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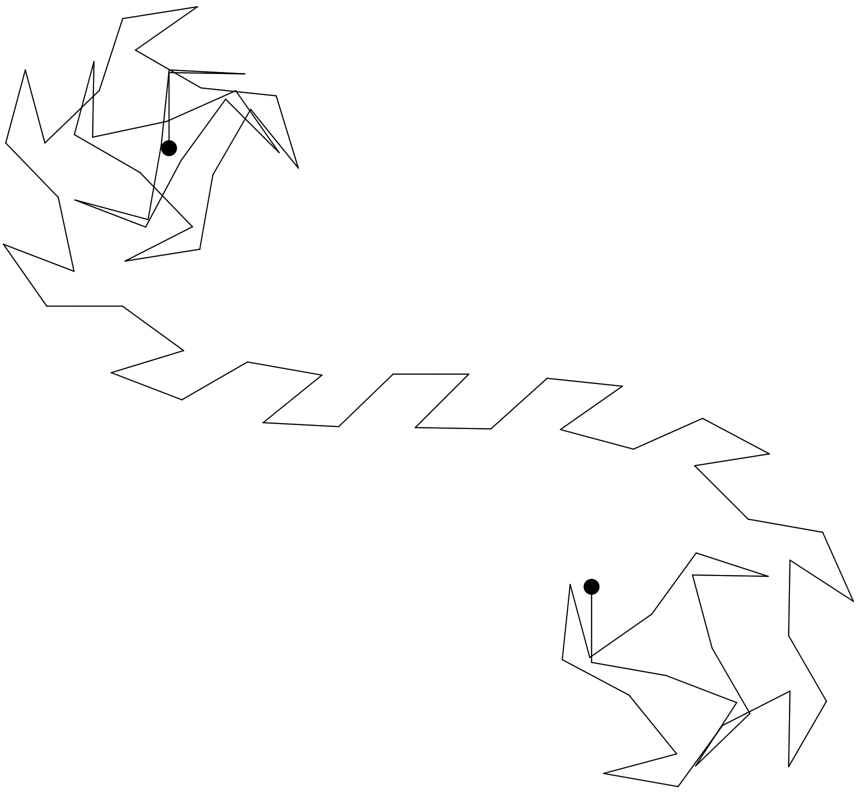

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Figurate numbers

Bob Bertuello

The following article aims to give just a flavour of the many types of numbers which were popular as long ago as *c.* 540BC and explored by Pythagoras. It does not purport to be comprehensive but gives a good sample of this type of numbers. Proof of the formulae are relatively simple (by induction or otherwise) and have only been included in one or two cases. Enjoy!

Figurate numbers are so called because they represent the number of points in a geometric figure. They are number sequences that are found by creating consecutive similar figures. The first member is always a single point. The figures may be flat polygonal (2-dimensional), solid polyhedral (3-dimensional) or polytopic (n-dimensional, $n > 3$).

Most well known are the polygonal numbers, which are found by making consecutive polygons, each successive one consisting of one more point in each side than in the previous one. The polygons may be nested or centred. Since each side consists of equally spaced points, each figure is regular.

Flat figures

Nested polygonal figures

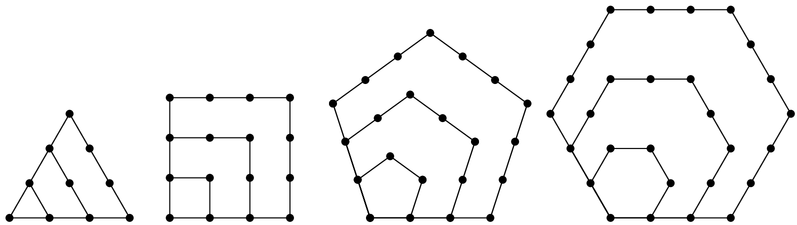


Figure 1 Figure 2 Figure 3 Figure 4

Let $\mathrm{PN}(m, n)$ represent the nth nested m-gonal number.

Figure 1 shows a set of nested triangles of sides zero, one, two and three and the number of points in each is 1, 3, 6 and 10. These are the triangular numbers. The number of points in the nth triangle, i.e. the nth triangular number, is $\mathrm{PN}(3, n) = n(n + 1)/2$.

Figure 2 shows a set of nested squares. The sequence of square numbers is 1, 4, 9, 16, ... and the nth square number is $\mathrm{PN}(4, n) = n^2$.

Figure 3 shows a set of nested pentagons. The sequence of pentagonal numbers is 1, 5, 12, 22, ... and the nth pentagonal number is $\mathrm{PN}(5, n) = n(3n - 1)/2$.

Figure 4 shows a set of nested hexagons. The sequence of hexagonal numbers is 1, 6, 15, 28, … and the nth hexagonal number is $\mathrm{PN}(6,n) = n(2n-1)$.

Similar figures may be constructed for polygons with more sides. Note that the polygons are nested in one vertex. Also, all sequences of figurate numbers start with the number 1, and the second number in the nested polygonal sequence is the number, n, of sides in the polygon. We may derive a general formula for all nested polygonal numbers thus.

