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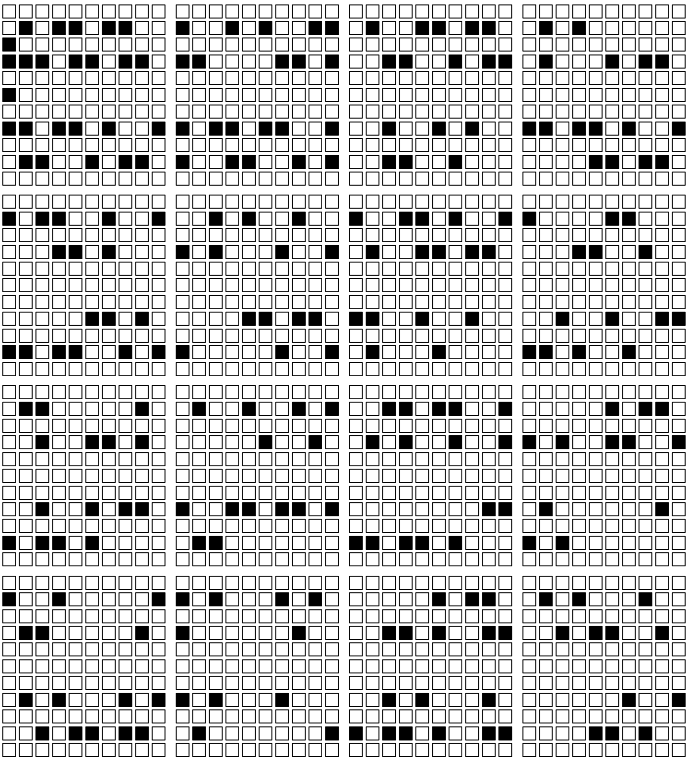
ISSN 1350-8539

M500 229



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Group cohomology: a simple set of examples

Tommy Moorhouse

Introduction

When a group acts on another, abelian, group we can often characterize the action using certain related groups of maps. In this article we will consider the multiplicative action of the group of primitive roots modulo N (which we denote by $S(N)$) on the additive group $\mathbb{Z}_N$. Actually $\mathbb{Z}_N$ has the structure of a ring, and $S(n)$ is the group of units in this ring. We will find the cohomology groups $H^0(S(n), \mathbb{Z}_n)$ and $H^1(S(n), \mathbb{Z}_n)$, to be defined below, for certain N. It is hoped that this introduction will lead the interested reader to further exploration. Some of the later material is based on exercises in [Lang].

Definitions Let $\mathbb{Z}_N$ consist of the integers 0, 1, …, $N-1$ with the operation of addition modulo N. Let $S(N)$ be the subset of positive integers in $\mathbb{Z}_N$ relatively prime to N, forming a group under multiplication. The action of $S(N)$ on $\mathbb{Z}_N$ is given by the pairing

$$(\sigma, m) \mapsto \sigma m.$$

Since $\sigma(m+m') = \sigma m + \sigma m'$, $1m = m$, and $\sigma(\theta m) = (\sigma\theta)m$ this defines a group action.

The zeroth cohomology group The cohomology group $H^0(S(N), \mathbb{Z}_N)$ is just the subgroup of $\mathbb{Z}_N$ fixed by the whole of $S(N)$. This subgroup can be found quite straightforwardly. If N is odd then $2 \in S(N)$. Since any element of $H^0(S(N), \mathbb{Z}_N)$, say m, is fixed by 2 we have $2m \equiv m \pmod N$, so $m \equiv 0 \pmod N$ and $H^0(S(N), \mathbb{Z}_N) = \{0\}$.

If N is even then all elements of $S(N)$ are odd. Consider $m = N/2$. This element is fixed by every $\sigma \in S(N)$ since

$$\sigma \frac{N}{2} \;=\; (2\sigma' + 1)\frac{N}{2} \;\equiv\; \frac{N}{2} \pmod N.$$

In this case, then, $\mathbb{Z}_2 \subset H^0(S(N), \mathbb{Z}_N)$. Now if $N/2$ is odd $S(N/2)$ has exactly the same number of elements as $S(N)$, since $\phi(2N) = \phi(N)$ for all odd N. If $\sigma m = m$ we have $\sigma' m' \equiv m' \pmod{N/2}$, where primed quantities are reduced modulo $N/2$. But, from the work above, we know that no element is fixed by every σ'.

The first cohomology group $H^1(S(N),\mathbb{Z}_N)$ is defined in terms of maps $\xi : S(N) \to \mathbb{Z}_N$. We require that these maps satisfy

$$\xi(\sigma\theta) \;=\; \sigma\xi(\theta) + \xi(\sigma). \tag{1}$$

The reason for this condition will be explored later. Note that given a fixed $m \in \mathbb{Z}_N$ the map $\sigma \mapsto \sigma m - m$ is a map from $S(N)$ to $\mathbb{Z}_N$, and this map satisfies condition (1) above. The elements of $H^1(S(N),\mathbb{Z}_N)$ are those maps satisfying condition (1) but not of this form. Now, from (1) we have

$$\xi(\sigma\tau) \;=\; \sigma\xi(\tau) + \xi(\sigma);$$

so

$$\sigma\tau\xi(\sigma\tau) \;=\; \sigma^2\tau\xi(\tau) + \tau\sigma\xi(\sigma).$$

We define $F(\sigma) = \sigma\xi(\sigma)$. Then

$$\begin{aligned} F(\sigma\tau) &= \sigma^2 F(\tau) + \tau F(\sigma) \\ &= \tau^2 F(\sigma) + \sigma F(\tau) \end{aligned}$$

by symmetry (since $S(N)$ is abelian). Thus $\sigma(\sigma-1)F(\tau) = \tau(\tau-1)F(\sigma)$. In the interesting case that τ and $\tau-1$ are both invertible we have

$$F(\sigma) \;=\; \tau^{-1}(\tau-1)^{-1}F(\tau)\sigma(\sigma-1).$$

Since the left hand side is independent of τ we must have $\tau^{-1}(\tau-1)^{-1}F(\tau) =$ a constant, say m, and so $F(\sigma) = m(\sigma-1)$, that is $\xi(\sigma) = \sigma m - m$ for all $\sigma \in S(N)$. This means that $H^1(S(N),\mathbb{Z}_N)$ is trivial.

In particular, if N is odd then both 2 and 1 are invertible in $S(N)$, so we take $\tau = 2$ to find that $H^1(S(2N+1),\mathbb{Z}_{2N+1})$ is trivial for all N.

