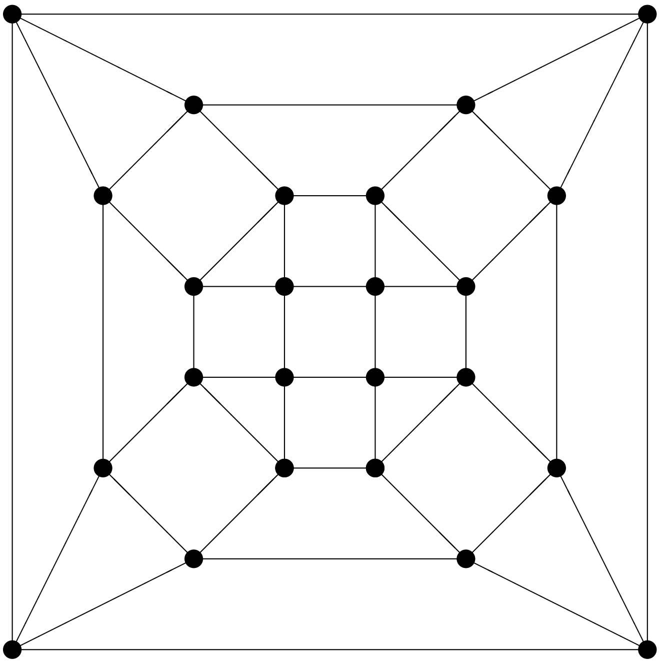
*

ISSN 1350-8539

M500 224



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file to tony@m500.org.uk.

Analogies between $Z(s)$ and $\zeta(s)$

Tommy Moorhouse

Introduction

An integer logarithm is a function ξ from the positive integers to the integers such that

$$\xi(nm) \;=\; \xi(n) + \xi(m)$$

for all integers n and m. Integer logarithms are completely determined by their values at prime numbers. In what follows we shall denote a general integer logarithm by ξ and, where we wish to restrict attention to the particular case $\xi(p) = p$, we shall use the letter κ instead of ξ; that is

$$\kappa(p) \;=\; p$$

for all primes p.

From the integer logarithm ξ we can construct a function $Z_\xi(s)$ defined by the series

$$Z_\xi(s) \;=\; \sum_{n=1}^{\infty} e^{-s\xi(n)}.$$

The deliberate similarity between this series and Riemann's ζ function (consider the substitution $\xi(n) = \log n$) is intended to suggest the application of $Z_\xi(s)$ to problems in number theory and perhaps give an alternative perspective on $\zeta(s)$.

The idea of this article is to present certain analogies between $Z(s)$ and $\zeta(s)$, examining where the analogies succeed and where they fail. This may point to methods for usefully applying Z to number theoretic problems. Some indications of applications to the study of partitions are also given.

1. An integral representation for $\log Z(s)$

Riemann made use of the integral representation, covered in detail in [2],

$$\log \zeta(s) \;=\; s \int_0^{\infty} J(x) x^{-s-1} dx$$

to examine the distribution of prime numbers. Here $J(x)$ is a function that 'jumps' by 1 at primes, by 1/2 at prime squares, by 1/3 at prime cubes and so on. The definition is, with an implied sum over the primes,

$$J(x) \;=\; \frac{1}{2}\left\{\sum_{p^n<x}\frac{1}{n}+\sum_{p^n\le x}\frac{1}{n}\right\}.$$

It is fairly easy to see that

$$J(x) \;=\; \pi(x)+\frac{1}{2}\pi(x^{1/2})+\frac{1}{3}\pi(x^{1/3})+\cdots$$

We can obtain an analogous function for Z, namely

$$M(x) \;=\; \frac{1}{2}\left\{\sum_{n\xi(p)<\log x}\frac{1}{n}+\sum_{n\xi(p)\le\log x}\frac{1}{n}\right\},$$

by breaking down the series for $\log Z(s)$. The function $M(x)$ jumps by 1 at $e^{\xi(p)}$, by 1/2 at $e^{\xi(p^2)}$ and so on. In the simple case $\xi(p)=p$ we have

$$M(x) \;=\; \pi(x)+\frac{1}{2}\pi(x/2)+\cdots.$$

Riemann went on to use Fourier inversion together with another expression for $\log\zeta(s)$ and a functional equation to find an alternative expression for $J(x)$. It is more difficult to find an analogy for Riemann's alternative expression, and we examine this problem further in a later section.

2. Use of the gamma function $\Gamma(s)$

Many of the most interesting properties of the Riemann ζ function follow from a functional equation relating $\zeta(1-s)$ to $\zeta(s)$. The analogous result for $Z(s)$ is elusive for reasons considered below.

To see what is involved we use an integral representation of the gamma function to find an alternative expression for $Z(s)$. We have

$$\Gamma(s) \;=\; \int_0^\infty t^{s-1}e^{-t}dt.$$

Substituting $\tau = te^{\xi(n)}$ we find

$$\Gamma(s) \;=\; e^{s\xi(n)}\int_0^\infty t^{s-1}e^{-te^{\xi(n)}}dt.$$

This tells us that

$$\sum_{n=1}^{\infty} e^{-s\xi(n)} \;=\; \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-1} e^{-te^{\xi(n)}} dt.$$

The integrals are clearly absolutely convergent by comparison with Γ, and we expect the sum to have a pole at $s = 0$. The other properties of this integral representation are harder to deduce.