Assume that the formula for $\mathrm{PN}(m,n)$ is quadratic, i.e. $\mathrm{PN}(m,n) = an^2 + bn + c$. Then consider the 0th, 1st and 2nd m-gonal numbers for some m. They are 0, 1 and m. Then $\mathrm{PN}(m,0) = 0$; therefore $c = 0$. Also $\mathrm{PN}(m,1) = 1$. Therefore $a + b = 1$; i.e.

$$a = 1 - b. \tag{1}$$

Moreover, $\mathrm{PN}(m,2) = m$; therefore

$$4a + 2b = m. \tag{2}$$

Inserting (1) into (2) gives $4(1-b) + 2b = m$, whence

$$b = -\tfrac{1}{2}(m-4). \tag{3}$$

Inserting (3) into (1) gives $a = 1 + (m-4)/2 = (m-2)/2$. And we find the formula

$$\mathrm{PN}(m,n) = \frac{n}{2}\big((m-2)n + (4-m)\big).$$

Different nested polygonal numbers may also be obtained by adding the first n terms of the following arithmetic progressions starting with 1.

Δ	arithmetic progression		
1	$1+2+3+4+5+\dots$	gives triangular numbers	1, 3, 6, 10, 15, …
2	$1+3+5+7+9+\dots$	gives square numbers	1, 4, 9, 16, 25, …
3	$1+4+7+10+13+\dots$	gives pentagonal numbers	1, 5, 12, 22, 35, …
4	$1+5+9+13+17+\dots$	gives hexagonal numbers	1, 6, 15, 28, 45, …
5	$1+6+11+16+21+\dots$	gives heptagonal numbers	1, 7, 18, 34, 55, …
6	$1+7+13+19+25+\dots$	gives octagonal numbers	1, 8, 21, 40, 65, …

In what follows, all polygonal numbers not referred to as ‘centred’ are to be taken as ‘nested’.

3.2 Centred polygonal figures

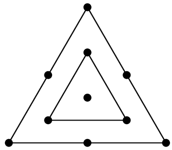


Figure 5

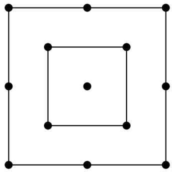


Figure 6

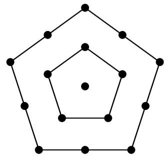


Figure 7

Let PC(m,n) represent the nth centred m-polygonal number. Centred polygonal numbers, except the first, are greater than the nested ones.

Figure 5 shows a set of centred triangles. Centred triangular numbers are 1, 4, 10, 19, $\ldots$, $3n(n-1)/2+1 = \mathrm{PC}(3,n)$.

Figure 6 shows a set of centred squares. Centred square numbers are 1, 5, 13, 25, $\ldots$, $4n(n-1)/2+1 = \mathrm{PC}(4,n) = n^2+(n-1)^2$.

Figure 7 shows a set of centred pentagons. Centred pentagonal numbers are 1, 6, 16, 31, $\ldots$, $5n(n-1)/2+1 = \mathrm{PC}(5,n)$.

A centred hexagonal number is also known as a hex number.

The general formula for the nth centred m-polygonal number is

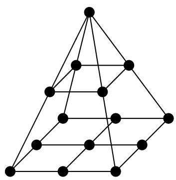
$$\mathrm{PC}(m,n) \;=\; \tfrac{1}{2}\, mn(n-1)+1.$$

Proof (by induction). The $(n+1)$th number is observed to be obtained by adding mn to the nth number. Therefore $\mathrm{PC}(m,n+1) = mn(n-1)/2+1+mn = mn(n+1)/2+1$, which can also be obtained by putting $(n+1)$ for n in the formula for PC(m,n). Therefore, if it is true for n, it is also true for $(n+1)$. But it is true for $n=1$; therefore it is true for all n.

Solid figures

Pyramidal numbers

Figure 8: a square-based pyramid



The nth m-sided pyramidal number is given by

$$\mathrm{PYR}(m,n) \;=\; \frac{n}{6}\,(n+1)((m-2)n+5-m).$$

Thus:

Triangular pyramidal (tetrahedral) numbers are 1, 4, 10, 20, ..., $n(n+1)(n+2)/6 = \mathrm{PYR}(3,n)$.

Square pyramidal numbers are 1, 5, 14, 30, ..., $n(n+1)(2n+1)/6 = \mathrm{PYR}(4,n)$.

Pentagonal pyramidal numbers are 1, 6, 18, 40, ..., $n^2(n+1)/2 = \mathrm{PYR}(5,n)$.

Hexagonal pyramidal numbers are 1, 7, 22, 50, ..., $n(n+1)(4n-1)/6 = \mathrm{PYR}(6,n)$.

Cubic numbers are 1, 8, 27, 64, ..., n^3.

Centred cube numbers are 1, 9, 35, 91, ..., $(2n-1)(n^2-n+1)$.

Octahedral numbers are 1, 6, 19, 44, 85, ..., $n(2n^2+1)/3$.

Polytope figures

A polytope is a generalization of a polygon or polyhedron to any number of spacial dimensions. The nth d-polytopic number is given by $\mathrm{PT}(d,n) = (d+n-1)!/(d!(n-1)!)$, where d is the dimension of the figure; e.g. for a tetrahedron, $d = 3$. Since the triangular and tetrahedral numbers fit into this pattern, they are included here. Thus:

Triangular numbers are 1, 3, 6, 10, ..., $n(n+1)/2 = \mathrm{PT}(2,n)$; i.e. 2-dimensional.

Tetrahedral numbers are 1, 4, 10, 20, ..., $n(n+1)(n+2)/6 = \mathrm{PT}(3,n)$; i.e. 3-dimensional.

Pentatope numbers are 1, 5, 15, 35, 70, ..., $n(n+1)(n+2)(n+3)/24 = \mathrm{PT}(4,n)$; i.e. 4-dimensional.

Hexatope numbers are 1, 6, 21, 56, 126, ... $n(n+1)(n+2)(n+3)(n+4)/120 = \mathrm{PT}(5,n)$; i.e. 5-dimensional, etc.