The case N even We will make some preliminary observations and leave further exploration to the reader. Now all $\sigma \in S(N)$ are odd, say $\sigma = 2\sigma'+1$. As before

$$\begin{aligned} \sigma(\sigma-1)F(\tau) &= \tau(\tau-1)F(\sigma), \\ (2\sigma'+1)2\sigma' F(\tau) &= (2\tau'+1)2\tau' F(\sigma), \\ \sigma'\sigma F(\tau) &= \tau'\tau F(\sigma). \end{aligned}$$

Again, if τ and τ' are both invertible, we can argue as in the case of odd N. We find that

$$\xi(\sigma) \;=\; \sigma' m \;=\; \frac{1}{2}(\sigma-1)m,$$

where m is the constant $\tau\tau' F(\tau)$. Clearly if m is odd there is no element of $\mathbb{Z}_N$ such that ξ is of the form $\sigma m - m$ for all σ. Thus $H^1(S(N), \mathbb{Z}_N)$ is not necessarily trivial. On the other hand, if m is even then $H^1(S(N), \mathbb{Z}_N)$ is trivial. We would therefore need to determine when there exist τ and τ' both invertible, and find the value of $\tau^{-1}\tau'^{-1}F(\tau)$ in each case.

Where do the formulae come from? Cohomology is an essential tool in many branches of mathematics. One area with which some readers may be familiar is the application to surfaces (manifolds) in differential geometry. There we have vector spaces of differential forms (0-forms, 1-forms etc.) and a map d (actually a collection of maps, one for each p) from p-forms to $(p+1)$-forms, the exterior derivative. It is shown that $d^2 = 0$ (i.e. that $d(d\omega) = 0$ for any form ω), and this leads to the study of those forms for which $d\omega = 0$ but $\omega \neq d\eta$ for any form η. This is the basis of the study of the structure of the surface in terms of cohomology.

In the case of group actions of a group G on a set M the theory is built up in exactly the same way. We have a collection of spaces $\{E_i\}$ analogous to the spaces of i-forms with maps between them, $d_i : E_i \to E_{i-1}$ such that $d_{i-1} \circ d_i = 0$. Here we choose our spaces E_i to be the sets of $(i+1)$-tuples of elements of G:

$$E_i \;=\; \{(x_0, x_1, \cdots, x_i)\}$$

with G acting through $g(x_0, x_1, \cdots, x_i) = (gx_0, gx_1, \cdots, gx_i)$. Here the tuples are to be viewed as objects in their own right, and cannot be added component-wise, although we can add n-tuples together to get another n-tuple (although we don't have an explicit formula). Our map d involves the operation of addition:

$$d_i(x_0, x_1, \cdots, x_i) \;=\; \sum_{j=0}^{i} (-1)^j (x_0, x_1, \cdots, \hat{x}_j, \cdots, x_i)$$

where the hat indicates that the element is omitted from the tuple. This is a sum of $(i-1)$-tuples.

We now consider the set $\mathrm{Hom}_G(E_i, M)$ of maps (homomorphisms) $f : E_i \to M$ satisfying $f(gx) = gf(x)$ for $g \in G, x \in E_i$. Given $f : E_i \to M$ we define $\delta^i f = f \circ d_{i+1}$, which is a map from E_{i+1} to M. That is, $\delta^i f \in \mathrm{Hom}_G(E_{i+1}, M)$. We see that δ works 'in the opposite direction' to d. It is straightforward to show that $\delta^{i+1} \circ \delta^i = 0$, and we build our cohomology groups from the complex

$$\cdots \to \mathrm{Hom}_G(E_i, M) \to \mathrm{Hom}_G(E_{i+1}, M) \to \cdots .$$

Now consider $H^0(G,M)$, the kernel of δ^0. This is the set of maps from E_0 into M, where E_0 is the set of 1-tuples (x) where $x \in G$, with the action of G on E_0 $x(x_0) = (xx_0)$. Now $\delta^0 f$ is a map from E_1 into M given by

$$\delta^0 f((x_0,x_1)) = f(d_1(x_0,x_1)) = f((x_1)-(x_0)) = f((x_1)) - f((x_0))$$

from the definition $d_1(x_0,x_1) = (x_1)-(x_0)$ and the fact that f is a homomorphism. But there is an element x of G such that $x_1 = xx_0$ (since G is a group) and so, by the group action,

$$0 \;=\; \delta^0 f((x_0,x_1)) = xf((x_0)) - f((x_0)).$$

This holds for any x_1 and, considering f as a map from G into M (via the isomorphism $(x) \to x$), we see that the first cohomology group $H^0(G,M)$ corresponds to the set of elements $m = f((x))$ of M fixed by G (sometimes denoted M^G).

To find $H^1(G,M)$ we consider the kernel of δ^1 modulo the image of δ^0. Here $f: E_1 \to M$, where E_1 is the collection of 2-tuples (x_0,x_1) and

$$\delta^1 f((x_0,x_1,x_2)) \;=\; f((x_1,x_2)) - f((x_0,x_2)) + f((x_0,x_1)).$$

Thus

$$\begin{aligned}
f((x_0,x_2)) &= f((x_1,x_2)) + f((x_0,x_1)) \\
f((1,x_0^{-1}x_2)) &= f((x_0^{-1}x_1, x_0^{-1}x_2)) + f(1,x_0^{-1}x_1)) \\
&= f((1,x_0^{-1}x_1)) + x_0^{-1}x_1 f((1,x_2x_1^{-1}))
\end{aligned}$$

making use of the group action. Setting $x = x_0^{-1}x_1, y = x_1^{-1}x_2$ and $g(x) = f((1,x))$ we can rewrite this result as $g(xy) = g(x) + xg(y)$. This is the condition we used above.

References and useful books

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1998.

[2] S. Lang, *Algebra*, Springer, 2002.

Problem 229.1 – Red and yellow vertices

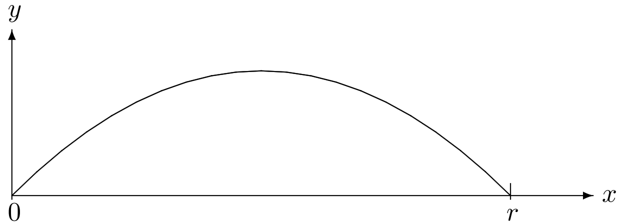
Each vertex of a simple graph has odd degree. At time 0 you colour each vertex either red or yellow. At time $t+1$ each vertex of the graph adopts the colour of the majority of its neighbours as they were at time t. Show that the system either stabilizes or goes into a 2-cycle.