The kernel of the integral $e^{-te^{\xi(n)}}$ lacks two crucial properties enjoyed by the kernel e^{-tn}: it is not periodic in t, and the sum of all such terms cannot be formed into a simple geometric series. These appear to be the main obstacles to developing a further level of analogy with the ζ function and in particular finding a functional equation for Z.

We can still obtain quite detailed results, at least on the half-plane $\sigma > 0$. To see this consider a p-th root of unity, ζ, and for each $1 \le i \le p-1$ form the series

$$W_p^{(i)}(s) \;=\; \sum_{n=1}^{\infty} \zeta^{ni} e^{-s\kappa(n)}.$$

It is straightforward to show, using the properties of the roots (essentially that $\sum_{m=1}^{p} (\zeta^i)^m = 0$ if p does not divide i, and is equal to p otherwise), that

$$Z(s) + \sum_{i=1}^{p} W_p^{(i)}(s) \;=\; pe^{-sp} Z(s).$$

We could actually write $Z(s)$ as $W_p^{(0)}(s)$. The simplest case is of course when $p = 2$, when $W_2^{(1)}(s)$ is the alternating sum

$$W_2^{(1)}(s) \;=\; \sum (-1)^n e^{-s\kappa(n)}.$$

For each p we can therefore write

$$Z(s) \;=\; \frac{\sum_{i=1}^{p-1} W_p^{(i)}(s)}{pe^{-sp} - 1}$$

and since the Ws have much nicer convergence properties than Z (the absolute values of the partial sums of the coefficients of $e^{-s\kappa(n)}$ are bounded) we effectively get an analytic continuation of Z with a singularity at the point $s = \log p/p$. Different primes p allow us to continue $Z(s)$ to these points.

3. Using $Z(s)$ and related series

We turn now to the application of series related to Z in the exploration of number theoretic problems, particularly partitions. Further background material will be found in [1]. First we recall that if

$$F(s) \;=\; \sum_{n=1}^{\infty} f(n)e^{-s\xi(n)}, \qquad G(s) \;=\; \sum_{n=1}^{\infty} g(n)e^{-s\xi(n)},$$

we have

$$F(s) \;=\; \sum_{n=1}^{\infty} E_f(n)e^{-sn}, \qquad G(s) \;=\; \sum_{n=1}^{\infty} E_g(n)e^{-sn},$$

where $E_f(n) = \sum_{\xi(m)=n} f(m)$. We can show that

$$E_{f*g} \;=\; E_f \circ E_g,$$

where

$$f * g(n) \;=\; \sum_{ij=n} f(i)g(j)$$

and

$$A \circ B(n) \;=\; \sum_{i+j=n} A(i)B(j).$$

This allows us to convert our functions into Dirichlet-like series. The map from $(\{f\}, *)$ to $(\{E_f\}, \circ)$ is a ring homomorphism with kernel I.

One interesting application of the above is in finding an explicit expression for the function $E_{(-1)^x}(n) \;=\; \sum_{\kappa(m)=n}(-1)^m$. This arises from the expression for

$$W_2^{(1)}(s) \;=\; \sum_{n=1}^{\infty}(-1)^n e^{-s\kappa(n)} \;=\; \sum_{n=0}^{\infty} E_{(-1)^x}(n)e^{-sn}.$$

It is straightforward to show, using the expression for W in terms of Z given above, that $E_{(-1)^x}(n) = 2P(n-2) - P(n)$, where $P(n)$ is the number of prime partitions of n.

We can also make use of series such as

$$\sum_{n=1}^{\infty} \frac{\kappa(n)}{n^s}$$

which are perhaps more familiar. Let $K(n) = p$ if $n = p^r$ for some $r > 0, 0$ otherwise. We note that, since $\kappa = K * u$,

$$\sum_{n=1}^{\infty} \frac{\kappa(n)}{n^s} \;=\; \zeta(s) \sum_{n=1}^{\infty} \frac{K(n)}{n^s} \;=\; \zeta(s) \sum_{p} \frac{1}{p^{s-1}}.$$

There are undoubtedly many more such identities to be found!

4 Conclusion

There are many analogies between Reimann's zeta function and the functions Z. It may be possible to develop a theory of the Z functions much further, and we have examined some of the obstacles that would need to be addressed.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer 1998.

[2] H. M. Edwards, *Riemann's Zeta Function*, Dover 1974.

On the operator $L(x)$

Sebastian Hayes

We introduce an operator $L(f(x))$ which obeys the following rule.

Definition 1. $L(f(x) \cdot g(x)) = L(f(x)) + L(g(x))$.

Assuming the normal rules for $\mathbb{Z}$, we have

$$L(1) \;=\; L(1 \cdot 1) \;=\; L(1) + L(1) \;=\; 2L(1).$$

A similar argument gives $L(0) = 0$. Thus

$$L(0) \;=\; 0, \qquad L(1) \;=\; 0. \tag{i}$$

Also, if N is composite, $L(N) = L(n \cdot m) = L(n) + L(m)$. Thus $L(f(x) \cdot 1/f(x)) = L(1) = 0$ from (i), and $L(f(x) \cdot 1/f(x)) = L(f(x)) + L(1/f(x))$ by Definition 1. Also