Pascal's triangle

If Pascal's triangle is set out as below, certain figurate numbers are displayed. The rows in Figure 9 form the polytopic numbers $\mathrm{PT}(d,n)$, where d is the dimension and n is the nth number in the sequence. Thus the pentatopic numbers are 1, 5, 15, 35, 70, ..., and the 6th pentatopic number is

$$\mathrm{PT}(4,6) \;=\; \frac{(4+6-1)!}{4!(6-1)!} \;=\; \frac{9!}{4!5!} \;=\; \frac{6\cdot 7\cdot 8\cdot 9}{24} \;=\; 126.$$

d	n 1	2	3	4	5	6	7		
0	1	1	1	1	1	1	1		
1	1	2	3	4	5	6	7	natural numbers	PT$(1, n)$
2	1	3	6	10	15	21	28	triangular numbers	PT$(2, n)$
3	1	4	10	20	35	56	84	tetrahedral numbers	PT$(3, n)$
4	1	5	15	35	70	126	210	pentatope numbers	PT$(4, n)$
5	1	6	21	56	126	252	462	hexatope numbers	PT$(5, n)$
6	1	7	28	84	210	462	924		
7	1	8	36	120	330	792	1716		

Figure 9

Note. Pentatope is the name of a specific geometric figure that human beings cannot directly visualize because it does not exist as a 3-dimensional object. The suffix '-tope' refers to the 'cells' which comprise geometric figures that exist in a greater number of dimensions than three. A 3-dimensionsl 'tetrahedron' is composed of four equilateral triangles whose planes provide its 'four faces.' A 'pentatope' contains 'five cells' in the form of five tetrahedrons enclosed within a hypersphere.

Relationships between some figurate numbers

Of the many relationships, here are just a few.

Referring to Figure 9, $\mathrm{PT}(d, n) = \mathrm{PT}(d, n-1) + \mathrm{PT}(d-1, n)$; e.g. $\mathrm{PT}(3, 6) = \mathrm{PT}(3, 5) + \mathrm{PT}(2, 6) = 35 + 21 = 56$.

Also, due to the symmetry of the table, $\mathrm{PT}(d, n) = \mathrm{PT}(n-1, d+1))$; e.g. $\mathrm{PT}(3, 6) = \mathrm{PT}(5, 4) = 56$.

The nth centred square number equals the sum of two nested square numbers, one of equal rank and one of one less.

The nth cubic number equals the square of the nth triangular number.

The sum of two consecutive triangular numbers is a square number.

Every pentagonal number is one third of a triangular number. Thus

$$\tfrac{1}{2}n(3n-1) \;=\; \tfrac{1}{3}\cdot\tfrac{1}{2}(3n-1)(3n).$$

Every odd-numbered triangular number is a hexagonal number.

The sum of two consecutive tetrahedral numbers is a square pyramidal number.

The nth tetrahedral number equals the sum of the first n triangular numbers.

Any pentagonal number equals the sum of the square number of same rank and a triangular number of the preceding rank. For example, $\mathrm{PN}(5, n) = \mathrm{PN}(4, n) + \mathrm{PN}(3, n-1)$.

Eight triangular numbers increased by unity produce a square, because

$$8\,\frac{n(n+1)}{2} + 1 \;=\; 4n^2 + 4n + 1 \;=\; (2n+1)^2.$$

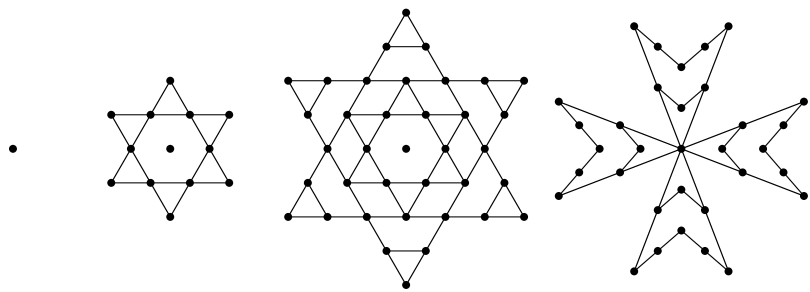
This property can be used to find square numbers that are also triangular. This requires the solution of an equation of the form $8x^2 + 1 = y^2$, a Pell equation. Solution gives the side of the square (x) as 1, 6, 35, 204, 1189, $\ldots$, $u(n)$, where $u(n) = 6u(n-1) - u(n-2)$.

The nth m-sided centred polygonal number equals the nth $(m-1)$-sided centred polygonal number plus the $(n-1)$th triangular number; i.e. $\mathrm{PC}(m, n) = \mathrm{PC}(m-1, n) + \mathrm{PN}(3, n-1)$.

An m-gonal number equals the sum of the $(m-1)$-gonal number of the same rank and the triangular number of the previous rank; i.e. $\mathrm{PN}(m, n) = \mathrm{PN}(m-1, n) + \mathrm{PN}(3, n-1)$.

Some other figures

A centred star consists of a central m-gon surrounded by m equilateral triangles. The nth star number is $S(m, n) = mn(n-1) + 1$. Thus the 5th heptagonal star number, $S(7, 5)$ is $7 \cdot 5 \cdot 4 + 1 = 141$.



1 13 37 33

A Maltese cross consists of four chevron arms connected to a common centre. Each arm of the nth Maltese cross consists of n^2 points. However, the central point is common to all four arms. Therefore the nth Maltese cross number, $\mathrm{MC}(n)$ is $4n^2 - 3$. Thus the tenth MC number is 397.