[Thanks to **Emil Vaughan** for this.]

Solution 223.2 – Gun

> If a gun has a maximum range of r on a level plain, what is it from the top of a cliff of height h? Find a construction for θ, the angle of elevation.

Steve Moon



On level ground, fire the projectile with initial velocity u at angle θ to the horizontal. It strikes the ground again at distance r. In the vertical (y) direction, the flight time t is obtained from $0 = u(\sin\theta)t - \frac{1}{2}gt^2$, giving

$$t = \frac{2u\sin\theta}{g}.$$

In the x direction,

$$x = u(\cos\theta)t = \frac{2u^2(\sin\theta)(\cos\theta)}{g} = \frac{u^2\sin 2\theta}{g}.$$

For maximum x,

$$\frac{dx}{d\theta} = 0 = \frac{2u^2}{g}\cos 2\theta \;\Rightarrow\; \theta = \frac{\pi}{4}.$$

Hence the maximum range is

$$r = \frac{u^2}{g}. \tag{1}$$

Now consider the projectile fired from height h at time 0, again with initial velocity u. Assume the projectile lands on the ground at time t.

Its vertical motion is given by $-h = u(\sin\theta)t - \frac{1}{2}gt^2$. Its horizontal motion is given by $x = u(\cos\theta)t$. Eliminating t, we obtain

$$-\frac{2hu^2}{g} = \frac{2u^2}{g}x\tan\theta - x^2\sec^2\theta. \tag{2}$$

Substituting (1) into (2) gives

$$-2hr = 2rx\tan\theta - x^2\sec^2\theta.$$

Hence

$$x^2 - rx\sin 2\theta - hr(\cos 2\theta + 1) = 0. \tag{3}$$

Let the maximum range be R. When $x = R$, $dx/d\theta = 0$. Differentiating (3) with respect to θ,

$$2x\frac{dx}{d\theta} - r\frac{dx}{d\theta}\sin 2\theta - 2rx\cos 2\theta + 2hr\sin 2\theta = 0.$$

But when $x = R$, $dx/d\theta = 0$; so

$$-2rR\cos 2\theta + 2hr\sin 2\theta = 0.$$

Therefore

$$\tan 2\theta = \frac{R}{h}, \qquad \cos 2\theta = \frac{h}{\sqrt{h^2+R^2}}, \qquad \sin 2\theta = \frac{R}{\sqrt{h^2+R^2}}.$$

Substituting into (3) with $x = R$, we have

$$R^2 - \frac{rR^2}{\sqrt{h^2+R^2}} - hr\left(\frac{h}{\sqrt{h^2+R^2}} + 1\right),$$

which simplifies to

$$R^2\sqrt{h^2+R^2} - hr\sqrt{h^2+R^2} - r(h^2+R^2) = 0.$$

Therefore $R^2 - hr = r\sqrt{h^2+R^2}$, and on squaring,

$$R^4 - 2hrR^2 + h^2r^2 = r^2(h^2+R^2).$$

Hence

$$R = \sqrt{r(2h+r)}.$$

Norman Graham

Given that $\tan 2\theta = R/h$ and $R = \sqrt{r(2h+r)}$ [from Steve's solution, above], we can compute $\tan\theta$ using the double-angle formula:

$$\tan 2\theta \;=\; \frac{2\tan\theta}{1-\tan^2\theta} \;=\; \frac{R}{h},$$

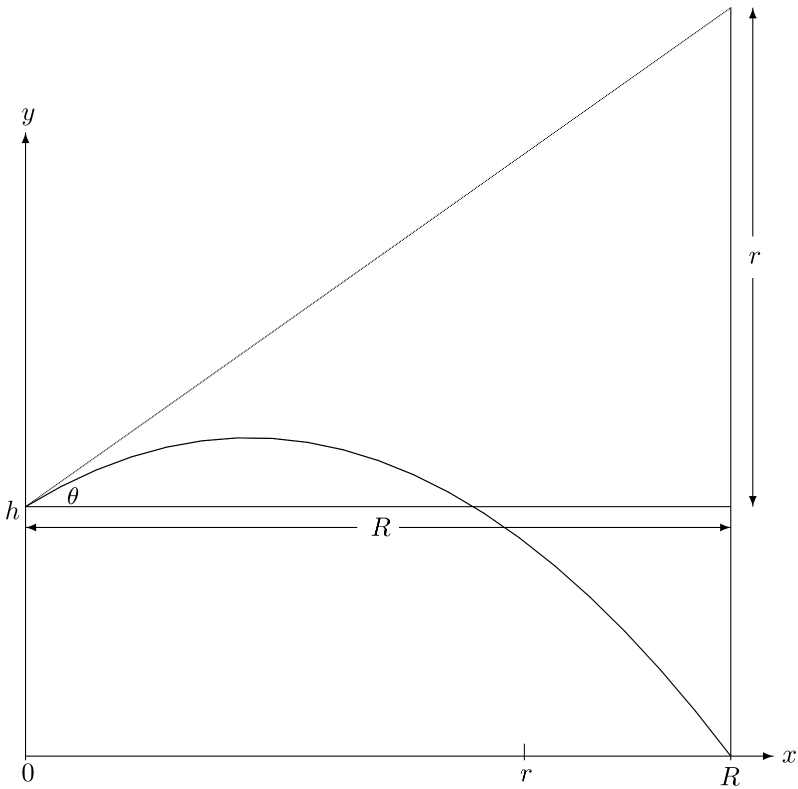
or

$$R\tan^2\theta + 2h\tan\theta - R \;=\; 0.$$

Solving yields

$$\tan\theta \;=\; \frac{-2h+\sqrt{4h^2+4R^2}}{2R} \;=\; \frac{-h+\sqrt{h^2+2hr+r^2}}{R} \;=\; \frac{r}{R}.$$