$$L(1/f(x)) \;=\; -L(f(x)) \tag{ii}$$

and in particular

$$L(-1) \;=\; 0. \tag{iii}$$

Hence

$$L(-f(x)) \;=\; L((-1)\cdot f(x)) \;=\; L((-1)) + L(f(x)) \;=\; 0 + L(f(x))$$

from (iii); so $L(-f(x)) = L(f(x))$. Furthermore,

$$L\left(\frac{f(x)}{g(x)}\right) = L\left(f(x)\cdot\frac{1}{g(x)}\right) = L(f(x)) + L\left(\frac{1}{g(x)}\right) = L(f(x)) - L(g(x))$$

from (ii). Thus

$$L(f(x)/g(x)) \;=\; L(f(x)) - L(g(x)). \qquad \text{(iv)}$$

If n is a positive integer then, by Definition 1,

$$L(f(x)^n) = L(f(x)\cdot f(x)\cdot\cdots\cdot f(x)) = L(f(x)) + L(f(x)) + \cdots + L(f(x)),$$

with n factors in the product and n terms in the sum. So

$$L(f(x)^n) \;=\; nL(f(x)). \qquad \text{(v)}$$

With n, m positive integral,

$$L(f(x)^{n-m}) = L((f(x)^n/f(x)^m) = L(f(x)^n) - L(f(x)^m$$

from (ii), (iv), and this is equal to $nL(f(x)) - mL(f(x))$ from (v). Hence

$$L(f(x)^{n-m}) \;=\; (n-m)L(f(x)).$$

Now, so far we have not determined $L(f(x))$ for any function except a constant function. We have thus to start by defining $L(x)$. If we make it 0 it will be no different from a constant function, which seems bizarre, while if we make it x, it does not change under the operation which is possible but somewhat uninteresting.

Definition 2. $L(x) = 1$.

It follows at once from (v) that $L(x^n) = n$, $L(1/x^n) = L(x^n\;) = -n$ and more generally $L(x^{n-m}) = (n-m)$, $n, m \in \mathbb{Z}$. With some care we can perhaps extend the index to any real number giving $L(x^r) = r$, $r \in \mathbb{R}$. It would seem that we cannot proceed further without a definition for the sum of two or more functions. There are, seemingly, various possibilities which do not lead to contradiction with what we have established so far. One possibility is the following.

Definition 3. $L(f(x)+g(x)) \;=\; \dfrac{f(x)L(f(x))+g(x)L(g(x))}{f(x)+g(x)}$.

This allows us to determine the value of $L(p)$ where p is prime, which we have not so far been able to do from the definitions: $L(2) = L(1+1) = \frac{1}{2}\bigl(1\cdot L(1) + 1\cdot L(1)\bigr) \;=\; 0$ and all further natural numbers can be worked up recursively from $k = 2$; $L(k+1) = 1/(k+1)\bigl(kL(k) + 1\cdot L(1)\bigr) \;=\; 0$.

It might be interesting to have a system which differentiates between $L(n)$ when $n = c$ and $n = p$ prime. One result is that $L(nf(x)) = L(n) + L(f(x)) = L(f(x))$, where n is an integer.

It will perhaps by now be obvious to many readers where all this is drifting. Some years ago I was put in contact (via M500) with a Mr Henry Jones, an OU student and contributor to M500, and he introduced me to a kind of calculus he had invented, named Logarithmic Calculus. He does not develop this form of calculus in the way I have, but in a manner much more analogous to the normal derivative dy/dx. However, I wanted to see how much we need to assume to get to this point; hence my more axiomatic treatment.

Basically, the relation between Jones's calculus (as I call it) and 'normal' calculus is via the master equation which appears on the frontispiece of the privately printed treatise *Logarithmic Calculus*:

$$L_x^1 f(x)\cdot D_x^1 f(x) \;=\; L_x^1 e^{f(x)}.$$

This brief introduction to his interesting mathematical invention may suffice for the moment. Mr Henry Jones was a Welsh civil engineer from Knighton, who subsequently became a good friend of mine and whom I visited several times towards the end of his life. As far as I know I am the only person who attempted to master his deviant form of calculus and I tried to interest professional mathematicians in it.

Henry Jones also has several mechanical inventions to his credit, some of which I have seen working, including a drip-feed mechanism based on Hook's law for use in hospitals, and a device for drawing an ellipse—he was amongst other things a surveyor and architect. Like many inventors he gave up bothering to patent his inventions and suspected that, in at least one case, a big company had simply introduced a slight variation and gone into production. Henry Jones also developed a personal theory of gravitation. At any rate, this brief note is a tribute to the many fascinating days I spent in his company during the late seventies.

Mathematics in the garden

Tony Forbes

Recently I found myself accidentally listening to one of those diabolical gardening programmes. I was particularly annoyed because I had just previously ordered members of my family to desist from putting kitchen waste on our compost heap. And then suddenly I was hearing an expert droning on about the benefits of composting, and how he makes a point of putting anything that can be vaguely described as organic on his heap—cardboard, carpets, dead animals, urine, and various other forms of undescribably biological material.

I have to say that his advice had the same credibility as that of the lady who once told us, 'No matter how small your garden, always leave a few acres to wild woodland.' With a modest suburban plot, I have difficulty dealing with the material created by just the garden itself without adding to it from other sources. If I were to follow the expert's instructions, the garden would be dominated by its compost heap. The problem is one of dimensionality. A large garden can cope with all of its waste. A small garden cannot. If you don't believe me, try maintaining a compost heap in a window-box.

Let $\mathcal{G}$ be a garden and let $\mathcal{C}$ be a compost heap. Suppose for simplicity $\mathcal{G}$ is circular, radius R, and $\mathcal{C}$ is a hemisphere of radius r in the centre of $\mathcal{G}$. Well, you never know, this kind of design might one day become fashionable—especially if it includes a water feature.

