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Integer logarithms and related zeta functions

Tommy Moorhouse

Introduction

We have previously considered a function from the positive integers to the natural numbers, denoted by κ, with the properties

$$\kappa(nm) = \kappa(n) + \kappa(m); \qquad \kappa(1) = 0.$$

This function has certain interesting properties, and we will see that we can use it to define a number of auxiliary functions with which to explore the properties of the integers. It is perhaps the simplest example of an integer logarithm, namely a function sharing some of the properties of the usual logarithm but having values in the integers. We will see how to generalize κ in a natural way to include all integer logarithms. The main aim of this article, however, is to introduce and very briefly explore some properties of certain related partition (zeta) functions.

One reason for considering partition functions is that they point to a link between number theory and physical systems. This link has been explored in the context of Riemann's ζ function and certain Dirichlet series, but the particular functions we will define may be novel. Although the link with physical systems is only touched on here it may well be that we can develop analogies and insights by pursuing this connection. However, in this article we complete our investigation of the use of integer logarithms to obtain certain recurrence relations.

1 Definitions

We define κ as follows: we can express any positive integer as a product of prime numbers in an essentially unique way (i.e. unique up to ordering). If we agree to order the prime factors of an integer n by magnitude, and label the smallest p_1 and so on up to the largest p_m say, we have

$$n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}.$$

Now we define $\kappa(n)$ as

$$\kappa(n) = \sum_{i=1}^{m} k_i p_i.$$

This gives a well-defined function with the stated property. The generalization to be used later in this article is to replace κ with the function, defined for some auxiliary function ξ (although not all choices of ξ will work),

$$L_\xi(n) = \sum_{i=1}^{m} k_i \xi(p_i).$$

In this notation $\kappa = L_{\text{id}}$ where id is the identity function (an alternative notation, based on that used in [1], is L_N). The reader may wish to check that the more general functions do indeed share the integer logarithm property.

We now consider solutions to the equation $L_\xi(u) = n$ given n. From the definition of L_ξ in terms of the prime factorization of u we immediately see that u corresponds to a unique ξ-partition of n: namely if $u = \Pi_{i=1}^{r} p_i^{k_i}$ then $n = L_\xi(u) = \sum \xi(p_i) k_i$ which is an expression for the prime partition of n:

$$n = \underbrace{\xi(p_1) + \xi(p_1) + \cdots + \xi(p_1)}_{k_1} + \cdots + \underbrace{\xi(p_r) + \xi(p_r) + \cdots + \xi(p_r)}_{k_r}.$$

Each prime partition of n corresponds to an integer u such that $L_\xi(u) = n$, so, denoting the number of ξ-partitions of n by $P_L(n)$, we have

$$|L_\xi^{-1}(n)| = P_L(n),$$

where the set theoretic definition of L_ξ^{-1} is the set of all elements mapped to n by L_ξ. Now consider the product $\Pi_{u \in L_\xi^{-1}(n)} u$. We denote this simply by $\Pi_L(n)$. Then the reader may quickly check that

$$L_\xi(\Pi_L(n)) = n P_L(n).$$

The set $L_\xi^{-1}(n)$ for given n is of particular importance in our study of the zeta functions to be defined below.

2 The zeta function $Z_\kappa(s)$

We define our first zeta function as

$$Z_\kappa(s) = \sum_{n=1}^{\infty} e^{-s\kappa(n)}.$$

In a sense this is the exact analogy of Riemann's $\zeta(n)$ with log replaced by our log-type function κ. We will show that Z_κ is analytic on a certain complex half-plane (from which we can deduce that the number of prime partitions of an integer n is exponentially bounded). We will not give precise definitions of these phrases, but the idea is that $P_L(n) = o(e^{\alpha n})$ in the standard 'little oh' notation [1].

First we note that $\max(\kappa^{-1}(n)) < 4^{n/3}$, which we prove below. From this we deduce that $\kappa(n) > \frac{3}{2\log 2}\log n$ and find a bound for $Z(s)$ in terms of $\zeta(s)$. In fact since $\kappa(N) > \log(n)$ we deduce that $Z_\kappa(s)$ is analytic whenever $\zeta(s)$ is, although we should be careful to check that the analytic continuation of $Z_\kappa(s)$ can be defined over the same domain as $\zeta(n)$.

Lemma $\max(\kappa^{-1}(n)) < 4^{n/3}$.

Outline of proof We consider the cases $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ and $n \equiv 2 \bmod 3$ separately. First let $n \equiv 0 \bmod 3$. Then $n = 3k$ for some k and $\kappa(3^k) = n$. We will show that $m_0 = 3^k$ is the largest such number. First write n as $n = 3(k-5) + 3\cdot 5$. This is the image of $m_1 = 3^{k-5}\cdot 5^3$, which is smaller than m_0 since $m_0/m_1 = 3^5/5^3 > 1$ (take logarithms).

Similarly for any prime $p > 3$ we can write $n = 3(k - rp) + 3rp$ which is the image of $m_s = 3^{k-rp}\cdot p^{3r}$. This is smaller than m_0 since $m_0/m_s = 3^{rp}/p^{3r} = (3^p/p^3)^r > 1$. This establishes the result for $p > 3$. Now write n as $n = 3(k-2t) + 2.3t = \kappa(3^{k-2t}\cdot 2^{3t}) = \kappa(m_2)$ and $m_0/m_2 = 3^{2t}/2^{3t} = (3^2/2^3)^t > 1$.

If $n = 3k+2$ then $m_3 = 2\cdot 3^k$ turns out to be the largest number such that $\kappa(m) = n$, and if $n = 3k+1$ then the largest number mapping to n is $4\cdot 3^{k-1}$. To obtain our upper bound, therefore, we choose $4^{n/3} > 4^k$. This could be refined further, but it is sufficient for our needs. It follows that, given an integer m, there is a lower bound on $\kappa(m)$ (that is to say, $\kappa(m)$ cannot be smaller than some well-defined bound).