Although the study of figurate numbers is mainly elementary and recreational, deeper study involves some advanced mathematics. For example, it has been established that every integer is the sum of at most three triangular numbers. Similar results have been discovered for other polygonal numbers.

Finally, here are some definitions of related terms that are not used herein. Gnomon: the part that can be added to a figure to produce the next larger similar figure. Pronic number: Twice a triangular number, $n(n+1)$; i.e. the product of two consecutive integers.

Reference: John Conway and Richard Guy, *The Book of Numbers*.

Problem 222.1 – Rectangle construction

Tommy Moorhouse

Two parallel lines are tangent to a circle $\mathcal{C}$ at its North and South poles N and S. A segment of length l is constructed, starting from S and terminating on the same line at a point A.

A second line segment is constructed as follows: the line NA is drawn, intersecting the circle at a point E distinct from N. The line SE is extended to meet the line tangent to N at B. The line segment in question is NB which has length m.

Show that, whatever l we start with, a rectangle with sides of length l and m constructed in this way has the same area as the smallest square that completely encloses $\mathcal{C}$ (i.e. the square enclosing $\mathcal{C}$ which touches $\mathcal{C}$ at exactly four points).

Problem 222.2 – Three powers

Find solutions in co-prime positive integers x, y, z and exponents $p, r, q \geq 2$ such that $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} \leq 1$ and $x^p + y^q = z^r$.

In particular, are there any solutions in which $p, q, r \geq 3$.

Closely related is a conjecture of Texan banker Andrew Beal. If $x^p + y^q = z^r$, where x, y, z, p, q, r are positive integers and $p, q, r > 2$, then x, y and z must have a common factor. We did this in M500 **160** (February 1998). And it looks like a tough one. After ten years it's still open, and the prize offered by Beal for resolving his conjecture has increased to $100,000.00.

Upon mathematical spaces

Dennis Morris

The set of objects that form a vector space (also called a linear space) are called vectors, but they are not the arrow-like things which we are used to calling vectors. Within the vector space axioms, there is no concept of length or of angle or of direction and hence these concepts cannot be appended to the objects (vectors) in a vector space.

If the mathematician imposes a norm on to the vector space, then each vector can be associated with a particular (unique) real number that is thought of as its length. However, that spatial concept is within the mathematician's mind, and a norm does no more than associate a real number with each vector. A vector space with a norm is called a *normed space*. It is possible to define different norms upon the same vector space thereby producing different normed spaces. The existence of a norm sometimes allows (it depends on the norm) the definition of convergence.

Cauchy sequences do not converge in every normed space, but they do in some. Normed spaces in which all Cauchy sequences converge are called *Banach spaces*. A normed space is said to be complete if, and only if, every Cauchy sequence in it converges; i.e. a complete normed vector space is a Banach space. A norm does not allow the concept of distance between two vectors (roughly—points in space); to do this, the mathematician imposes upon the vector space a metric (distance function) whose arguments are the two vectors whose distance apart it defines. (A norm takes only one vector as its argument.) A vector space together with a metric is called a *metric space*. Such a space has no concept of angle or direction. Strangely, it has no concept of distance between vectors—this is in the mathematician's mind. All a metric does is associate a real number with two vectors. However, the axioms of the metric are such as to make us think of this real number as a distance.

A mathematician might impose an inner product (dot product) upon a vector space. Such spaces are known as *inner product spaces*. An inner product takes two vectors for its arguments and produces a real number (that is usually associated with the angle between the vectors). A metric does only the same! All inner product spaces are normed spaces (a norm can be the inner product with the same vector for both arguments), but not all normed spaces are inner product spaces.

Thinking of vectors as arrow-like objects, there is one aspect (a real number) of a vector that stays the same as the basis is changed—its length. As well as their lengths, there is another aspect to a system of two vectors that stays the same as the basis is changed—the angle between the vectors. The real numbers produced by norms, inner products, and metrics also have this property of invariance under change of basis.

An inner product space whose norm is the inner product with the same vector as both arguments and which is complete is called a *Hilbert space*. Although Hilbert spaces are thought of as having the concept of length and the concept of angle in them, they have neither. They are not what we would call geometric spaces. However, when the normalized inner product is set equal to a trigonometric function (usually the cosine function), the concept of (euclidean in the case of cosine) angle is inserted into the space. With the concept of angle comes the concept of projection from one vector on to another, which is the concept of distance. If euclidean trigonometric functions are used (sine, cosine), then the innerproduct can be zero, and, in this case, the vectors are said to be orthogonal; this cannot be done with the cosh function since it never equals zero.

The reader might take the view that, with all these different possible inner-products, trigonometric functions, metrics, and norms, and that they can be pick-and-mixed together at will, just about anything is possible; traditionally, it is. An alternative view, and the view to which I hold, is that a space is a particular set of objects, a distance function, a rotation matrix, and a set of trigonometric functions, and that there can be no pick-and-mix. Such spaces are called natural spaces. They are geometric by nature having both their own concept of distance and their own concept of angle within them. Algebraically, the natural spaces are all Hilbert spaces, but they are more than that since they have within them a set of trigonometric functions, a rotation matrix, and a distance function; they are true geometric spaces.

Russell's attic

Eddie Kent

> [Recall from M500 **214** that Russell's attic is a room containing countably many pairs of shoes and countably many pairs of socks. It is easy to see that there are countably many shoes, for instance by matching the left shoes to the odd numbers and the right shoes to the even numbers. But can you say how many socks there are?]

