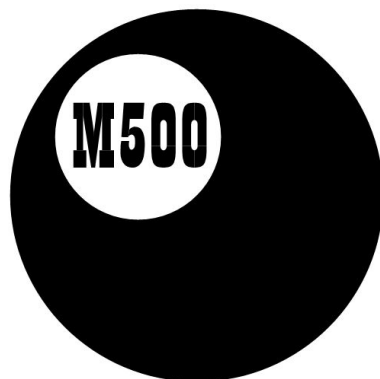
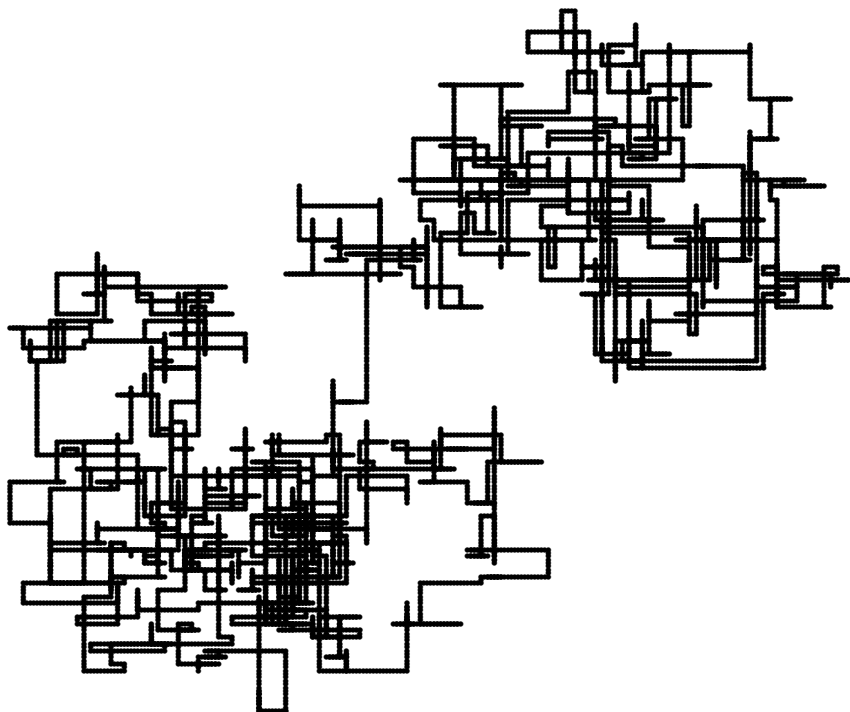


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M500 230



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

Editor – *Tony Forbes*

Editorial Board – *Eddie Kent*

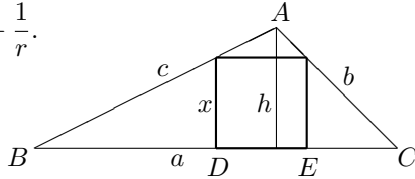
Editorial Board – *Jeremy Humphries*

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file to tony@m500.org.uk.

Solution 226.4 – Three squares

Let \mathcal{T} be a triangle with sides a, b, c and in-circle radius r . Let x be the side of the square such that (i) one side of the square shares a common border with side a of \mathcal{T} , (ii) the other two vertices of the square lies on sides b and c of \mathcal{T} . Define y and z similarly in terms of sides b and c respectively. Show that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{r}.$$



Norman Graham

This proof uses the following properties of the triangle.

- (i) Height $h = c \sin B = b \sin C$ (by the sine rule);
- (ii) Area of triangle $\Delta = \frac{1}{2}ah = \frac{1}{2}ab \sin C$;
- (iii) Area $\Delta = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc$;
- (iv) Since $A + B + C = 180^\circ$, we have $\sin(B + C) = \sin A$.

To prove the stated assertion, we have

$$x = DE = a - x \cot B - x \cot C.$$

Therefore $x(1 + \cot B + \cot C) = a$ and hence

$$\frac{1}{x} = \frac{1}{a} + \frac{1}{a}(\cot B + \cot C).$$

But

$$\cot B + \cot C = \frac{\cos B \sin C + \sin B \cos C}{\sin B \sin C} = \frac{\sin(B + C)}{\sin B \sin C} = \frac{\sin A}{\sin B \sin C}$$

from (iv). Therefore

$$\frac{1}{x} - \frac{1}{a} = \frac{\sin A}{a \sin B \sin C} = \frac{1}{b \sin C} = \frac{a}{ab \sin C} = \frac{a}{2\Delta}$$

using (i) and (ii). The corresponding result for y and z gives

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} = \frac{a + b + c}{2\Delta} = \frac{1}{r}$$

using (iii).