We readily find that $\kappa(m) > \alpha\log m$, where $\alpha = 3/2\log 2$. This tells us that $|Z_\kappa(s)| < |\zeta(\alpha s)|$ and that $Z(s)$ is therefore analytic for a large range of complex s. We will not pursue the details further here.

3 The transform $E_f(n)$

Of importance in what follows is the mapping (or κ-transform) defined for any integral function f by

$$E_f(n) = \sum_{m\in\kappa^{-1}(n)} f(m),$$

where the sum is taken over all integers mapping to n under κ. For example, if we have $u(n) = 1$ for all n then $E_u(n) = P(n)$, the number of elements of the set $\kappa^{-1}(n)$. Any integral function will have an associated 'transform', namely the sum of the function to be transformed over the pre-image of (or fibre over) the argument.

We will also make use of the Dirichlet transform (see [1] for the idea behind this expression) of a function f defined by

$$\widehat{f}(n) = \sum_{d|n} f(d).$$

The reader is invited to prove that for any ω we have

$$\kappa * \omega = \nabla\widehat{\omega} - \widehat{\nabla\omega}.$$

This identity will form the basis of another exercise for the reader. We will also need to consider functions related to Z_κ, generally written as

$$F(s) = \sum_{n=1}^{\infty} f(n)e^{-s\kappa(n)}.$$

We define a derivative on integer functions satisfying the Leibnitz rule over the Dirichlet product, namely

$$\nabla f(n) = \kappa(n)f(n).$$

This will be used in the generalized form

$$\nabla f(n) = L_\xi(n)f(n)$$

below. It is an exercise for the reader to show that Leibnitz's rule does indeed hold for all integer logarithms L_ξ (and for $\log(n)$). From this point on we will generally shift our attentions away from κ to the more general functions L_ξ. We note that

$$F(s) = \sum_{n=1}^{\infty} f(n)e^{-s\kappa(n)} = \sum_{n=0}^{\infty} E_f(n)e^{-sn}$$

which we verify by collecting terms in e^{-sm} for each m.

Some interesting identities are needed next. The three identities below are simple consequences of the properties of κ, as the reader is invited to show.

1 κ-Transform of Dirichlet products $E_{f*g}(n) = \sum_{i=0}^{n} E_f(i)E_g(n-i)$.

2 κ-Transform of Dirichlet transforms $E_{\widehat{f}}(n) = \sum_{i=0}^{n} P(i)E_f(n-i)$

3 κ-Transform of derivatives $E_{\nabla f}(n) = nE_f(n)$.

We prove these in a slightly more general context, namely when, instead of κ, we have the integer logarithm function $L_\xi(n) = \sum_i k_i \xi(p_i)$. It is easily checked that L_ξ has many of the same properties as κ and, crucially, the characteristic logarithm properties. The reader who wishes to continue thinking in terms of the concrete example of the function κ will lose little by doing so.

Identity 1 The result follows readily from the following lemma.

Lemma

$$\sum_{n=1}^{\infty} f(n)e^{-sL_\xi(n)} \sum_{m=1}^{\infty} g(m)e^{-sL_\xi(m)} = \sum_{n=1}^{\infty} f*g(n)e^{-sL_\xi(n)},$$

where $f*g(n) = \sum_{d|n} f(d)g(n/d)$ is the Dirichlet product.

Proof. First we note that the product can be rewritten as

$$\sum_{n=1}^{\infty} f(n)e^{-sL_\xi(n)} \sum_{m=1}^{\infty} g(m)e^{-sL_\xi(m)} = \sum_{n=1}^{\infty} \sum_{lm=n} f(l)g(m)e^{-sL_\xi(n)},$$

since $L_\xi(l) + L_\xi(m) = L_\xi(lm)$ by the logarithm property. The product on the right-hand side is just $\sum_{n=1}^{\infty} f*g(n)e^{-sL_\xi(n)}$.

The next step is to look at the definition of $E_f^{L_\xi}$, where the notation is intended to show which transform is being used. The superscript will be dropped in what follows for ease of notation. It is clear that

$$\sum_{n=1}^{\infty} f(n)e^{-sL_\xi(n)} = \sum_{m=0}^{\infty} E_f(m)e^{-sm}$$

by a counting argument. Then we must have, by the lemma,

$$\sum_{m=0}^{\infty} E_f(m)e^{-sm} \sum_{n=0}^{\infty} E_g(n)e^{-sn} = \sum_{k=0}^{\infty} E_{f*g}(k)e^{-sk}.$$

The final step is to note that

$$\sum_{m=0}^{\infty} E_f(m)e^{-sm} \sum_{n=0}^{\infty} E_g(n)e^{-sn} = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} E_f(i)E_g(k-i) \right) e^{-sk}.$$

Identity 2 We observe that $\widehat{f} = f * u$, where $u(n) = 1$ for all n. The result follows at once from Identity 1.

Identity 3 We have $\nabla f(n) = L_\xi(n) f(n)$ so that

$$E_{\nabla f}(n) = \sum_{m \in L_\xi^{-1}(n)} \nabla f(m) = n \sum_{m \in L_\xi^{-1}(n)} f(m) = n E_f(n).$$

The second equality follows from the fact that for all $m \in L_\xi^{-1}(n)$ we have $L_\xi(m) = n$.

In our further explorations we can call on a catalogue of number-theoretic functions including those listed for illustration below. The notation largely agrees with Apostol (except for τ and κ).

$$I(n) = \delta_{1i}; \tag{1}$$
$$u(n) = 1; \tag{2}$$
$$N(n) = n; \tag{3}$$
$$\tau(n) = \sum_{d|n} 1; \tag{4}$$
$$\sigma(n) = \sum_{d|n} d. \tag{5}$$

We have, for example, $\tau = \widehat{u}, u = \widehat{I}$ and lots of other identities that we can use to find the κ-transforms of these functions. The reader may like to show that the κ-transform of the identity $\kappa * \omega = \nabla \widehat{\omega} - \widehat{\nabla \omega}$ is consistent (that is, it gives an expression on each side of the equals sign that is true for all relevant arguments.)