What we are asked to do is count the shoes and socks in Russell's attic. How do we go about counting the elements of a set? One obvious way is by matching. If we can associate the elements of one set with those of another, we can calculate without going near the second set. We know, for instance, that each cow in a certain field has four legs; we know there are n cows, therefore we can be clear that there are $4n$ legs.

What is n? It has to be a natural number, because we have used it for counting. The natural numbers are ideal for this job, because the successor function applies (for every number m there exists the number $m + 1$) and it is also well-ordered (has a least member). A set must be well-ordered or capable of having a well-ordering applied to it for it to be countable. The number of counting numbers is called $\aleph_0$.

As an example, the rational numbers are not well-ordered—what is the smallest rational number? However various techniques have been developed for imposing a well-ordering on it. Cantor used a grid method where the number m/n was placed in the nth column of the mth row. They can then be counted diagonally. Alternatively one could add together the numerator and denominator of each number to give an integer, and the integers are easily well-ordered (squaring, for instance).

Any finite number of elements in a set is well-ordered; sheep in a field can be well-ordered by choosing. Taking all this it is clear that the shoes in Russell's attic can be counted—they are in two distinct matching sets, one can be associated with the odd positive integers and the other with the evens. Since the odds and the evens add up to exactly $\aleph_0$, we are done. The number of shoes is countable and there are $\aleph_0$ of them.

Unfortunately the socks are indistinguishable one from another; there is no easy way of imposing a well-ordering on them. One can of course see them as a collection of pairs. Each pair would then be a set, and one could use the Axiom of Choice to select an element from each set, to give another set, which could then be counted. Perhaps we are getting somewhere, but the Axiom of Choice is a pretty blunt instrument. It turns out to be equivalent to the statement that every set can be well-ordered.

What it says is that for a collection of sets one can choose one element from each of them to form another set. This seems pretty obvious and innocuous, and for finite sets it works well enough. But for infinite sets problems arise. The most famous is the paradox whereby one can take a solid ball, cut it into pieces and reassemble it twice (or a thousand times) as big; see Stan Wagon, *The Banach–Tarski Paradox*, for example.

So there we have our dilemma. Count the socks and bring down confusion upon your head, or admit you can't. I won't actually say the socks are countable if and only if they aren't, but that is buzzing around in my head, since it's Russell's attic we are talking about.

OK then, what's the speed of dark?

The Antisocial Club

Gareth Harries

[Adapted from a puzzle on the Braingle web site.]

The members of the antisocial club go to a bar where there are n stools along the bar and sit according to the following rules.

(a) No member will sit next to another member.

(b) Upon entering the bar a member will sit as far away as possible from any other member there subject to rule (a).

(c) If there are no available seats without breaking rule (a) then the member leaves without buying a drink.

Clearly the maximum number of members sitting is when there is just one seat between each member and the minimum is when there are two seats between each member (assuming enough members arrive to reach this situation).

Whether one reaches the maximum or minimum depends on n and on where the first member sits. For example, if $n = 13$ and member 1 sits at seat 1 (where seats are numbered 1, 2, ..., 13 along the bar), then member 2 sits at seat 13. Member 3 will now sit at seat 7 and members 4 and 5 will sit at seats 4 and 10 thus reaching the minimum of 5 members. However, if member 1 sits at seat 5, then member 2 sits at seat 13 and members 3 and 4 sit at seats 1 and 9 which allows members 5, 6 and 7 to sit at seats 3, 5 and 11, giving the maximum number of 7 members.

Question. For any particular n, is it always possible to find a position for member 1 to sit to achieve the maximum and another to achieve the minimum number?

Problem 222.3 – Consecutive composite numbers

Ian Adamson

Is there a set S_n which is guaranteed to contain at least $n > 1$ consecutive composite numbers where $\min(S_n) < (n+1)! + 2$?

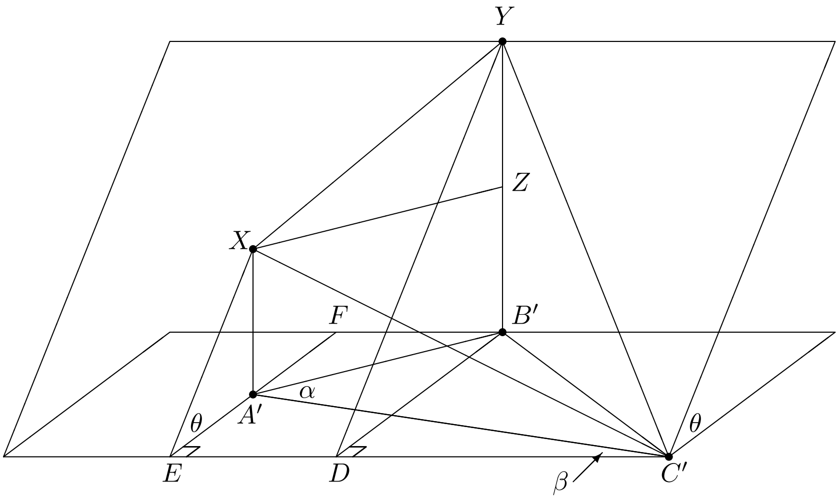
Yes, please *construct*, and not just show the existence of, S_n.

Think of two words which have opposite meanings, such that if you add the same letter to the front of each one you make two new words which also have opposite meanings. — **Jeremy Humphries**

Solution 210.4 – Coal

There is a coal deposit that occurs underground in a plane inclined at angle θ to the horizontal. You make vertical holes at points A, B and C in a horizontal plane on the surface, detecting the coal seam at depths a, b and c respectively. If $AB = x$, $AC = y$ and $\angle BAC = \alpha$, what is θ?