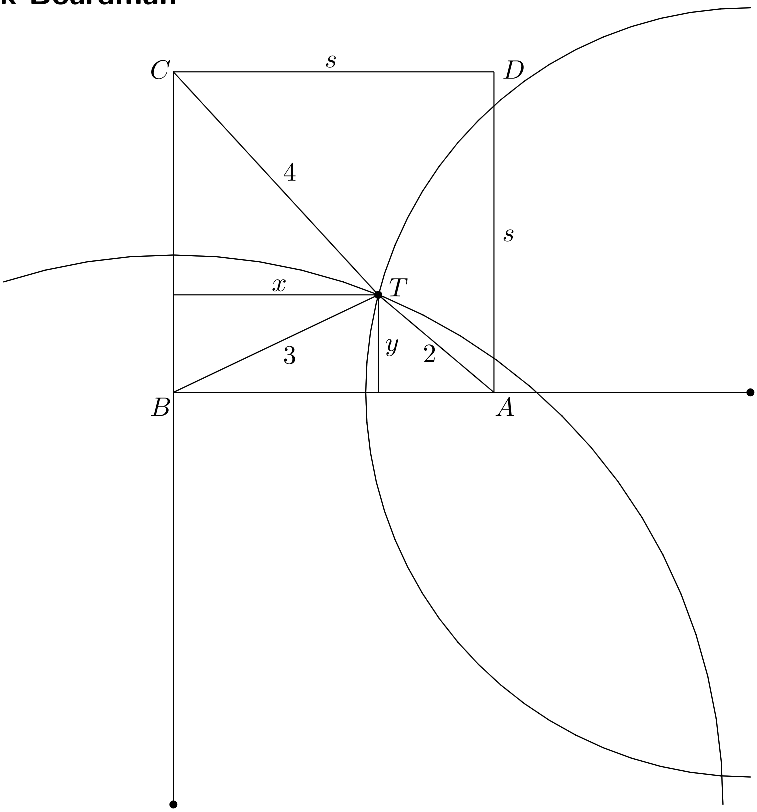
Hence the angle of elevation has the very simple construction indicated in the diagram.



Solution 224.2 – Buried treasure

> (i) Treasure has been buried at T, distances $a = 2$, $b = 3$ and $c = 4$ from three successive corners of a square field. Determine s, the length of a side of the field. (This is No. 66 of *The Canterbury Puzzles* by H. E. Dudeney.) For $a = 2$ and $c = 4$, find the range of values of b if T is (ii) anywhere, or (iii) somewhere within the field. (iv) Devise geometrical constructions to obtain the answers.

Dick Boardman



Let the corners of the square be A, B, C, D. Let the distance from T to BC be x. Let the distance from T to AB be y and let s be the side of the square. Then

$$\begin{aligned} x^2 + y^2 &= 9, \\ (s - x)^2 + y^2 &= 4, \\ (s - y)^2 + x^2 &= 16. \end{aligned}$$

On eliminating x and y we obtain

$$37 - 20s^2 + s^4 = 0.$$

Hence

$$s = \sqrt{10 + 3\sqrt{7}} \approx 4.23524, \qquad x \approx 2.70791, \qquad y \approx 1.29122$$

and

$$s = \sqrt{10 - 3\sqrt{7}} \approx 1.43623, \qquad x \approx 2.45879, \qquad y \approx -1.71883$$

are the solutions for T inside and outside the square. Graphically the problem is solved by dividing all of the lengths by s and using the following lemma, which is proved at the end.

Lemma *Given two fixed points A and B and a variable point P such that AP/BP, call it t, is constant. The locus of P is a circle whose centre C is on AB such that $AC/AB = t^2/(t^2 - 1)$ and whose radius is $t/(t^2 - 1)$.*

For the construction, we construct a figure with a unit square and a point T (the treasure) so that $AT = 2k$, $BT = 3k$ and $CT = 4k$ and then scale the figure to find the required side of square.

Applying the lemma to points A and B we get that the locus of T is a circle with centre on AB. In this case $t = 3/2$; so we need lengths of $9/5$ for the centre and $6/5$ for the radius. These are easily constructed using ruler and compasses. Applying the lemma to points C and B we get that the locus of T is a circle with centre on BC. Now $t = 4/3$, so that we need lengths of $12/7$ for the radius and $16/7$ to find the centre. Again, these are easily constructed.

These two circles cut at two points, one inside and one outside the square, and these are the solutions for T. Re-scaling this diagram gives the side of the square to be $\sqrt{10 + 3\sqrt{7}}$ for the point inside the square and $\sqrt{10 - 3\sqrt{7}}$ for the point outside the square.

Next we consider the maximum and minimum values for TB while TA and TC remain at 4 and 2. Applying the lemma to points A and C, the locus of T is a circle whose centre is on AC extended and the ratio is 2. This circle cuts the sides BC and CD. Within the square, the treasure lies on this arc, so that the maximum TB is where it cuts CD and the minimum where it cuts BC.

If T is on BC then the triangle TCA has sides $4/s$, $2/s$ and $\sqrt{2}$ and an angle of 45 degrees at C. Applying the cosine rule gives $s = 1 + \sqrt{7}$. Thus the minimum value for TB is $\sqrt{7} - 1$.

When T is on CD, the triangle TCA has the same sides and angles so that s is also $1 + \sqrt{7}$. In the triangle BCT, $BC = s$ and $CT = 2$ and there is a right angle at C, hence the maximum of BT is $\sqrt{s^2 + 4}$.

In order to find the maximum and minimum values for TB outside the square, we need to return to the original equations (1)–(3) to replace the right hand side of (1) with $(TB)^2$. If we consider a 3-dimensional space with x, y and s axes, then each of these equations is a cylinder and the points we require are the intersections of these cylinders. The intersection of (2) and (3) is a solid. A point on the surface of the solid is a solution to one equation, a point on the edge of the solid is a solution to two equations. Suppose we use a value for TB less than the minimum. This will not intersect any of the edges and hence there is no real solution. A hole has been cut through the solid, corresponding to such a cylinder. We now increase the value of TB until it just touches one of the edges. There will now be two real solutions and they will be the same. We now increase the value of TB and the cylinder will cut the edges twice, giving two real solutions. If we further increase the value of TB, the two solutions will converge until they coincide again. At this point, the cylinder will completely enclose the solid. This will be the maximum value. These values occur where $s = \sqrt{10}$ and the maximum and minimum values for TB are $\sqrt{18}$ and $\sqrt{2}$.

Proof of the lemma Let $A = (0,0)$, $B = (1,0)$ and $P = (x,y)$; AB is on the x-axis. Let $AP/BP = t$. Now $(AP)^2 = x^2 + y^2$ and $(BP)^2 = (1-x)^2 + y^2$. Hence

$$\frac{x^2 + y^2}{(1-x)^2 + y^2} = t^2,$$

$$x^2 + y^2 - \frac{2xt^2}{t^2 - 1} + \frac{t^2}{t^2 - 1} = 0.$$

This is the equation of a circle whose centre is on the x-axis such that $AC/AB = t^2/(t^2 - 1)$ and whose radius is $t(AB)/(t^2 - 1)$. QED.