The rate at which a garden produces compostable material is proportional to its surface area. Let $G(t)$ be the surface area of the remaining part of $\mathcal{G}$ after excluding the circle occupied by the compost heap. If we regard r as a function of time, t, $r(t)$, then $G(t) = \pi(R^2 - r(t)^2)$. The rate of production of garden waste will be $gG(t)$ for some constant g. This should be interpreted as a nett figure, because we must remember that composted material is being removed from the heap to replenish the garden, again at a rate which is proportional to $G(t)$.

Now let as advance the clock from t to $t+\delta t$. The volume of garden waste produced by $\mathcal{G}$ is $gG(t)\delta t$, and this will be added to $\mathcal{C}$, thereby increasing its volume to

$$\tfrac{2}{3}\pi r(t)^3 + \pi g(R^2 - r(t)^2)\delta t.$$

Hence the radius of $\mathcal{C}$ is now

$$\left(r(t)^3 + \tfrac{3}{2}g(R^2 - r(t)^2)\delta t\right)^{1/3},$$

which by expanding using the binomial theorem and discarding $(\delta t)^2$ and higher powers simplifies to

$$r(t)\left(1+\frac{g(R^2-r(t)^2)}{r(t)^3}\,\delta t\right).$$

Hence

$$\delta r(t) \;=\; \frac{g(R^2-r(t)^2)}{r(t)^2}\,\delta t.$$

Letting $\delta t \to 0$ we have

$$\frac{dr(t)}{dt} \;=\; \frac{g}{2}\left(\frac{R^2}{r(t)^2}-1\right). \tag{1}$$

Since $r(t) < R$, this simple result is going to tell us that inevitably the compost heap will occupy almost all of the garden. But note that the constant g is universal. Approximately speaking, it is the same for all gardens, minute or huge, and, moreover, one can even try to estimate it. Thinking about the amount of grass cuttings you might collect after mowing a square metre of lawn suggests a figure of about 0.001 m/year.

If we consider the garden at the time when the compost heap occupies such a small part that we can ignore the minus-one term in (1), we can actually solve (1) exactly to get

$$r(t) \;\approx\; \left(\tfrac{3}{2}gR^2t\right)^{1/3}.$$

This tells us that the rate of growth of a compost heap nearly depends on the two-thirds power of the radius of the garden and, since $\frac{2}{3} < 1$, we have (I think) an extremely clear explanation as to why compost heap growth becomes more and more significant as garden size decreases.

Problem 224.1 – Three rolling spheres

Norman Graham

If the times to roll down an inclined plane are t_1 for a hollow sphere, t_2 for a solid sphere and t_3 for a ‘semi-solid’ sphere (solid except for a central hole of half the radius), prove that

$$t_1 : t_2 : t_3 \;=\; \sqrt{\frac{5}{3}} : \sqrt{\frac{7}{5}} : \sqrt{\frac{101}{70}} \;\approx\; 1.291 : 1.183 : 1.201.$$

Dr Panic's improved jam sandwich

Ralph Hancock

It was with reluctance that I accepted the invitation of the notorious food topologist Dr Urban Panic to visit him at his villa in Antibes, where he had retired in suspicious affluence after the failure of his company Toposnax. But I discovered the doctor in high spirits. "I've finally cracked it," he said. "The perfect jam sandwich, and no jam on your fingers, ever."

Remembering my previous experience with his Klein-bottle jam doughnut, and the resulting dry cleaning bill which Dr Panic had not offered to pay, I may have looked sceptical. But, full of reassurances, he bustled me over to his laboratory.

The large room was dominated by what appeared to be a huge toy gyroscope mounted in massive gimbals encrusted with coils, the whole structure about three metres in diameter. Closer inspection showed that the rotor was actually a ring, apparently edge-driven, and where the axle would have been, there was another set of gimbals and a smaller gyroscope; and inside this was a third set. This too had a ring-shaped rotor, but the small space inside it was empty. The outer two rotors were horizontal, and white-coated assistants were nudging the inner one into alignment with wooden poles.

"You will know about the phenomenon of gyroscopic precession," the doctor said. Seeing my blank look, he went on, "To put it simply, if you try to rotate a spinning gyroscope, the force you apply will be, as it were, carried round 90 degrees in the direction the rotor is spinning. Push the gyroscope away and it will go sideways, you understand that?"

"You see three gyroscopes here. As you will have realized, each gimbal and rotor assembly is an electric motor—we can only get power to all of them when they are horizontal, but that does not matter. What you have not seen is . . . the fourth rotor!"

He waved a hand, and an assistant brought him a loaf of bread, some butter, a pot labelled 'Jam No 71', which appeared to contain bitumen, a bread knife and what turned out to be a cutting jig, with which the doctor cut a precise eight cm cube of bread. He speared the cube on a fork, buttered the entire outside, and finished it off with a sticky black coating from the pot. "It's our high-iron, high-viscosity blackcurrant jam," he said. "We needed the iron for the magnetic field, of course."

The doctor went on, "Now, when you spin all those gyros, and you rotate the outer gimbals by 180 degrees, what do you think happens?" He pointed at a stout hydraulic arm connected to the outermost frame.

I said, "I suppose the second one turns over too, but in a plane at right angles to the first. And the third one turns over at right angles to both of the other rotors."