The zeta function ... is the generating function We can easily see by the rearrangement of $Z(s)$ into

$$Z(s) = \sum_{m=0}^{\infty} P(m) e^{-sm}$$

that Z is the generating function for P, but we are now in a position to take an excursion into the use of the κ-type transforms that will help us see how this generating function works and leads to a well-known alternative form due to Euler. From now on we will use the more general logarithm type functions L_ξ, of which κ is a special case. In this case we write P_L to indicate that we are considering the more general functions.

Take as our zeta function

$$Z_\xi(s) = \sum_{n=1}^{\infty} e^{-sL_\xi(n)}$$

and consider

$$\frac{d}{ds} Z_\xi(s) = -\sum_{n=1}^{\infty} L_\xi(n) e^{-sL_\xi(n)}$$

which we know (from our lemmas above) is equal to

$$-\sum_{n=0}^{\infty} P_L(n) e^{-sn},$$

writing $P_L(n)$ instead of $|L_\xi^{-1}(n)|$. Now we need a new result.

Lemma There is a recurrence relation for P_L of the form

$$nP_L(n) = \sum_{i=0}^{n} c(i) P_L(n-i),$$

where the function $c(i)$ is defined as a sum over all prime p as

$$c(i) = \sum_{\xi(p)|i} \xi(p).$$

Proof. This follows by considering the set of all partitions of n as $\sum_j k_j \xi(p_j)$. Then there are $P_L(n-\xi(2))$ partitions in which $\xi(2)$ appears at least once, $P_L(n-2\xi(2))$ in which it appears twice and so on. Continuing in this way we see that the power of 2 appearing in $\Pi_L(n)$ is

$$P_L(n-\xi(2)) + P_L(n-2\xi(2)) + \cdots$$

with a similar expression for the powers of each prime. Applying L_ξ to $\Pi_L(n)$ we have

$$\begin{aligned} nP_L(n) \quad = \quad & \xi(2)(P_L(n-\xi(2)) + P_L(n-2\xi(2)) + \cdots) \\ & + \xi(3)(P_L(n-\xi(3)) + P_L(n-2\xi(3)) + \cdots) \\ & + \cdots \\ & + \xi(p)(P_L(n-\xi(p)) + P_L(n-2\xi(p)) + \cdots). \end{aligned}$$

Collecting terms in $P_L(n-r)$ we see that the coefficient is just the sum of those $\xi(p)$ dividing r, counted once. This is the required recurrence relation.

Using the lemma we have

$$\frac{d}{ds}Z_\xi(s) = -\sum_{n=0}^{\infty}\sum_{i=0}^{n} c(i)P_L(n-i)e^{-sn}.$$

From our earlier work we see that the term in the second sum is of the type $E_{f*L_\xi}(n)$, where $c(i) = E_f(i)$. It is not difficult to show that the function $f(p^r) = \xi(p)$ (non-zero only on prime powers) has the property $E_f(n) = c(n)$.

We now have

$$\frac{d}{ds}Z_\xi(s) = -A(s)Z(s)$$

with

$$A(s) = \sum_{n=0}^{\infty} f(n)e^{-s\xi(n)}.$$

Since f is non-zero only for arguments that are prime powers we can rewrite $A(s)$ as

$$A(s) = \sum_p \xi(p)\sum_{r=1}^{\infty} e^{-srL_\xi(p)},$$

where the first sum is over all primes and the second follows from the fact that only powers of primes are represented in the sum since f vanishes for other values of n. The crucial point is that $L_\xi(p^r) = rL_\xi(p)$, giving the terms in the exponentials in the second sum. The second sum then reduces to

$$e^{-sL_\xi(p)}\big/\big(1-e^{-sL_\xi(p)}\big)$$

by summing the geometric series.

The solution to the differential equation for $Z(s)$ is

$$Z(s) = Ke^{-\int A(s)ds}$$

and

$$\int A(s)ds = \int ds \sum_p \frac{\xi(p)e^{-sL_\xi(p)}}{1-e^{-sL_\xi(p)}}.$$

Noting that $L_\xi(p) = \xi(p)$ directly from the definition we can integrate at once to find that

$$Z(s) = \prod_p \frac{1}{1-e^{-s\xi(p)}}.$$

We can now 'plug in' any suitable function ξ to obtain the generating function $Z(s)$. Interestingly, although some of our arguments are not strictly reproducible for non-integral functions, setting $\xi(n) = \log(n)$ reproduces the usual product representation of Riemann's ζ function.

The zeta function as a partition function

Statistical thermodynamics makes use of a function, defined for a system (or for an ensemble of systems), from which many of the thermodynamic properties of the systems can be deduced. This is called the partition function [5] (a name which is particularly relevant in this context) and it can be directly compared with our zeta functions. In this context the partition function may be written as

$$q(\beta) = \sum_{n=0}^{\infty} e^{-\beta\epsilon_i}.$$

Here, β is the conventional symbol for $1/kT$, where k is the Boltzmann constant, T is the 'absolute' temperature and ϵ_i is the energy of the i-th state of the system in question. If $\epsilon_i = \epsilon_0\kappa(i)$ (taking care to shift the sum to avoid the awkward argument 0) we begin to see a link between our zeta function and the partition function of a 'physical' system. In fact if we consider a system of non-interacting harmonic oscillators, one for each prime p with energy spacing $\epsilon_0 p$, it is tempting to suggest that the statistical occupancy of a state of energy ϵn is just the prime partition function $P(n)$. This may be of limited practical use in the physics world, but the link between basic thermodynamics and the κ-transform (and its relatives) begs the question: is there scope for more exploration? The reader may wish to start with [6].