Also solved by **Norman Graham**, **Tony Moulder**, **Chris Pile** and **Steve Moon**.

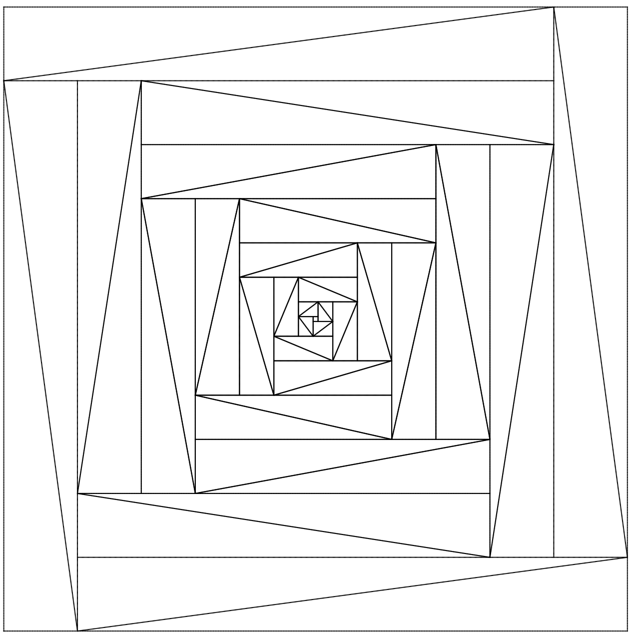
Pythagorean squares

Chris Pile

As a postscript to Solution 224.2 – Buried treasure, even though it has nothing to do with treasure, buried or otherwise, except that the numbers 3 and 4 form the two short sides of a Pythagorean triangle, I offer this construction.

The triangles are created as $(a+b)^2 - 4ab = (b-a)^2$ to give an infinite sequence of squares. The small square in the centre has side 1.

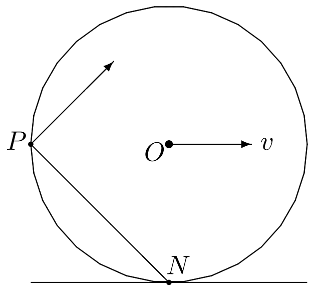
$7 = 4 + 3$	$7^2 - 4 \cdot 4 \cdot 3 = (4-3)^2 = 1^2$	$4^2 + 3^2 = 5^2$
$17 = 12 + 5$	$17^2 - 4 \cdot 12 \cdot 5 = (12-5)^2 = 7^2$	$12^2 + 5^2 = 13^2$
$31 = 24 + 7$	$31^2 - 4 \cdot 24 \cdot 7 = (24-7)^2 = 17^2$	$24^2 + 7^2 = 25^2$
$49 = 40 + 9$	$49^2 - 4 \cdot 40 \cdot 9 = (40-9)^2 = 31^2$	$40^2 + 9^2 = 41^2$
$71 = 60 + 11$	$71^2 - 4 \cdot 60 \cdot 11 = (60-11)^2 = 49^2$	$60^2 + 11^2 = 61^2$
$97 = 84 + 13$	$97^2 - 4 \cdot 84 \cdot 13 = (84-13)^2 = 71^2$	$84^2 + 13^2 = 85^2$
$\dots$	$\dots$	$\dots$



Solution 223.2 – Mud

Mud flies off the hindmost point of a wheel rolling at a uniform speed. Will it hit the wheel again if it leaves the wheel in the same direction as the hindmost point (a) at the same speed, (b) slower, (c) faster? As usual, all this takes place in a vacuum.

Norman Graham



(a) Let a be the radius of the wheel and v the speed of its centre, O. Then the angular velocity of O about N, the point of contact with the ground, is $\omega = v/a$. If P is the hindmost point of the wheel, then $PN = a\sqrt{2}$, the velocity of P is $a\sqrt{2}$ and its angular velocity about N is $v\sqrt{2}$.

Since PN is at 45° to the horizontal, the mud has components of velocity v horizontally and v vertically. hence the path of the mud will be a parabola with constant horizontal velocity v. This is the same as that of the hindmost point of the wheel. Therefore the mud will always be directly above the hindmost point, which it will strike again!

(b) The horizontal component of the mud's velocity will be less than that of the hindmost point, so it will drop behind the wheel.

(c) Let v_1 $(> v)$ be the initial vertical component of the mud's velocity. After time t the height above level of O is $v_1 t - \frac{1}{2}gt^2$. This is zero when $t = 2v/g$. The distance the mud travels to a point at the level of O is $2v_1^2/g$. Hence the mud will strike the front of the wheel if $\dfrac{2v_1^2}{g} = 2a + \dfrac{2v_1 v}{g}$; that is, when $v_1 = \frac{1}{2}v + \sqrt{(\frac{1}{2}v)^2 + ag}$. If v_1 is less than this critical value, the mud will strike the wheel again, but if it is greater, the mud will fall in front of the wheel.

Problem 229.2 – Tank

I am driving a tank and I have to make a circular tour of various military bases along a given route. I can arrange to have my tank transported to a starting point of my choice. Initially my (fuel) tank is empty, but distributed along the route there is sufficient fuel to complete my tour. Show that I can choose my starting point so that I can complete the whole journey and return to the waiting tank-transporter without running out of fuel.

Problem 229.3 – Harmonic triangle

Norman Graham

Behold an array of fractions.

$$
\begin{array}{ccccccc}
 & & & \frac{1}{1} & & & \\
 & & \frac{1}{2} & & \frac{1}{2} & & \\
 & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & \\
\frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} \\
 & & & \cdots & & &
\end{array}
$$

The first fraction in row n is $\dfrac{1}{F(n,1)} = \dfrac{1}{n}$, and the rth fraction is

$$\frac{1}{F(n,r)} = \frac{1}{F(n-1,r-1)} - \frac{1}{F(n,r-1)}$$

for $r = 1, 2, \ldots, n$. Find a general formula for $F(n,r)$. Hence show that each row is symmetrical about the centre; i.e. $F(n,r) = F(n, n-r+1)$.

Problem 229.4 – Balls

There are bn balls, n each of b different colours. They are arranged in a line in b blocks of n.

$$\odot\odot\odot\odot\odot\odot\otimes\otimes\otimes\otimes\otimes\otimes\oplus\oplus\oplus\oplus\oplus\oplus$$

A *move* is to take a ball from the line, place it somewhere else in the line and close up the gap.

$$\odot\odot\otimes\odot\odot\odot\odot\otimes\otimes\otimes\otimes\otimes\oplus\oplus\oplus\oplus\oplus\oplus$$

What is the minimum number of moves necessary to create a line with no two adjacent balls having the same colour? Prove that the answer is $\geq \frac{1}{2}b(n-1)$. So that in the example, where $b = 3$ and $n = 6$, we are asserting that at least 7.5 moves are required. See if you can do it in eight.