Steve Moon



Triangle $A'B'C'$ is a map of triangle ABC on the surface to a parallel plane at a distance c below C'; so C' is the datum level. Points C', X and Y lie on the inclined plane. Hence

$$B'Y = c - b, \qquad A'B' = x, \qquad \angle B'A'C' = \alpha,$$
$$A'X = c - a, \qquad A'C' = y, \qquad YZ = a - b.$$

Also $\angle XEA' = \angle YDB' = \theta$.

Let $\beta = \angle EC'A'$. Then $\angle EA'C' = 90^\circ - \beta$ and $\angle FA'B' = 90^\circ - \alpha + \beta$. Hence

$$\tan\theta \;=\; \frac{c-a}{A'E} \;=\; \frac{c-a}{y\sin\beta}.$$

Therefore

$$\sin\beta \;=\; \frac{c-a}{y\tan\theta}, \qquad \cos\beta \;=\; \sqrt{1 - \frac{(c-a)^2}{y^2\tan^2\theta}}.$$

Also

$$\begin{aligned}
\tan\theta &= \frac{c-b}{B'D} = \frac{c-b}{A'F + A'E} \\
&= \frac{c-b}{y\sin\beta + x\cos(90^\circ - \alpha + \beta)} \\
&= \frac{c-b}{y\sin\beta + x\sin(\alpha - \beta)} \\
&= \frac{c-b}{y\sin\beta + x(\sin\alpha\cos\beta - \cos\alpha\sin\beta)} \\
&= \frac{c-b}{\dfrac{c-a}{\tan\theta} + x\sin\alpha\sqrt{1 - \dfrac{(c-a)^2}{y^2\tan^2\theta}} - \dfrac{x(c-a)\cos\alpha}{y\tan\theta}}.
\end{aligned}$$

Hence

$$y(c-a) + x\sin\alpha\sqrt{y^2\tan^2\theta - (c-a)^2} - x(c-a)\cos\alpha \;=\; y(c-b).$$

Therefore

$$x\sin\alpha\sqrt{y^2\tan^2\theta - (c-a)^2} \;=\; y(c-b) + x(c-a)\cos\alpha - y(c-a).$$

Hence

$$y^2\tan^2\theta \;=\; \left(\frac{y(a-b) + x(c-a)\cos\alpha}{x\sin\alpha}\right)^2 + (c-a)^2$$

and finally we have the answer to the problem:

$$\theta \;=\; \arctan\left(\frac{1}{y}\sqrt{\left(\frac{y(a-b) + x(c-a)\cos\alpha}{x\sin\alpha}\right)^2 + (c-a)^2}\right).$$

Problem 222.4 – Eleven

Find all solutions in positive integers x and n of

$$x^2 \;=\; 3^n - 11.$$

Solution 214.1 – River crossing

> There is a river and a rowing boat which can carry at most two people. A number of married couples are on one bank and they want to cross to the other side of the river. For the usual reason a woman must never be in the presence of a man who is not her husband unless her husband is also present.
> (i) Arrange a crossing schedule for one married couple.
> (ii) Arrange a crossing schedule for two couples.
> (iii) Arrange a crossing schedule for three couples.
> (iv) Can four couples cross the river?
> (v) Show that any number of couples can cross if there is an island in the middle of the river.

[If you are confused, the problem number above is correct; somehow we managed to get it wrong in M500 **219**. Apologies.]

Rob Evans

This article presents a solution to part (iv). That problem, together with important clarifications, was reprinted in M500 **219**.

Not surprisingly(?), the answer to part (iv) turns out to be 'no'! In order to justify this answer we proceed on the basis of the following *reductio ad absurdum* argument.

Assume that the answer to part (iv) were 'yes'. Then, irrespective of the constraint implied by the jealousy of the husbands it is (as readers can confirm) obvious that however the transfer is effected there must be a stage at which there are three people on each of the two riverbanks and the boat has two people in it and is somewhere between the two riverbanks. Moreover, there must be a stage of this description that is immediately preceded/succeeded by a stage in which the two people in the boat are brought together with the three people on the departure/arrival riverbank. For ease of reference, we shall label our three stages in chronological order as stages (1), (2), (3) respectively

Next, consider the transfer to be at a stage (2). Taking into account the constraint implied by the jealousy of the husbands, it is (as readers can again confirm) now obvious that one of the following three compound statements is true.

(a). There are two wives in the boat. There is a married couple plus a husband on each of the two riverbanks.

(b). There is a married couple in the boat There are three wives on one riverbank and three husbands on the other riverbank.

(c). There are two husbands in the boat. There are three wives on one riverbank and a married couple plus a husband on the other riverbank.

However, whichever of (a), (b), (c) is true at stage (2) it is (as readers can again confirm) a straightforward matter to show that at stage (1) and/or at stage (3) the constraint implied by the jealousy of the husbands is not satisfied. In other words, our assumption that the answer to part (iv) is 'yes' has lead to a contradiction. Hence, we are justified in asserting that the answer to part (iv) is 'no'!

What's wrong?