"Not bad for a journalist," he said. "But what happens if there's a fourth rotor inside all the others?"

He transferred the cube to a plastic clamp on the end of a wooden rod. "Levitation field on," he said, and carefully placed the bread in the middle of the inner rotor; it bobbed lazily in the air and he released the clamp. "Run up the gyros." The massive mechanism gained speed. I noticed that the bread cube was spinning even faster than the rotor surrounding it.

"Rotate!" shouted Dr Panic, and the hydraulic arm shot out. The floor trembled as the heavy rotors shifted in unimaginable directions. But as they slowed down, I caught a glimpse of the bread at the centre, and it was white. Had all the jam spun off? There was none on my suit this time.

"Clean fingers at last!" said Dr Panic. "I have inverted the sandwich in the fourth dimension." He reached into the rotors with a small net on a pole, neatly retrieved the cube, and handed it to me. There was no trace of butter or jam. It looked and felt like bread, but had an unfamiliar acrid smell. "Enantiomorphy," he explained. "The molecules are reversed, and have different effects on the olfactory sensors."

Leaving that remark well alone, I said, "You mean, the jam's on the inside?"

"Not exactly," he replied, and sliced the cube in half, into quarters and then eighths. The cut surface was as white as before. "You might say that the jam is on the back. Here, have a bite." He gave me a small cube, which I cautiously put into my mouth. It tasted slightly bitter. I chewed it experimentally.

"Now you see," he said, "that when the three-dimensional matrix is sufficiently distorted, the object will, ah, evert ... " There was a sharp crack as if a mousetrap had gone off on my tongue, and an instant flood of the vile taste of rusty iron ill disguised with sugar. I spat out the noxious gobbet.

"We're working on the flavour," said Dr Panic.

Three easy steps to get your dividend payments paid direct into your bank account: (1) Fill in the sort code; (2) fill in your account number; (3) sign it. [Instructions on share dividend payment form]

Solution 219.3 – Circumcircle

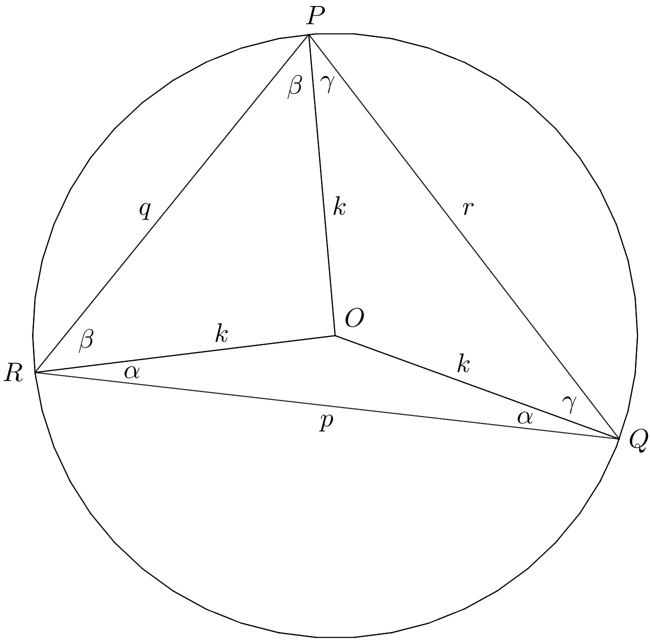
A triangle has sides which are the three roots of the cubic

$$x^3 - ax^2 + bx - c.$$

Show that its circumcircle has radius

$$\frac{c}{\sqrt{4a^2b - 8ac - a^4}}.$$

Steve Moon



Let the radius of the circumcircle of triangle PQR be k, centre O. Let the roots of $x^3 - ax^2 + bx - c$ be p, q, r, the sides of $\triangle PQR$. Then $(x-p)(x-q)(x-r) = 0$, and therefore

$$x^3 - (p+q+r)x^2 + (pq+qr+rp) - pqr \;=\; 0.$$

Hence

$$a = p+q+r, \quad b = pq+qr+rp, \quad c = pqr.$$

Let

$$\alpha = \angle ORQ = \angle OQR, \quad \beta = \angle ORP = \angle OPR, \quad \gamma = \angle OPQ = \angle OQP.$$

Using the cosine rule on $\triangle ORQ$,

$$k^2 = p^2 + k^2 - 2pk\cos\alpha; \quad \text{hence } \cos\alpha = \frac{p}{2k}.$$

Using the cosine rule on $\triangle ORP$,

$$k^2 = q^2 + k^2 - 2qk\cos\beta; \quad \text{hence } \cos\beta = \frac{q}{2k}.$$

Using the cosine rule on $\triangle PQR$,

$$r^2 = p^2 + q^2 - 2pq\cos(\alpha+\beta); \quad \text{hence } \cos(\alpha+\beta) = \frac{p^2+q^2-r^2}{2pq}.$$

But $\cos(\alpha+\beta) = (\cos\alpha)(\cos\beta) - (\sin\alpha)(\sin\beta)$. Therefore

$$\frac{p^2+q^2-r^2}{2pq} = \frac{pq}{4k^2} - \frac{\sqrt{4k^2-p^2}\sqrt{4k^2-q^2}}{4k^2}.$$