Erratum. The first display on page 4 of M500 **228** should read

$$(L_g + L_{g'})(1) \;=\; L_g(0) + L_{g'}(0) \;=\; 0 + 0 \;=\; 0,$$

not $(L_g + L_{g'})(0) \ldots$.

Professor Pile's prime pathway

Chris Pile

When I last visited the eponymous self-styled professor I found him supervising the building of a pathway around his country estate. The pathway was about 1m wide and already more than 400m long. The professor was beside a large stack of buff-yellow tiles and a smaller stack of orange terracotta tiles. Each tile was 10cm square and the path was 11 tiles wide, constructed mainly of yellow. The professor explained that the rows of tiles were effectively digit positions from 0 to 9 with an extra position 10 (duplicating 0) to give a symmetrical edge along the path. All the orange tiles that I could see were in two lines along the path in positions 1, 3, 7 and 9, but there were many gaps between them. He explained that every tile represented a number, N, from the start of the path in rows, and whenever N was prime it was an orange tile. Every ten rows there was a wider joint between the tiles to make a 1m 'century block'.

The first century block was exceptional in that there was an orange tile in positions 2 and 5 of the first row, but thereafter these positions were yellow. The first tile was yellow because 1 is not considered a prime. I strolled along the path, noting that after the first block there were never more than two orange tiles together and the number of orange tiles in each block tended to decrease. There were some centuries that had no orange tiles in one of the lines. Every century block appeared to be different and I wondered whether the pattern within a block would ever be repeated. There was one block with only four orange tiles but I could not find any block totally devoid of them. I asked the professor if that would occur. "Not for about 4km," he replied. I could see that the path had a long way to go and the professor was keen to progress the task.

(1) When will the pattern within a century block be repeated?

(2) When will a century block consist of all yellow tiles?

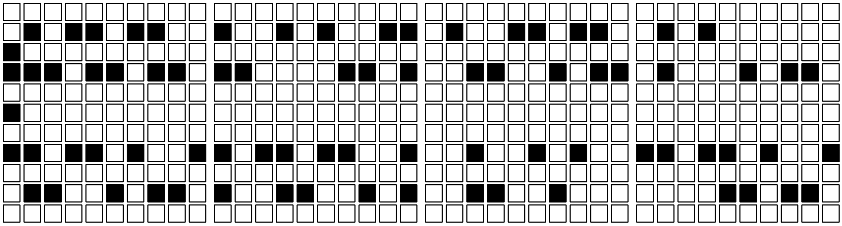
(3) Is there any patterned block that is symmetrical about the 5-line?

(4) Is there a block which is the same when rotated 180°?

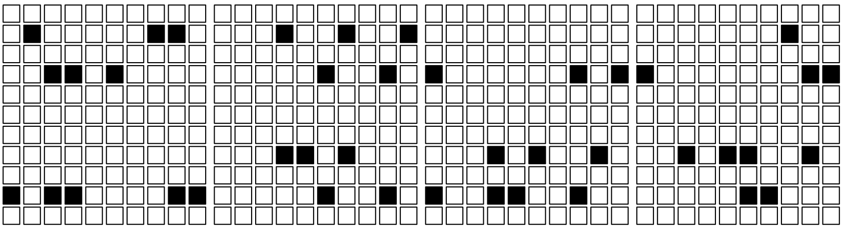
(5) Apart from the first two, are there any further blocks with more than 17 orange tiles?

(6) Apart from the first, which is in some sense rather special, what is the maximum possible number of orange tiles that can appear in a century block?

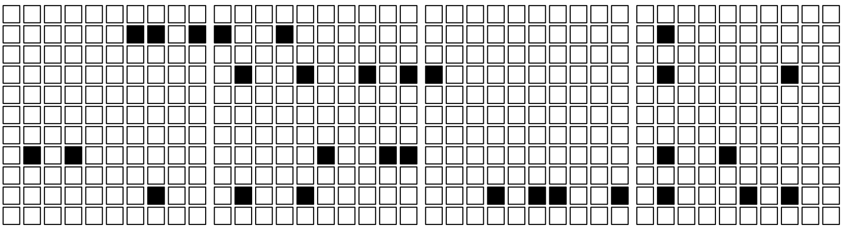
The start of the 'prime pathway'. Century-blocks 1–4:



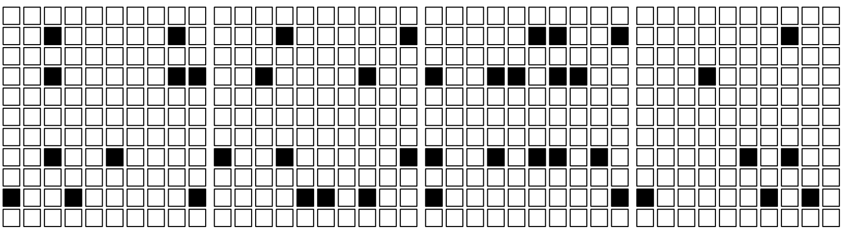
A paucity of primes after 107m. Centuries with no prime ending in 7 and no prime ending in 1. Blocks 108–111:



Relatively sparse centuries with only five or six primes. Blocks 187–190:



Relatively dense region after 400m. Blocks 423–426:



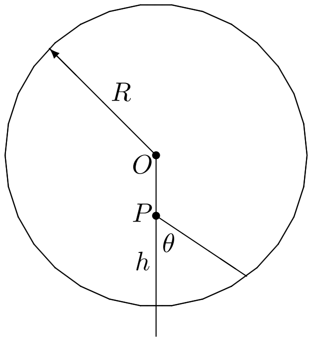
Problem 229.5 – Red and yellow points

Given $2n$ points in the plane, no three collinear, n red and n yellow. Can you connect the reds to the yellows in pairs by non-crossing straight lines?

Solution 225.1 – Toroidal planet

> There is a planet which has the shape of a (solid) torus. You are standing somewhere on its innermost circle. Depending on the parameters of the torus, do you stay attached to the ground, or do you drift upwards, attracted towards the rest of the planet arched out above you?