Dennis Morris

We have

$$\begin{aligned}&(a^2c^2+a^2d^2+b^2c^2+b^2d^2)(z^2x^2+z^2w^2+y^2x^2+y^2w^2)\\ &\quad = (aczx-adzw-bcyx+bdyw)^2+(aczw+adzx-bcyw-bdyx)^2\\ &\qquad + (acyx-adyw+bczx-bdzw)^2+(acyw+adyx+bczw+bdzx)^2.\end{aligned}$$

Substituting

$$ac=A,\ ad=B,\ bc=C, bd=E, zx=Z,\ zw=Y,\ yx=X,\ yw=W$$

gives

$$\begin{aligned}&(A^2+B^2+C^2+E^2)(Z^2+Y^2+X^2+W^2)\\ &\quad = (AZ-BY-CX+EW)^2+(AY+BZ-CW-EX)^2\\ &\qquad + (AX-BW+CZ-EY)^2+(AW+BX+CY+EZ)^2.\end{aligned}$$

The trouble is that these are not equal. When we calculate the difference between the two sides of the second equation, we get that difference to be

$$4BYCX+4AZEW-4AYEX-4BZCW.$$

Substituting

$$ac=A,\ ad=B,\ bc=C, bd=E, zx=Z,\ zw=Y,\ yx=X,\ yw=W$$

makes this zero.

Numbers written on restaurant bills within the confines of restaurants do not follow the same mathematical laws as numbers written on any other pieces of paper in any other parts of the Universe. **Douglas Adams**

Solution 219.1 – Walk

> You start facing North, you walk a mile then turn through d degrees, walk another mile, then turn through $2d$ degrees, walk another mile then turn through $3d$ degrees, and so on. If d is a prime greater than 5, how far have you travelled by the time you next face North?

Steve Moon

In order to face north again you turn through some integral multiple k_0 of $360°$. Let the distance travelled in total be n. Having turned the final time, you don't walk, but since you walk one mile initially before turning,

$$d(1 + 2 + \cdots + n) \;=\; 360k_0.$$

Since d is a prime greater than 5, d does not divide 360; so d divides k_0. Let $k = k_0/d$. Hence

$$\frac{n(n+1)}{2} \;=\; 360k, \qquad n \;=\; \frac{-1 \pm \sqrt{2880k+1}}{2}.$$

For n to be an integer, take the positive root and the discriminant must be a square of an odd integer. Indeed it is:

$$2880k + 1 \;=\; 4n^2 + 4n + 1 \;=\; (2n+1)^2.$$

Resorting to trial and error, we get $k = 9$ and $n = 80$. The answer to the problem is therefore 80 miles.

If you carry on the process of walking a mile then turning through the appropriate number of degrees, you will eventually face north again and again. On the computer I found the following solutions for k and n for $n \leq 720$.

k	9	29	70	341	460	568	719	721
n	80	144	224	495	575	639	719	720

Also, if n is a solution (i.e. if n is such that $k = n(n+1)/720$ is an integer), then so is $n + 720$. Hence the eight solutions in the table generate all solutions by adding multiples of 720 to the given values of n. In each case you turn through kd degrees in total.

Basil Thompson

The answer [by reasoning similar to that of Steve Moon, above] is 80 miles, or 81 if you actually walk the last mile.

Which values of d need only one 360° turn?

d	3	8	10	24	36	60	120	360
n	15	9	8	5	4	3	2	1

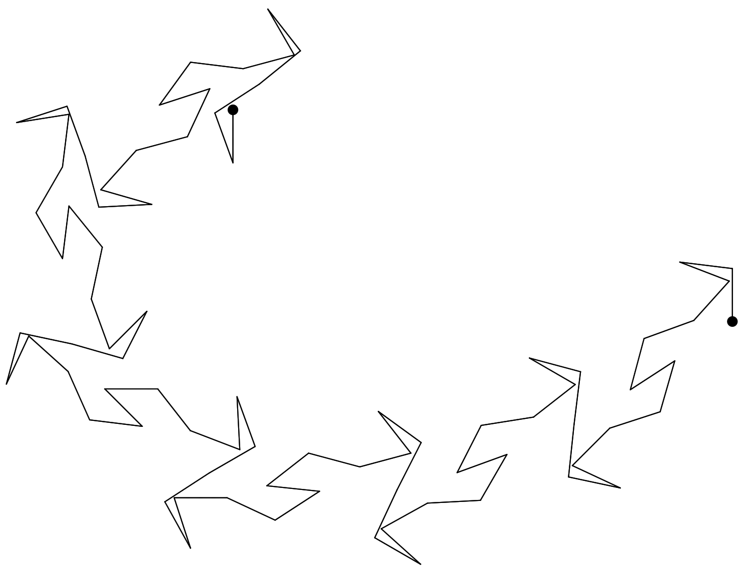
And if d is a multiple of three, we have the following.

d	3	6	9	12	15	18	21	24	27	30	33	36
n	15	15	15	15	15	15	15	5	15	8	15	4
k	1	2	3	4	5	6	7	1	9	3	11	1

Are there any other patterns?

Tony Forbes

Some of the walks create interestingly pretty patterns. Here, for example, is the one for $d = 83$, and I have put $d = 269$ on the cover.



Mondegreens

ADF

I see that my desperate attempt to fill the last page of M500 **119** has resulted in the opening of a small flood-gate. Behold, your efforts so far. Judging by the number of contributions relating to church material, our religious leaders, if by chance any of them are reading this, will be pleased to see that the message of Christianity is alive and well in M500—even if it has been a little misunderstood.

Jim James

Dear Tony,

How refreshing to read your excellent article on mondegreens. Marion Stubbs, founder of the M500 Society and well-known advocate of publishing non-mathematical articles in the magazine (what she always referred to as 'rubbish'), would have been proud of you! Here is a small contribution from my years of experience in the subject.