Conclusion

Although the final results set out above can be readily obtained by other means we have been able to explore some interesting properties of integer logarithm functions. There is much entertainment to be had exploring these functions further and hopefully the reader will feel ready to do so. The books listed below fill in the required background. Apostol [3] in particular is useful for those who want a quick introduction to number theory, while [1] goes further and makes use of complex analysis to explore the ζ function and its relatives. Although it could now be considered rather old fashioned in its style, [2] is a thorough introduction to the techniques of complex analysis; [4] is a more modern treatment with applications. This list is intended only as a glimpse at the kind of material that can be used as a reference for the mathematics in this article.

The physical chemistry text [5] is an excellent introduction to several essential areas of chemistry, including a nice exposition of quantum mechanics in familiar terms, with a decent guide to thermodynamical ideas. Knauf [6] is a quite recent overview of some of the links between number theory (via the ζ function and its relatives) and statistical mechanics. Born [7] is a classic text covering a wealth of classical and quantum physics including quantum statistics at a very accessible level, although it may seem dated in parts.

References and Useful Books

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1998.

[2] Whittaker and Watson, *A Course of Modern Analysis* (fourth edition), Cambridge University Press, 1927.

[3] D. M. Burton, *Elementary Number Theory*, McGraw–Hill, 1997.

[4] J. W. Dettman, *Applied Complex variables*, Dover Publications, 1984.

[5] P. W. Atkins, *Physical Chemistry*, Oxford University Press, 1978.

[6] A. Knauf, *Number Theory, Dynamical Systems and statistical mechanics: Lecture Notes*, Max Planck Institute, 1998.

[7] A. Born, *Modern Physics* (out of print?)

Solution 217.2 – Chords and regions

> We have n points situated irregularly on the circumference of a circle. They are joined by straight lines in all possible ways. What is maximum number of regions into which the lines divide the circle?

A. J. Moulder

Method 1. My U3A Mathematics Group leader has passed the following solution to this problem.

Let R_n be the number of regions for n points on the circumference. By drawing, the first few values we obtain are

n	1	2	3	4	5
R_n	1	2	4	8	16

but R_6 is 31 instead of the expected 32. Proceeding further we get the following values from which we can construct a difference table.

n	0	1	2	3	4	5	6	7	8
R_n	1	1	2	4	8	16	31	57	99
D_1		0	1	2	4	8	15	26	42
D_2			1	1	2	4	7	11	16
D_3				0	1	2	3	4	5
D_4					1	1	1	1	1

Thus a quartic expression is suggested. Suppose it is $an^4 + bn^3 + cn^2 + dn + e$. For $n = 0$, $R_n = 1$; so $e = 1$, and from the other values of n we get

$$n = 1\colon\ a + b + c + d = 1, \qquad n = 2\colon\ 16a + 8b + 4c + 2d + 1 = 2,$$
$$n = 3\colon\ 81a + 27b + 9c + 3d = 4, \qquad n = 4\colon\ 256a + 64b + 16c + 4d + 1 = 8.$$

These can be expressed in matrix form $\mathbf{Ax} = \mathbf{h}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 \\ 81 & 27 & 9 & 3 \\ 256 & 64 & 16 & 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 7 \end{pmatrix}.$$

The system of equations can be solved by Gaussian elimination to give

$$a = \frac{1}{24}, \quad b = \frac{-6}{24}, \quad c = \frac{23}{24}, \quad d = \frac{-18}{24}, \quad e = 1.$$

Hence

$$R_n = \frac{1}{24}\,(n^4 - 6n^3 + 23n^2 - 18n + 24).$$

As a check on this result, **Method 2** uses combinatorics as follows.

The circle with no points has one region. Each time a chord between two points is added, the number of extra regions is one more than the number of intersections the chord makes with those already there. The total number of regions is then 1 + number of chords + number of intersections; i.e.

$$R_n = 1 + \binom{n}{2} + \binom{n}{4}, \qquad n > 0.$$

This can be resolved by using the definition $\displaystyle\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to give the same expression as above.

His parting comment was that this is a very old question.

Stuart Walmsley

Preamble

The maximum possible number of regions occurs when the number of intersections is minimized; that is, when no more than two lines intersect at a point. The problem is analysed assuming that the configuration has this property. The irregular placing of the points helps to ensure this.

Networks and Euler's relation

The figure which results from the construction in the problem is a network with three kinds of components:

vertices: points of intersection;

edges: line segments (not necessarily straight) joining two vertices;

regions: areas bounded by perimeter of edges.

If the numbers of vertices, edges and regions in the system are denoted by V, E and R, Euler has shown that

$$V - E + R \;=\; 1.$$

Here it is R that is required, and the relation becomes

$$R \;=\; 1 + E - V.$$

The strategy followed here is to derive formulae for V and E and hence obtain R.

The number of vertices V

There are two sets of vertices: those on the circle and those within it. Their numbers will be denoted by C and I respectively. Clearly $C = n$ since the problem is defined by the n points on the circle.

The straight lines in the system join all possible pairs of points on the circle. Each internal vertex is characterized by being the intersection of two of these straight lines and so can be labelled uniquely by the set of the four points at the ends of these lines. The number of such vertices is then the number of ways of choosing four objects from a set of n objects; that is,

$$I \;=\; \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}.$$

The total number of vertices is then given by

$$V \;=\; C + I = n + I.$$

The number of edges E

The number of edges can be found by counting the number of edges at each vertex and halving, on the basis that each edge is terminated by two vertices.

The n vertices on the circle are each the terminals of straight edges from the directions of the other $n-1$ points, together with two circular edges from the two adjacent points on the circle. This gives a contribution to the total number of edges of $n(n+1)/2$.