Tommy Moorhouse



My initial expectation of how gravity behaves in the toroidal system was that gravity would be nullified in the central plane as it is inside a uniform hollow sphere. However, a little thought shows that this is not the case. Gravity obeys the inverse square law in the sense that two masses separated by a distance x experience an attractive force of

$$F(x) \;=\; -\,\frac{GMm}{x^2}.$$

In the interior of a uniform hollow sphere the gravitational attraction of any part of the sphere wall upon a mass is exactly balanced by the attraction from the parts of the shell on the opposite side. This involves areas of the internal surface, and this introduces a quantity of dimension $[\text{length}]^2$, essentially cancelling the $1/x^2$ factor in the force law. In the case of the torus there are no areas involved, and the masses on opposite sides of the ring can only introduce a factor of dimension [length], which cannot cancel the $1/x^2$ dependence.

It is not too hard to carry out the complete calculation by integration. We make the simplifying assumption that all the mass of the torus can be considered to lie in a ring of radius R defining the axis of the torus. This can be rigorously justified, but for our purposes it is a convenient modelling assumption. We choose the line from the centre of the circle through the particle to be the vertical. The set-up shown in the diagram makes it clear that only the net vertical force is required, because there is no lateral force, by symmetry. The first step is to calculate how the distance from an internal point to the ring varies with the angle θ made by the line joining the point to the ring and the line joining the point to the centre. This is just plane geometry. The result is

$$l(\theta) \;=\; (h-R)\cos\theta + \sqrt{R^2\cos^2\theta + (2hR-h^2)\sin^2\theta}$$

and the mass of the ring element is $\mu l(\theta)d\theta$, where μ is the mass per unit length of the ring. Next the force on the particle due to the mass of the ring element is calculated (it is actually easier to calculate the net force due to diametrically opposite elements). This gives (wrapping up all the constants G, μ, etc. into K)

$$\frac{K}{l(\theta)} - \frac{K}{l(\theta+\pi)} \;=\; \frac{2K(R-h)\cos\theta}{h(2R-h)}.$$

Projecting onto the vertical (i.e. multiplying by $\cos\theta$) and integrating from 0 to π one finds that inside the torus, in the same plane, the force on a particle is given by

$$\vec{F}(\vec{r}) \;=\; \frac{K\vec{r}}{R^2-r^2}.$$

This is a purely radial force which is zero at the centre of the circle defined by the torus axis and increases steadily towards the ring. Objects in the ring plane within the ring will fall towards the torus.

In the ring plane but outside the ring the calculation is similar, but the θ integration runs from 0 to $\theta_M = \arcsin(R/r)$ and the result is

$$\vec{F}(\vec{r}) \;=\; \frac{K\vec{r}}{R^2-r^2}\left\{\arcsin\left(\frac{R}{r}\right) + \frac{R}{r^2}\sqrt{r^2-R^2}\right\}.$$

Note that since $r > R$ the sign of the force shows that it is attractive, and objects will fall towards the ring. As $r \to \infty$ this reduces to

$$\vec{F}(\vec{r}) \;=\; -\frac{K\vec{r}}{r^3},$$

which is the usual asymptotic result that a body exerts a gravitational force as if it were a point of the same mass situated at its centre of mass.

The force has no angular component; so objects moving in orbits will tend to spiral down towards the ring. Meteor showers would be quite interesting events on the toroidal planet!

'Twenty per cent of people are habitually late. If there are ten people in a room, two of them won't be there yet.' — Man on R4 [sent by **JRH**]

Solution 226.2 – Eight sins

Show that

$$\sin^4\frac{\pi}{20}+\sin^4\frac{3\pi}{20}+\sin^4\frac{7\pi}{20}+\sin^4\frac{9\pi}{20}+\sin^4\frac{11\pi}{20}+\sin^4\frac{13\pi}{20}+\sin^4\frac{17\pi}{20}+\sin^4\frac{19\pi}{20} = \frac{13}{4}.$$

Stuart Walmsley

The strategy is to express the left-hand side of the basic expression in terms of $\cos 2\pi/5 = \cos 8\pi/20$ and $\cos 4\pi/5$, whose values are known:

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}, \qquad \cos\frac{4\pi}{5} = -\frac{\sqrt{5}+1}{4}.$$

Since $\sin^4 j\pi/20 = \sin^4(20-j)\pi/20$, the left-hand side reduces to

$$2\left(\sin^4\frac{\pi}{20}+\sin^4\frac{3\pi}{20}+\sin^4\frac{7\pi}{20}+\sin^4\frac{9\pi}{20}\right).$$

Since $\sin^4 j\pi/20 = \cos^4(10-j)\pi/20$, it becomes

$$2\left(\sin^4\frac{\pi}{20}+\sin^4\frac{3\pi}{20}+\cos^4\frac{3\pi}{20}+\cos^4\frac{\pi}{20}\right).$$

But

$$\sin^4 x + \cos^4 x + \sin^2 2x = \sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x = 1,$$

so that the left-hand side becomes

$$\begin{aligned} 4 - \sin^2 2\pi/20 - \sin^2 6\pi/20 &= 4 - \cos^2 2\pi/5 - \cos^2 4\pi/5 \\ &= 4 - \left(\frac{\sqrt{5}-1}{4}\right)^2 - \left(\frac{\sqrt{5}+1}{4}\right)^2 = \frac{13}{4} \end{aligned}$$

and the result is proved.

Also solved by **Steve Moon**

Problem 229.6 – 50 coins

Fifty 50-pence pieces lie on a circular table-top with no overlap and maximally, in the sense that there is no room for a 51st coin. Show that 200 coins will cover the table.

Algebra for beginners

Chris Pile

(1) Solve each of the following equations for x.

$$
\begin{array}{rrcl}
\text{(i)} & \mathrm{m}x\mathrm{y} & = & \text{cash.} \\
\text{(ii)} & x\mathrm{day} & = & \text{now.} \\
\text{(iii)} & \mathrm{fr}x & = & \text{cargo.} \\
\text{(iv)} & \mathrm{ca}x & = & \text{dog.} \\
\text{(v)} & \mathrm{kit}x & = & \text{cat.} \\
\text{(vi)} & \mathrm{ba}x & = & \text{fundamentals.} \\
\text{(vii)} & x\mathrm{t} & = & \text{jumbo.} \\
\text{(viii)} & x'\mathrm{t} & = & \text{does not.} \\
\text{(ix)} & x\mathrm{d} & = & \text{strength.} \\
\text{(x)} & \mathrm{w}x & = & \text{heavy.} \\
\text{(xi)} & x\mathrm{ous} & = & \text{devout.} \\
\text{(xii)} & \mathrm{f}x & = & \text{rhubarb tart.}
\end{array}
$$

(2) Solve each of the following equations for roman numeral r.

$$
\begin{array}{rrcl}
\text{(i)} & r & = & \mathrm{F}r\mathrm{E} - \mathrm{I}. \\
\text{(ii)} & r - 1 & = & \mathrm{ELE}r\mathrm{EN} - \mathrm{SE}r\mathrm{EN}. \\
\text{(iii)} & r & = & \mathrm{S}r + \mathrm{III}. \\
\text{(iv)} & r & = & \mathrm{si}r + \mathrm{iv}. \\
\text{(v)} & \mathrm{e}r\mathrm{t} & = & \text{way out.}
\end{array}
$$

ANSWERS. In the interests of neatness they have been sorted into numerical order.