Hence

$$4k^2(p^2+q^2-r^2) = 2p^2q^2 - 2pq\sqrt{(4k^2-p^2)(4k^2-q^2)}.$$

Therefore

$$pq\sqrt{(4k^2-p^2)(4k^2-q^2)} = p^2q^2 - 2k^2(p^2+q^2-r^2).$$

Squaring gives

$$16p^2q^2k^4 = 4p^2q^2r^2k^2 + 4k^4(p^4+q^4+r^4) + 8k^4(p^2q^2 - q^2r^2 - p^2r^2),$$

which becomes, on dividing by $4k^2$,

$$4p^2q^2k^2 = p^2q^2r^2 + k^2(p^4+q^4+r^4) + 2k^2(p^2q^2 - q^2r^2 - p^2r^2);$$

hence

$$k = \frac{pqr}{\sqrt{2(p^2q^2+p^2r^2+q^2r^2) - (p^4+q^4+r^4)}}.$$

We can then substitute $pqr = c$ in the numerator. For the denominator we will need

$$\begin{aligned}(p+q+r)^4 &= (p^4+q^4+r^4)+6(p^2q^2+q^2r^2+p^2r^2)\\ &\quad + 12pqr(p+q+r)\\ &\quad + 4(p^3q+pq^3+p^3r+q^3r+pr^3+qr^3).\end{aligned}$$

So in the denominator, $2(p^2q^2+p^2r^2+q^2r^2)-(p^4+q^4+r^4) = D$, say, becomes

$$\begin{aligned}D &= 8(p^2q^2+p^2r^2+q^2r^2)\\ &\quad + 4(p^3q+pq^3+p^3r+q^3r+pr^3+qr^3)+12ac-a^4.\end{aligned}$$

Prompted by the form of the solution, we note that

$$\begin{aligned}a^2b &= (p+q+r)^2(pq+qr+rp)\\ &= 2(p^2q^2+p^2r^2+q^2r^2)+(p^3q+pq^3+p^3r+q^3r+pr^3+qr^3)\\ &\quad + 5(p^2qr+pq^2r+pqr^2),\end{aligned}$$

which allows us to simplify the terms of D involving p, q and r. Thus

$$D = 4a^2b-20(p^2qr+pq^2r+pqr^2)+12ac-a^4 = 4a^2b-8ac-a^4$$

and hence

$$k = \frac{c}{\sqrt{4a^2b-8ac-a^4}}.$$

Problem 224.2 – Buried treasure

Norman Graham

(i) Treasure has been buried at T, distances $a = 2$, $b = 3$ and $c = 4$ from three successive corners of a square field. Determine s, the length of a side of the field. (This is No. 66 of *The Canterbury Puzzles* by H. E. Dudeney.)

For $a = 2$ and $c = 4$, find the range of values of b if T is (ii) anywhere, or (iii) somewhere within the field.

(iv) Devise geometrical constructions to obtain the answers.

There is a temporary unavailability of rabies pre-exposure vaccine due to finite supplies.—*The Times* [Sent by John Williamson.]

Solution 191.8 – Infinite exponentiation

> I want to know the value of $y = x^{x^{x^{x^{\cdots}}}}$ (where the powers go on for ever) when $x = 1.1$. Writing it as $y = x^y$ I get $\log y = y \log x$; so $\log x = (\log y)/y$. Putting $x = 1.1$, I find this has two solutions: $y = 1.111782011...$ and $y = 38.22873285....$ They can't both be right. What is the explanation?

Hugo Touchette

The infinite exponential $x^{x^{x^{\cdots}}}$ is the infinite iterate of the map $f(y) = x^y$ starting at $y = x$, and converges, as such, to the *stable* fixed-point of f satisfying $y = x^y$. For $x = 1.1$, there are two fixed-points, namely, $y = 1.111782011\ldots$ and $y = 38.22873285\ldots$, as mentioned in the problem, but only the first one is stable; the second is unstable. Thus, denoting by $f^{(n)}(x)$ the n-fold composition of the map f starting at x, we must have

$$x^{x^{x^{\cdots}}} = \lim_{n\to\infty} f^{(n)}(x) = 1.111782011\ldots$$

for $x = 1.1$. The convergence is quite rapid, as can be checked by calculating the first few iterates:

$$\begin{aligned} 1.1^{1.1} &= 1.110534241\ldots \\ 1.1^{1.1^{1.1}} &= 1.111649800\ldots \\ 1.1^{1.1^{1.1^{1.1}}} &= 1.111768002\ldots. \end{aligned}$$

Incidentally, if we start off with any number for the initial value of the iterate, we end up with the same infinite exponential because $y = 1.111782011\ldots$ is the only stable fixed point of f. Thus

$$\lim_{n\to\infty} f^{(n)}(x_0) = 1.111782011\ldots$$

for any real x_0. In words this means that the infinite exponential is insensitive to the number put at the top (infinite) level of exponentiation.

Problem 224.3 – Inspecting the column

Norman Graham

A column of soldiers of length a is marching steadily along a road, when an officer on horseback rides at uniform speed from the rear to the front and back again while the column moves distance b.

How far does the officer move?

