined by Sylvia Wright to describe an unintentional misrepresentation of a poem, song or other literary work. Apparently the term originates from her own mishearing of the Scottish ballad *The Bonnie Earl of Murray*:

*Ye Highlands and ye Lowlands,
O where have ye been?
They have slain the Earl of Murray,
And Lady Mondegreen.*¹

Note that a mondegreen is supposed to be the result of a genuine misunderstanding rather than a deliberate parody such as *While shepherds washed their socks*² by night. Here are some more, including some of mine, and all genuine.

From the hymn *Keep Thou My Way*:

*Kept by Thy tender care,
Gladly, the cross-eyed bear.*³

From another Scottish ballad:

*Speed bonnie boat, like a bird on the wing,
"Onward," the sailors cry.
"Larry the Lamb*⁴ *that's born to be king
Over the sea to Skye."*

From another song about ships and the sailing thereof:

*Somebody calls you, you answer quite slowly;
A girl with colitis goes by.*⁵

From a long-running TV advert for washing-up liquid:

*Now hands that judicious*⁶ *can feel soft as your face
With mild green Fairy Liquid.*

And for years I was convinced that Elvis Presley sang

*Warden threw a party in the county jail.
Prison van's siren*⁷ *began to wail.*

Now it's your turn. ...

¹laid him on the green

²watched their flocks

³Gladly the Cross I'd bear

⁴Carry the lad

⁵kaleidoscope eyes

⁶do dishes

⁷The prison band was there and they

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Cover: The thirty-nine planes in $AG(3,3)$, the affine geometry of dimension three over the finite field $GF(3)$.