The vertices within the circle (I in number) are each at the intersection of two straight lines and therefore four edges. In this way, the associated edges total $4I/2 = 2I$, and the total number of edges is

$$E = \frac{n(n+1)}{2} + 2I.$$

The number of regions R

The number of regions can now be obtained from the Euler relation:

$$\begin{aligned} R &= 1 + E - V \\ &= 1 + \frac{n(n+1)}{2} + 2I - n - I \\ &= 1 + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)}{24}. \end{aligned}$$

It will be seen that this can conveniently be rewritten in terms of the binomial coefficients

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

as

$$R = \binom{n}{0} + \binom{n}{2} + \binom{n}{4},$$

which gives a neat form for the result.

The other results, V and E, can also be written in this form:

$$\begin{aligned} V &= \binom{n}{1} + \binom{n}{4}, \\ E &= \binom{n}{1} + \binom{n}{2} + 2\binom{n}{4}. \end{aligned}$$

Expressions that maintain their form under multiplication

Dennis Morris

Quadratic expressions of the forms

$$x^2 + y^2, \qquad x^2 - y^2$$

maintain their form when multiplied together:

$$(a^2 + b^2)(c^2 + d^2) = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 = (ac - bd)^2 + (ad + bc)^2,$$

$$(a^2 - b^2)(c^2 - d^2) = a^2c^2 - a^2d^2 - b^2c^2 + b^2d^2 = (ac + bd)^2 - (ad + bc)^2.$$

So do cubic expressions of the form

$$x^3 + y^3 + z^3 - 3xyz:$$

$$\begin{aligned}(a^3 + b^3 + c^3 - 3abc)&(d^3 + e^3 + f^3 - 3def) \\ &= (ad + bf + ce)^3 + (ae + bd + cf)^3 + (af + be + cd)^3 \\ &\quad - 3(ad + bf + ce)(ae + bd + cf)(af + be + cd).\end{aligned}$$

There are such non-trivial expressions that maintain their form under multiplication in all powers. They are easy to find.

Choose any group and write its Cayley table with the identities on the leading diagonal. For groups of order four and above, there is more than one way to do this, but all ways work. Copy that Cayley table into a matrix in which each element of the group becomes an independent variable. For example, the group C_4:

$$\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ d & c & a & b \\ c & d & b & a \end{bmatrix},$$

where we have put the variable a in place of the identity. (The groups do not have to be commutative; the matrices do not have to be commutative.) Each variable configuration is effectively one of the permutation matrices that form the particular group. In the case of C_4, these permutation matrices are

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right\},$$

and such a set of permutations is closed under multiplication—it is a group. Thus, such matrices maintain their form under matrix multiplication. For example:

$$\left\{\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ d & c & a & b \\ c & d & b & a \end{bmatrix}\begin{bmatrix} e & f & g & h \\ f & e & h & g \\ h & g & e & f \\ g & h & f & e \end{bmatrix} = \begin{bmatrix} W & X & Y & Z \\ X & W & Z & Y \\ Z & Y & W & X \\ Y & Z & X & W \end{bmatrix}\right\},$$

where

$$W = ae + bf + ch + dg, \quad X = af + be + cg + dh,$$
$$Y = ag + bh + ce + df, \quad Z = ah + bg + cf + de.$$

The determinant of such a matrix and the determinant of the product of two such matrices are of the same form, and the determinant of a product is equal to the product of the determinants. Thus, the determinants are expressions whose form is maintained under multiplication. In the case above, we have

$$\begin{aligned} &(a^4 + b^4 - c^4 - d^4 - 2a^2b^2 + 2c^2d^2 - 4a^2cd + 4abc^2 + 4abd^2 - 4b^2cd) \\ &\times (e^4 + f^4 - g^4 - h^4 - 2e^2f^2 + 2g^2h^2 - 4e^2gh + 4efg^2 + 4efh^2 - 4f^2gh) \\ &\quad = W^4 + X^4 - Y^4 - Z^4 - 2W^2X^2 + 2Y^2Z^2 \\ &\qquad - 4W^2YZ + 4WXY^2 + 4WXZ^2 - 4X^2YZ. \end{aligned}$$

That this expression will factorize to

$$\begin{aligned} &(a^4 + b^4 - c^4 - d^4 - 2a^2b^2 + 2c^2d^2 - 4a^2cd + 4abc^2 + 4abd^2 - 4b^2cd) \\ &\quad = \big((a+b)^2 - (c+d)^2\big)\big((a-b)^2 + (c-d)^2\big) \end{aligned}$$

wherein we find the quadratic expressions at the start of this article is because the group C_4 contains a C_2 subgroup.

Although the expressions above have unity coefficients, this is not necessary. For example the matrix with quadratic determinant:

$$\begin{bmatrix} a & b \\ jb & a \end{bmatrix}$$

maintains its form under matrix multiplication, and the matrices with cubic determinant have two such free parameters (but we must not divide by zero), and so the expression

$$a^3 + jkb^3 + \frac{j^2}{k}c^3 - 3jabc$$

maintains its form under multiplication.

Such expressions are the distance functions of n-dimensional geometric spaces where n is the power of the expressions. (By geometric space, we mean a rotation matrix containing n different trigonometric functions and a distance function (not necessarily a metric).)

Reference: Dennis Morris, *Complex Numbers—The Higher Dimensional Forms: The Unification of Groups, Algebra, and Geometry & The Nature of Space*, ISBN: 978-0-955600-30-2.

Thirty-five years of M500

ADF

Good grief, it really has been that long. M500 began life as the *Solent M202 Newsletter*. No. 1 was published in 1973; according to founding editor **Marion Stubbs**: At 2300 hrs., precisely, 16 February, 1973, Southampton, England, 51° N, 1°25′ W (born out of despair)—24 copies, together with an application form to join the Solent OU Mathematics Self-Help Scheme, dashed off in four hours flat for an M202 tutorial.

Of course, it was an instant success. M202, *Topics in Pure Mathematics*, was, to say the least, challenging; perhaps the most difficult course the Maths Faculty had to offer at the time, and the *Newsletter* was just the kind of thing that struggling, geographically isolated students needed.