Rounder than the sphere

Dennis Morris

The unit circle in 2-dimensional euclidean space is usually defined to be the set of points that are distance unity from the origin. It is also the set of points defined by

$$\{x = \cos\theta,\ y = \sin\theta\}.$$

We have

$$\left\{\frac{d(\cos\theta)}{d\theta} = -\sin\theta,\ \frac{d(\sin\theta)}{d\theta} = \cos\theta\right\}.$$

Thus, a defining property of the circle is

$$\left\{\frac{dx}{d\theta} = -y,\ \frac{dy}{d\theta} = x\right\}.$$

We have a similar scheme in the case of the hyperbolic circle, which is the set of points that are distance unity from the origin in hyperbolic space. In this case, we have

$$\left\{\frac{dx}{d\theta} = y,\ \frac{dy}{d\theta} = x\right\}.$$

It is often thought that the sphere (ball) is a general form of the 2-dimensional circle in 3-dimensional space. However, the rotation matrix associated with the sphere is

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This contains the two euclidean trigonometric functions and the 1-dimensional trigonometric function—unity. Although the 2-dimensional trigonometric functions have a differentiation cycle, that cycle does not include the 1-dimensional trig function. Therefore, the set of points at distance unity from the origin associated with this rotation matrix is not the general form of the circle in 3-dimensional space.

The rotation matrix of the natural geometric space inside the group C_3 is

$$\begin{bmatrix} \nu A(\theta,\phi) & \nu B(\theta,\phi) & \nu C(\theta,\phi) \\ \nu C(\theta,\phi) & \nu A(\theta,\phi) & \nu B(\theta,\phi) \\ \nu B(\theta,\phi) & \nu C(\theta,\phi) & \nu A(\theta,\phi) \end{bmatrix}.$$

This contains the three 3-dimensional trigonometric functions, and these functions have a differentiation cycle. This 3-dimensional 'circle' is the set of points that are distance unity from the origin and it also has the defining property

$$\left\{x = \frac{dy}{d\theta} = \frac{dz}{d\phi},\ y = \frac{dz}{d\theta} = \frac{dx}{d\phi},\ z = \frac{dx}{d\theta} = \frac{dy}{d\phi}\right\}.$$

This, it seems, is the proper general form of the circle in 3-dimensions. It is, of course, rounder than the sphere.

Palindromes in morse

Jeremy Humphries

Find some English palindromes that are also palindromes in morse code.

Find some morse code palindromes that make sense, but not palindromes, in English.

A	· —	N	— ·
B	— · · ·	O	— — —
C	— · — ·	P	· — — ·
D	— · ·	Q	— — · —
E	·	R	· — ·
F	· · — ·	S	· · ·
G	— — ·	T	—
H	· · · ·	U	· · —
I	· ·	V	· · · —
J	· — — —	W	· — —
K	— · —	X	— · · —
L	· — · ·	Y	— · — —
M	— —	Z	— — · ·

Also we would be interested in any examples of morse occurring in *songs*.

Normally I (**TF**) would have dismissed this last idea as far too daft to consider, even for M500. But that was before I discovered the German radio operators' song, *Funkerlied*, popular amongst the Wehrmacht in the 1940s. The chorus goes something like this.

Denn wir sind ja von der Funker Kompanie,
Und wir geben stets: Ich liebe, liebe Sie!
Und alle Mädchen hören mit:
Di-da-di-dit di-da di-dit!

Go to *You Tube* if you want to hear what it sounds like.

Problem 224.4 – Integers

John Bull

Given that $u = (p+1)(p+2n)/2$, where p and n are positive integers, show that values of p and n can be chosen to produce every positive integer except for those that take the form 2^{m-1} where m is a positive integer.

Solution 220.1 – Marbles and fruit

In the interests of brevity I've restated the problem as follows.

> Let F be the number of distinct rectangular arrays of n marbles. Show that F is even if $n \equiv 2 \pmod 3$. What if $n \equiv 1 \pmod 3$?

See M500 **220** for further information about *twins*, *fall out*, *fruit*, *kitty*, *tray* and *Bilal's second rule of thumb.* — **TF**

Sally Metcalfe

The twins will only fall out [F odd] if n is a square number. For every $k \times m$ rectangle produced, an $m \times k$ rectangle may also be produced so long as m and k are different; so an even number of fruit pieces [F] will be put into the kitty unless an $m \times m$ rectangle can be produced. We need to show that two marbles will never be left in the tray [$n \not\equiv 2 \pmod 3$] if n is a square number. Clearly if n is divisible by 3, no marbles will be left [$n \equiv 0 \pmod 3$].

Now consider the possibility that n is not divisible by 3. It follows that $m = \sqrt{n}$ is not divisible by 3 either. We may rewrite n as $n = m^2 = (m+1)(m-1)+1$ and, as one of $m+1$ and $m-1$ is divisible by 3, n is a multiple of 3 plus 1, and hence one marble will be left in the tray [$n \equiv 1 \pmod 3$].

One marble left in the tray will not necessarily mean the twins falling out. For example if $n = 7$, only two rectangles 1×7 and 7×1 are possible; so there will be two pieces of fruit and one marble left over at the end. However if Bilal's hands are not large enough to scoop up seven (or more) marbles then 1 left over would mean $n = 1$ or $n = 4$ which are both square numbers and would cause a fall out.

Tommy Moorhouse

The total number of fruit pieces given to the twins is just the number of divisors of n, which will be denoted by $\tau(n)$.

The number of marbles left in the tray is the remainder after dividing n by 3, which is usually called $n \pmod 3$. The problem reduces to proving that $\tau(3m+2) \equiv 0 \pmod 2$ (i.e. $\tau(3n+2)$ is even).

If we write n as a product of prime factors as follows $n = p_1^{k_1} \cdots p_m^{k_m}$, then $\tau(n) = (1+k_1)(1+k_2)\cdots(1+k_m)$. Since $m^2 \equiv 1 \pmod 3$ for all $m \not\equiv 0 \pmod 3$ while $n \equiv 1 \pmod 3$, we deduce that at least one of the k_i is odd, and thus $\tau(n)$ is even. If $n \equiv 1 \pmod 3$ we could still have two of the k_i odd and hence $\tau(n)$ even, so Bilal's second rule of thumb is false.