1, $\sqrt{\pi}$, 2, π, IV, V, 6, 8, 9, IX, 10, x, 11, xi, 12, 42, 80.

Concerning the Golden Ratio of Fibonacci numbers, $\phi = \frac{1}{2}(\sqrt{5}+1)$ $= 1.6180339887498948482\ldots$. As is well known, we have

$$\cos\frac{\pi}{5} = \frac{\phi}{2}, \qquad \sin\frac{\pi}{5} = \frac{\sqrt{3-\phi}}{2}, \qquad e^{i\theta} = \cos\theta + i\sin\theta.$$

Combining these gives

$$e^{i\pi/5} = \frac{\phi}{2} + i\,\frac{\sqrt{3-\phi}}{2}.$$

Is this useful?

Dennis Morris

Magic moments

Tony Huntington

Once upon a time, a long time ago, there was a magic clock. I know that it was a magic clock as there was a sign next to it in the watchmaker's shop window which read: **Magic Clock**. As befitted its name, it was a strange looking device ... it had but a single hand. No face, no case, no mechanism, no pendulum; nothing but a solitary hand mounted on a central pivot. Slowly the hand rotated as its point showed the passing of the hours.

Of course, as befits a magic clock, this was no ordinary hand. It was, maybe 6 inches long (I said this was a long time ago—it was in my childhood many years BM[1] and had a cross-section that was perhaps $\frac{3}{4}$ of an inch square. As I have previously remarked, the end which indicated the hours came to a point. The other end remained a mere $\frac{3}{4}$ of an inch thick, but blossomed out into a squat cylinder of perhaps 2 inches diameter. The whole was covered with mystical, silvery tracery patterns wherein, for all I know, maybe the magic itself dwelt.

That such a clock could move at all (given that it had no apparent means of motivation) may seem remarkable enough, but there is still more wonder and magic to relate. As I have said, the hand was mounted on a central pivot, but what I have omitted so far to mention was that the hand was freely pivoted. Just a gentle push from an index finger would set it spinning around and around. And yet ... no matter how often, or how hard, or in which direction it was spun, it always eventually settled down to point to the correct time again and resume its slow, patient rotation.

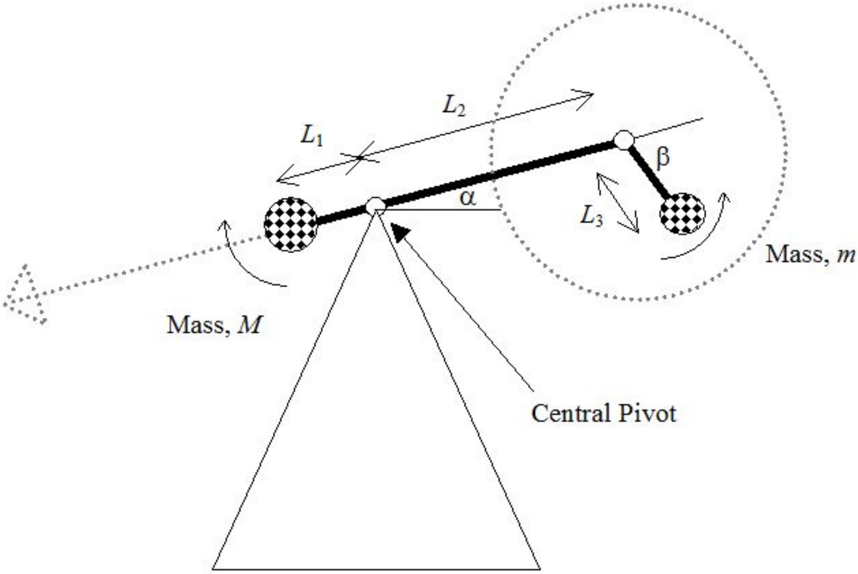
In later life as the innocence of childhood faded, I devised an explanation of what I had witnessed (although I was never able to check it with the clockmaker). My explanation (you may well have thought of others involving Oofle Dust, phases of the moon, or auras from pyramids that are equally valid) is something like this.

First, imagine that the hand is pivoted exactly at its centre of gravity. In such a circumstance, the slightest touch would set it spinning until air resistance and friction in the bearings of the pivot eventually brought it to a halt, but in a random orientation.

Now consider what might be inside the cylindrical blunt end. Suppose there was something like a watch mechanism (I am talking about a proper, wind-up watch mechanism—a digital watch won't do the trick) with just an hour hand, and there was a small weight on the end of the hour hand. This small weight is enough to perturb the centre of gravity of the hand away from the central pivot. If the hand is set spinning, then eventually it will settle down with its centre of gravity directly below the pivot.

[1]BM: Before Metrication

Now I ask you to accept my proposed mechanism, and your quest is to determine under what circumstances this mechanism could keep good time. To help (???) you in your quest, consider that the clock is actually made out of weightless beams whose mass is concentrated at single points. The mechanism then reduces to something like this.



Assuming that mass, m, moves with a uniform angular velocity relative to its pivot point, what is the relationship between L_1, L_2, L_3, M, m, a, and b which results in beam (L_1+L_2) also revolving with a uniform angular velocity (or the best approximation to uniform that can be achieved)?

Tony Forbes writes—Some time ago, before Kirsty Young started running things, I found myself listening to *Desert Island Discs* on Radio 4. On the island was Nobel prize-winning physicist Sir Peter Mansfield, FRS, famous for his ground-breaking work on imaging by nuclear magnetic resonance. After six classical pieces his seventh choice of music was to be: “A popular song from the 1950s ....” As soon as those words were uttered I made a guess at the title and, to the astonishment of all those present (including myself), I got it right! However, Sue Lawley and Sir Peter decided not to explore the technical details of magnetic moments, the obvious link between nuclear magnetism and Perry Como singing *Magic Moments*.

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