The self-help scheme became 'MOUTHS' (Mathematics OU Telephone Help Scheme) and the newsletter spread in all directions. When issue 6 went out in July, 1973, readers were invited to supply a new title as it was no longer restricted to the Solent, nor to M202. **Peter Weir** suggested 'M500' for reasons: (1) Why not? (2) A top-level course in communications. Full credit. (3) It's an overview of OU maths. (4) Why not? [*sic*] (6) [*sic*] I thought of it. By Issue 7, the first to bear the name 'M500', readership had risen to about 200, and later it rose to over 500 when the Faculty allowed M500 to be publicized in the maths stop presses.

The distinctive logo, shamelessly modelled on the OU shield, first appeared on M500 **59**.

The editorship has remained remarkably stable throughout the last 35 years. Marion did everything for the first few issues. Eddie joined her from Number 25 and thereafter Eddie edited while Marion published. Jeremy was recruited as Problems Editor and later took over from Eddie at M500

68. Seventeen years later Jeremy handed the job to me at M500 **157**.

"Out of all the [undergraduate mathematics] magazines I've seen, you're the best," was the unsolicited comment of an eminent mathematician. It's because you the readers are the contributors. If you look at other similar publications, you will often notice a pretty obvious division: authors are superior omni-cognate beings, readers are inferior mortals. However, there's no such class distinction in M500. Readers and writers are equal.

We will continue to flourish if you keep up the good work. You have done very well to keep the thing going for such a long time against all the competition—many, many thanks. You have demonstrated that the entire resources of the Internet can never provide an adequate substitute for a real paper journal dropping through your letter-box. But do keep the articles coming; as usual, be as informal as you like and write to us about anything to do with mathematics and at any level.

Solution 218.4 – Repeated differentiation

Show that $$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n\, n!}{x^{n+1}}\left(\log x - \sum_{r=1}^{n}\left(\frac{1}{r}\right)\right).$$

Steve Moon

We use induction. Let $\mathcal{P}(n)$ be the proposition

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n\, n!}{x^{n+1}}\left(\log x - \sum_{r=1}^{n}\left(\frac{1}{r}\right)\right).$$

Clearly $\mathcal{P}(1)$ is true. Suppose $\mathcal{P}(k)$ is true. Then we have

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}\left(\frac{\log x}{x}\right) &= (-1)^k k!\,\frac{x^{k+1}/x - (k+1)x^k\left(\log x - \sum_{r=1}^k\right)}{(x^{k+1})^2}\\ &= (-1)^k (k+1)!\,\frac{1/(k+1) - \left(\log x - \sum_{r=1}^k\right)}{x^{k+2}}\\ &= \frac{(-1)^{k+1}(k+1)!}{x^{k+2}}\left(\log x - \sum_{r=1}^{k+1}\right).\end{aligned}$$

Dingbats

It wasn't all fun and games at the recent M500 Winter Weekend. To give you some idea of the challenging tasks that were presented, here's a selection of puzzles contributed by **Tracey Cool**. See how many you can get before looking up the answers.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$t\cos 0$

$$\sum_{n=0}^{\infty}\frac{2}{2^n} \qquad \sum_{n=0}^{\infty}\frac{1}{2^n}$$

$17\sqrt{64}$ $7(9-1)$

47×2^3

3×8

$3(3+5)$

$19\left(\frac{16}{2}\right)$ $37\times 4^{3/2}$

$$\frac{d\left[4\int_0^1\sqrt{1-x^2}dx\right]}{dt}$$

π^2 $\Gamma(1/6)$ $2^{\sqrt{2}}$ e

Essential:

$$\frac{1}{n}\sum_{j=1}^{n}x_j$$

$$\frac{1}{n-1}\sum_{j=1}^{n}(x_j-\overline{x})^2$$

$$\begin{bmatrix}x\\y\end{bmatrix}\mapsto\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}p\\q\end{bmatrix}$$

$\heartsuit$

$$\sum_{n=0}^{\infty}\frac{1}{n!} \quad - \quad ♂$$

$$\sum_j x_j \, \mathbb{P}(X = x_j)$$

$$\int_{-\infty}^{\infty} x f(x) \, dx$$

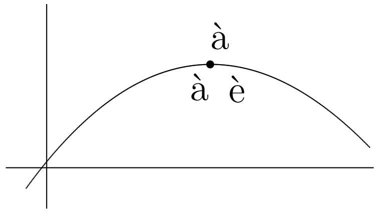
$$(g \circ f)'(\alpha) \;=\; g'(f(\alpha))f'(\alpha)$$

$$\begin{aligned} &\mathrm{CHOH(CH_2OH)_2 + 3\,HNO_3} \\ &\quad \rightleftharpoons \; \mathrm{CHONO_2(CH_2ONO_2)_2} \\ &\qquad + \; \mathrm{3\,H_2O} \end{aligned}$$

Winter Weekend Organizer

Winter Weekend Organizer

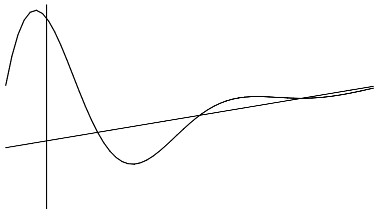
Winter Weekend Organizer



$$f(x) \;=\; \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{(x-\mu)^2}{2\sigma^2}\right)$$