Problem 224.5 – Digit reversal

Tony Forbes

Find integers $a, b, c \geq 2$, a not divisible by 10, such that $a^b = c$ and $(a')^{b'} = c'$, where x' is obtained from x by reversing the order of its decimal digits.

For example, the first two that my computer found are the familiar

$$12^2 = 144,\ 21^2 = 441, \quad \text{and} \quad 13^2 = 169,\ 31^2 = 961.$$

And just to demonstrate that this is not a trivial problem, here is a solution with $b = 3$:

$$1110001^3 = 1367634696303330001,\ 1000111^3 = 1000333036964367631.$$

Notice that if this reversal property works for a, b and c, then it holds for $10a$, b and $10^b c$. Hence the additional restriction on a.

Are there any examples with $b > 3$ if we discount instances where all three of $a' = a$, $b' = b$ and $c' = c$ hold, as is the case with $a = 11$, $b = 4$, $c = 14641$?

Problem 224.6 – Consecutive integers

Tony Forbes

Find eleven consecutive integers all of which have been completely factorized into primes.

If the numbers have over 500 digits, report the result to **Jens Kruse Andersen**, who maintains a Web site at

http://hjem.get2net.dk/jka/math/consecutive_factorizations.htm.

The current record for ten consecutive numbers is 509 digits, due to **David Broadhurst**. The numbers are

$$f(d) = \frac{x^2(x^2 - 23)(x^2 - 41)(x^2 - 64)}{55440} - 4 + d,$$

$d = 0, 1, \ldots, 9$, where $x = 13860(10^{60} + 1898683)$. The first two factorize as

$$f(0) = 2^2 \cdot 97 \cdot P_1, \quad f(1) = 3 \cdot 19 \cdot 443 \cdot P_2,$$

where P_1 and P_2 are large primes. The others I leave for you to do.

Problem 221.4 revisited – Eleven bottles

> Three people, A, B, C, are stuck in a lift over the weekend. They have 11 bottles of water, four supplied by A and seven by B, which are to be shared equitably. C donates £11 for the water. How is it to be divided between A and B?
>
> The same thing happens the following weekend, but this time A has three bottles and B has eight. Again, £11 is to be split between A and B.

Tony Forbes

This is not a solution (at any rate not one that I am happy with), just some ramblings by me.

A first solution: A and B share the money evenly, £5.50 each.

Well, no; B is going to complain bitterly because he contributed more bottles than A. (The sexes of A, B and C are not relevant. I mention this because someone I tested the problem on did ask.)

So let's share the money according to the number of bottles of water supplied: A £4.00 and B £7.00. I think it's safe to assume that the bottles contain the same amounts of water; otherwise the problem would elevate to an additional level of unnecessary complexity.

But B is still complaining because A drank most of the water he contributed. So he suggests A and B share the money according the actual amount of water contributed for quenching the thirst of persons other than the provider. Thus A gets £1.00 and B gets £10.00.

This is fine. Everyone is happy, especially B, who has no further grounds for complaint. And if you can find the book from which I ripped this problem, you will see that in the back pages this is precisely the answer they give.

We move to the second weekend.

Note that B contributes $8 - 3\frac{2}{3} = 4\frac{1}{3}$ bottles of water (for the enjoyment of others), but A's contribution is negative, $-\frac{2}{3}$ bottles. Applying the algorithm we devised for the first week, we see that A must donate £2.00 to make a total of £13 for B.

We are in serious trouble. Poor A is embarrassed by C's generosity. So the solution we developed cannot possibly be correct, even for the first weekend.

What now?

M500 Winter Weekend 2009

A Weekend of Mathematics and Socializing

Join with fellow mathematicians for a weekend of fun and a look at some unusual, interesting and recreational mathematics. The 28th M500 Society Winter Weekend will be held at **Florence Boot Hall**, **Nottingham University** from Friday evening to Sunday afternoon **9th–11th January 2009**. Provisional programme:

Friday		
8.00 p.m.	Ice breaker: mathematical investigations	Mel Starkings
9.00 p.m.	Pub quiz	Rob Rolfe
Saturday		
morning	Data structures for family trees	Tony Huntington
	Dyslexia and mathematics	Sonia Rochford
afternoon	Non-euclidean metrics	Tom Roper
evening	Something	Dennis Morris
Sunday	Mathematical knots	
		Lynda Goldenberg, Elaine Smith, John Hoskinson

Cost: £190 to M500 members, £195 to non-members. You can obtain a booking form from the M500 site.

http://www.m500.org.uk/winter/booking.pdf

Alternatively, send a stamped addressed envelope to

Diana Maxwell.

I (TF) had to visit the Waitrose store in Kingston one day to replenish our supplies of soya milk and especially to take advantage their special offer: *Buy three cartons for the price of two.*

So I bought eight.

Fortunately the cashier knew a little number theory, and she pointed out that $8 \not\equiv 0 \pmod 3$. But there was nothing else she could do except refer me to Customer Services. There was no problem; all I had to do was claim the ninth carton for free. The man in charge was very sympathetic. "It's all right," he said, "I, too, am useless at maths."

Contents   M500 224 – October 2008

Cover: The rhombicuboctahedron graph.