As a choirboy from aged eight to the teens, I was exposed to a vast collection of sung parodies of both the intended and the unintended types. Since we choristers were always provided with scores and were expected to be able to read the words, if not always the music, they were mostly of the intended variety. These ranged from the banal *Fling wide the gates for the Player's Weights*[1] from Stainer's *Crucifixion* to my favourite which still makes me smile every time I hear the Christmas hymn *Hark the hairy*[2] *angels sing.*

But the true, unintended, mondegreens are much to be preferred. As an example, an elderly lady of my acquaintance once told me that in her childhood and before she knew any better, she always sang the first two lines of the popular evening hymn as *Now the day is over / Nighties*[3] *drawing nigh*, which gave her a happy and comforting mental image as bed-time drew near.

And then, as a fifteen years old youth at school, my greatest clanger occurred when I was asked to sing a solo audition for a house music competition. Priding myself now as budding tenor, I had a go at one of Handel's simplest and loveliest songs, unaccompanied and without the aid of a score. My version of the first line ran *Where'er you walk, school girls*[4] *shall fan the glade.* Our music teacher soon put me right afterwards; but I still preferred my version.

Jeremy Humphries

A girlfriend of mine, a pop music fan at school, thought it was:

Our Father, we chart[5] *in Heaven ...*

And Rose at school was mystified by the imprecation:

And lead us snotting to[6] *temptation ...*

Chris Woodhouse

May your petrol lighter[7] *shine upon them.*

Blessed art thou a monk swimming.[8]

Dick Boardman

A couple of religious 'mondegreens' for you.

The piece of cod[9] *which passeth all understanding ...*

Pity Mice in Plicity.[10] *Suffer me to come to thee.*

Me again

Having got this far we might as well finish the page. So let's dig out a few more. Possibly Jimi Hendrix was misunderstood when he sang:

'Scuse me while I kiss this guy.[11]

Now imagine you are strolling along a path on the outskirts of a Surrey town. Whilst watching the trees swaying in the breeze you might be moved to comment:

Oh! A tree[12] *in motion;*
Woking[13] *by my side.*

And when your husband challenges the communist meteorologist's latest weather forecast you can support him by responding:

Rudolph the Red knows rain, dear.[14]

Enough!

---

[1] Saviour waits
[2] herald
[3] Night is
[4] cool gales
[5] which art
[6] not into
[7] perpetual light
[8] amongst women
[9] peace of God
[10] my simplicity
[11] the sky
[12] Poetry
[13] Walking
[14] red-nosed reindeer

Letters to the Editor

Tarts

Dear Mr Forbes,

I have just applied to join M500 and have been sent M500 **196**. Perhaps I can add to what was said on Solution 193.2 – Thirteen tarts, or perhaps the subject has already been done to death.

It has been proved and published that n weighings are sufficient to find the faulty tart among $(3^n - 1)/2$ tarts. Hence two weighings for four tarts, three for 13 tarts, ... In most of the weighing results it will also be possible to tell whether the faulty tart is light or heavy—but not always.

In the case $n = 3$, I think Mr Boardman's assertion that when four balance with four it is not possible with two weighings to find the faulty tart among five remaining tarts ignores the fact that we have additional information that eight tarts are good.

If four balance with four, say ABCD against EFGH, then the tactic is to weigh ABC against IJK.

If they balance, then L or M is faulty and we simply weigh L with A. (If they balance, we do not know whether it is heavy or light.)

If ABC is heavier, then IJK contains the light tart. Weighing I against J will determine which. Similarly we can deal with ABC lighter than IJK.

Mathematical Snapshots by H. Steinhaus gave me the above. I hope it is useful. (He uses 1–13 but there are confusing printing errors.) Steinhaus also deals fully with ABCD not balancing EFGH.

Yours sincerely

James Elsey

ADF — Mr Boardman need not lose any sleep over this. I must admit that initially I was puzzled. The solution space of one heavy or light tart amongst five has size 10, but two weighings provide only nine bits of information. So how can one possibly resolve five tarts in two weighings? Then I noticed the parenthetical remark after weighing L with A, above. Boardman and Elsey are attacking different problems!

A good question to ask at this point is, 'Are there any n for which we can always determine the relative weight of the bad tart amongst $(3^n - 1)/2$ tarts in n weighings?'

Soap

Three things I have avoided all my life: soap operas (my mother was addicted to *The Archers*), musicals, and Harry Potter. I find that now I am totally unemployed but haven't yet broken into the pension scheme proper and am not yet used to being without deadlines, that I watch a lot of television—films mainly. One had finished and I didn't fancy what was coming next so I idly flicked the change button while I headed for the kitchen. When I returned we were well into *The Boyfriend*—a superb Buzby Berkely set. Brilliant. But the dialogue that followed was dire and Twiggy trying to sing was unforgivable. The next set piece was the entire cast dressed as dice. Would you believe that in every case four and three were adjacent. Can the general public be so ignorant?

Eddie Kent

The reason for my own total lack of interest in Harry Potter and similar constructs is that nobody has created a reasonably consistent theory of magic. If I am wrong, can someone enlighten me? — **ADF**

Mathematics Revision Weekend 2008

The thirty-fourth **M500 Society Mathematics Revision Weekend** will be held at

Aston University, **Birmingham**

over

Friday 12**th** – **Sunday** 14**th** **September 2008**.

The cost, including accommodation (with *en suite* facilities) and all meals from bed and breakfast Friday to lunch Sunday is £226 – £268. The cost for non-residents is £115 (includes Saturday and Sunday lunch). M500 members get a discount of £10. For full details and an application form, see the Society's web page at www.m500.org.uk, or send a stamped, addressed envelope to

Jeremy Humphries, M500 Weekend 2008.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. Tutorial sessions start at 19.30 on the Friday and finish at 17.00 on the Sunday. We plan to present most OU mathematics courses.

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