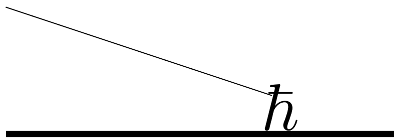
$$-\infty \;<\; x \;<\; \infty$$

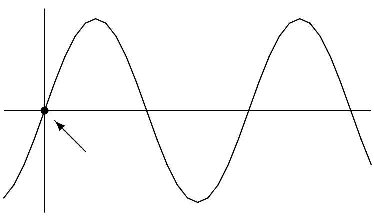
$$2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}}}}$$



$$\int \left[v\frac{du}{dx} \;+\; u\frac{dv}{dx} \right] dx$$

☼





$$f(x) \;=\; \frac{dx}{dt}$$

$$\lim f(x) \;=\; c$$

Problem 220.1 – Marbles and fruit

Tommy Moorhouse

Twins Alia and Amjad love fruit but, being twins, hate it when they don't have the same amount. Every week they stay with their grandfather Bilal. Their favourite game at Grandad's is 'making rectangles', which Bilal devised himself. He sets a large tray on the floor, the tray having a rectangular array of indentations, each of which can hold a marble. He scoops out at random a number (which we will call n) of marbles from a large bag. Then each of the twins in turn puts all the marbles into the tray to form a perfect rectangle. The first twin always starts with the rectangle consisting of a single row $(1 \times n)$ and the last always finishes with the $(n \times 1)$ rectangle. $(k \times m)$ rectangles are distinguished from $(m \times k)$ rectangles if k and m are different. Every time one of the twins produces a new rectangle a piece of fruit goes into the kitty. This continues until all the possibilities have been exhausted (Bilal knows how many possibilities there are given n).

Bilal then uses his patent marble collector to return the marbles to the bag. It works as follows: if there are three or more marbles in the tray the collector returns three to the bag and starts again. Otherwise it leaves the marbles in the tray and Bilal has to put them back by hand. Thus after each game there will be one, two or no marbles left in the tray.

At the end of the game the fruit is shared out as equally as possible. The twins are happy if they both get the same number of pieces of fruit but always fall out if there is an odd piece left over.

Bilal has noticed that the twins never seem to fall out if the patent marble collector leaves exactly two marbles in the tray. Prove that this 'rule of thumb' is true for all n. Bilal suggests that whenever there is one marble left in the tray the twins will fall out. What can you say about this?

Problem 220.2 – Two ingots

I [ADF] found this in a book of lateral thinking mind-bending puzzles.

A woman wishes to transport two gold ingots and herself across a bridge. She weighs 100 kg. The ingots weigh 10 kg each. However, the bridge will support only 117 kg. How does she do it?

The problem here in M500 is not to get the answer that was given in the back of the book—she juggles the gold bars whilst she walks across the bridge—but to explain why this solution works, or not, if indeed it does, or doesn't.

Problem 220.3 – Three integers

Tony Forbes

Find all solutions in positive integers a, b, c of

$$\frac{a(a-1)}{2}+\frac{b(b-1)}{2}+\frac{c(c-1)}{2} \;=\; ab+ac+bc \;=\; \frac{(a+b+c)(a+b+c-1)}{4},$$

where $a \equiv 1 \pmod 6$ and $b \equiv c \equiv 3 \pmod 6$.

The problem occurs in the determination of possible parameters for the existence of type B 3-colourable Steiner $S(2,4,v)$ systems. See *Zoe's Design* at http://anthony.d.forbes.googlepages.com/ZoeDes.pdf.

Problem 220.4 – RATS

This is like that problem where the number 196 plays a significant role. See, for example, 196 revisited, M500 **205**.

Take any positive integer, p. Reverse its decimal digits to get q. Add to get $p+q$ and then sort the digits to get p^*. Repeat.

Thus RATS: Reverse, Add, Then Sort. For example, if you start with 77, you get 77, 145, 668, 1345, 6677, 13444, 55778, 133345, 666677, 1333444, 5567777, 12333445, 66666677, 133333444, 556667777, 1233334444, 5566667777, 12333334444, 55666667777, 123333334444, and thereafter the terms continue to expand with 5566…667777, 1233…334444, ….

Investigate John Conway's conjecture: Either the sequence goes into a closed cycle, or it enters the divergent sequence 1233334444, 5566667777, 12333334444, 55666667777, ….

Problem 220.5 – Biseptic

Tony Forbes

I'm a bit worried about the title of this problem because the *Shorter Oxford* gives as the only meanings of 'septic' the ones that are familiar to pathologists and plumbers. But that's irrelevant—all you have to do is solve

$$x^{14} + 508x^4 \;=\; 9171655.$$

Answers to pp 18–19: {transcendental meditation, *A Fine Romance*, three dimensions, chain reaction, identity parade, Max Bygraves, e-mail, speed limit, pirate, bell tower, uv lamp, *Rising Damp*, primates, *Great Expectations*, vital statistics, original sin, walk the plank, *Tea for Two*}.

Letters to the Editor

Non-commutative algebras and quantum entanglement

Dear M500,

I bring to the attention of readers interested in QM, and who are rather more *au fait* with non-commutative algebras than I am, the important article 'Quantum Untanglement' by Mark Buchanan, *New Scientist*, 3 November 2007. One of the problems with quantum mechanics is that, seemingly, it allows what Einstein called 'spooky action at a distance'. If we have a pair of so-called entangled photons, which are quite easy to produce in the laboratory, quantum mechanics dictates that the total spin must cancel out, so if you measure one of them and its spin is 'up', the other one's spin will always be 'down' and vice-versa. So what? The odd thing is that in principle the two paired photons can be light years apart and experiments have confirmed the predictions of QM when measuring paired photons which could not 'communicate' with each other except by sending information at a speed greater than that of light.

Not only this, QM says that, prior to an 'act of measurement', all physical systems exist in a superposition of possible states, only collapsing into one, and only one, state when they interact with a different system. 'Hidden variable' theories, however, claim that there is a deeper level of reality which is unambiguous and more like what we are used to.

Where does algebra come in? Because Bell famously 'proved' that 'hidden variable' theories could not yield results compatible with experiment, while orthodox QM could and did. But now, Joy Christian has challenged Bell's theorem. In brief, the argument seems to be that Bell assumed that the supposed 'hidden variables' would behave according to the rules of commutative algebras and did not consider the possibility of non-commutative algebras—Hamilton's quaternions, for example, are non-commutative. If we do this, Christian argues, a lot of the spookiness of QM disappears. If you're up to that level, see 'Disproof of Bell's theorem by Clifford algebra dots' by Joy Christian (www.arxiv.org/abs/quant-ph/0703179) or check into the more readable *NS* article via http://archive.newscientist.com.

Yours,

Sebastian Hayes

Re: Problem 212.3 – 100 seats

Dear Tony,

In the interests of brevity I've developed a C program to replace the Fortran program in M500 issue **218** [Solution 212.3 – 100 seats, pages 8–11]. It runs the 100 seats simulation 2^{20} times.

```
int c=0,s[100],i,j,a;
r(){int a=100;for(;a>99;a=rand()&127);return a;}
main(){srand(time(c));for(i=1<<20;i--;)
{for(j=0;j<100;s[j++]=0);s[r()]++;
for(j=98;j;j--){for(a=j;s[a];a=r());s[a]++;}c+=!*s;}
printf("%g\n",c*1.0/(1<<20));}
```

It will compile on any ANSI compiler, but there might be some warnings.

Sincerely,

Emil Vaughan

M500 218

Thanks for M500 **218**. Most interested in the story of Dr Robert Bohannon and his caffeinated doughnut. In this, he is merely following the work of Dr Urban Panic, accounted a promising topologist in his youth but whose later career was dedicated to the development of novel snacks.

In 1995 Dr Panic, irritated by having dropped a piece of toast on the floor butter side down, conceived the idea of Möbius toast, which would always fall on the same side. He devised a laser slicer-toaster which would cut as many as four Möbius slices from a large loaf, toasting each one as it sculpted the form; once early problems of crumb fires had been disposed of with a vacuum extractor, the device was widely hailed in the catering press. However, buttering experiments failed to produce the hoped-for result, and the project was abandoned.

After this setback, Dr Panic turned his attention to doughnuts. His first creation was a jam doughnut which would always keep the jam clear of the consumer's fingers. The resulting design, in the form of a jam-filled Klein bottle, did not live up to expectations and was never marketed.

In 2001 Dr Panic began another series of experiments along the line later pursued by Dr Bohannon: combining coffee and a doughnut in one easy-to-consume snack. He reasoned that, since a ring doughnut is topologically the same as a coffee cup, it should be possible to fill an ordinary doughnut (suitably protected with edible varnish) with coffee. Again, the early promise of this radical concept was never fulfilled, and Dr Panic has now wound up his company, TopoSnax plc, and has retired to his villa in Antibes. It is good to know that he has a worthy successor.

Best wishes,

Ralph Hancock

Equation 216.5

Dear Tony,

Somehow I missed this problem when it was published [Problem 216.5–Equation: Solve $x = 3\,\exp(x^2/214)$].

Since an exponential is present, I realized that a second solution could be possible, $x = 20.2$ approx. This illustrates one of the pitfalls which I learned about in M101 (about 1986), whereby solutions tend towards the 'dominant' value.

Regards,

Ken Greatrix

Dear Tony,

This equation has two solutions, $x \approx \pi$ and $x \approx 20.24245$. I solved the equation graphically with $y = x$ and $y = \exp(x^2/214)$ to find the two intersection points. Wrongly, possibly, I assumed there must be a 'mathematical' solution, and hence did not offer my solution. Is there a rigorous way to get the two answers?

Regards,

Basil Thompson

No. I am at least 100 percent certain that there is no solution in terms of rational numbers and elementary functions thereof. There is nothing particularly special about $x = 3\,\exp(x^2/214)$ except that ... well, see if you can guess. It is merely one of a small collection that I have built up over a number of years. To make these interesting equations public I am drip-feeding them into M500 on a semiregular basis. You have seen some before. Look out for the next one! — **ADF**

Latin squares

Tony,

Solution [to the latin square puzzle in M500 **218**]? No problem! Or so I thought. I put the square into my solving program and pressed the 'go' button—it failed. Then I realized that it's not a sudoku puzzle—you did warn me, but I took no notice. Even so ....

I've noted on these occasions that you seem to be reluctant to use the name 'Sudoku' for your puzzles. So, thinking that the name may be copyrighted or patented I did a search on the 'ole wibbley-wobbley-way'. ... The modern version of Euler's latin square using the term 'Sudoku' was coined by Kaji Maki, but he failed to copyright or patent the idea—wishing it to be spread very widely in a short space of time.

If you wish to spread your analytical expertise to another number-in-cell type of problem then attached is a 'new' idea called *number workout*. It's published every Saturday in the *Daily Mail Weekend* magazine.

Regards,

Ken Greatrix

The rest of the page was originally meant for the puzzle sent by Ken. It's a rectangularish array of 78 triangles grouped into hexagons. Some of the triangles have numbers in them and you must fill in the others. Then I [ADF] became afraid of getting clobbered by one of the *Mail*'s henchmen for stealing it. So in its place you will have to put up with the following.

Late Arrivals at the Mathematicians' Ball

Ladies and gentlemen, will you please welcome

Mr and Mrs Micks and their children Hamil, Tony and Dinah;
Mr and Mrs Lizing–Transformation and their daughter Norma;
Mr and Mrs d'Oku–Puzzle and their daughter Sue;
Mr and Mrs Ear–Dependence and their daughter Lynne;
Mr and Mrs Strophe–Theory and their daughter Katie;
Mr and Mrs Isis and their children May, Trix and Al;
Mr and Mrs Nomical–Unit and their daughter Astra;
Mr and Mrs Centric–Coordinates and their son Barry;
Mr and Mrs Imum–Likelihood and their son Max;
Mr and Mrs Lation–Coefficient and their daughter Corrie;
Mr and Mrs Stick–Modulus and their daughter Ella;
Mr and Mrs O'Lute and their daughter Eve.

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Cover: The first member of an amazing 6223-digit *prime triplet*,

$$33672313734 \prod_{\substack{7 \le p < 14500,\\ p \text{ prime}}} p \; + \; d, \qquad d = -1, 1, 5,$$

discovered by **Norman Luhn** in December 2007.