Also solved by **Dick Boardman**.

Reversible Numbers

Dennis Morris

Theorem *In any non-trivial positive integer number-base, B , the number*

$$1 \cdot B^3 + 0 \cdot B^2 + (B - 2)B + (B - 1)$$

will have the order of its digits reversed when multiplied by $B - 1$.

Example In number-base 10:

$$1089 \cdot 9 = 9801.$$

Proof We have

$$\begin{aligned} & (1 \cdot B^3 + 0 \cdot B^2 + (B - 2)B + (B - 1))(B - 1) \\ &= B^4 - 2B^2 + 1 \\ &= B^3(B - 1) + B^3 - 2B^2 + 1 \\ &= B^3(B - 1) + B^2(B - 2) + 0 \cdot B + 1. \end{aligned}$$

Theorem *In any non-trivial positive integer number-base, B , the number*

$$2 \cdot B^3 + 1 \cdot B^2 + (B - 3)B + (B - 2)$$

will have the order of its digits reversed when multiplied by $(B - 2)/2$.

Example In number-base 10:

$$2178 \cdot 4 = 8712.$$

Proof Clearly,

$$\begin{aligned} & (2 \cdot B^3 + 1 \cdot B^2 + (B - 3)B + (B - 2)) \frac{B - 2}{2} \\ &= B^4 - B^3 - 3B^2 - B + 2 \\ &= (B - 2)B^3 + B^3 - 3B^2 - B + 2 \\ &= (B - 2)B^3 + (B - 3)B^2 + 3B^2 - 3B^2 - B + 2 \\ &= (B - 2)B^3 + (B - 3)B^2 - 1 \cdot B + 2. \end{aligned}$$

Although we are interested only in multiplication by integers, the formula also works for non-integers; in this case, when the number-base $B = 3$ we have

$$2101 \cdot \frac{1}{2} = 1012.$$

The above are part of a general formula. We have

$$\begin{aligned}
 & (\alpha B^3 + (\alpha - 1)B^2 + (B - \alpha - 1)B + B - \alpha) \frac{B - \alpha}{\alpha} \\
 &= (\alpha B^3 + \alpha B^2 - B^2 + B^2 - \alpha B - B + B - \alpha) \frac{B - \alpha}{\alpha} \\
 &= (\alpha B^3 + \alpha B^2 - \alpha B - \alpha) \frac{B - \alpha}{\alpha} \\
 &= (B^3 + B^2 - B - 1)(B - \alpha) \\
 &= (B^4 + B^3 - B^2 - B) - (\alpha B^3 + \alpha B^2 - \alpha B - \alpha) \\
 &= B^4 + B^3 - \alpha B^3 - \alpha B^2 - B^2 + \alpha B - B + \alpha.
 \end{aligned}$$

Using $B^4 = (B - \alpha)B^3 + \alpha B^3$, the above is equal to

$$(B - \alpha)B^3 + B^3 - \alpha B - B^2 + \alpha B - B + \alpha.$$

Using $B^3 = (B - \alpha - 1)B^2 + \alpha B + B^2$, this is equal to

$$(B - \alpha)B^3 + (B - \alpha - 1)B^2 - (\alpha - 1)B + \alpha,$$

an example of which, when $\alpha = 1$ and $B = 8$ is $1067 \cdot 7 = 7601$.

In some number bases there are two-digit reversible numbers. Stringing these together will make four-digit or six-digit or $2n$ -reversible numbers. Of course four-digit reversible numbers can be strung together to make eight-digit or twelve-digit reversible numbers. Inserting a zero (or several) between the parts of such a string produces a reversible number. Inserting a $(B - 1)$ (or several) in the centre of a reversible number will produce a reversible number; inserting a reversible number in the centre of a reversible number also works. Most reversible numbers are either the product of such manipulations or are given by the general formula above. However, there are sporadic exceptions. We have in number-base $B = 8$:

$$1015 \cdot 5 = 5101, \quad 11165 \cdot 5 = 56111.$$

Most two digit reversible numbers are generated by the formula

$$\left(nB + (B - (n + 1)) \right) \frac{B - n}{n + 1}.$$

Again, there are sporadics. In number-base $B = 11$ we have $14 \cdot 3 = 41$, although this is simply half of the number generated by the formula.

Cubic equations

Norman Graham

Following on from the article by Tony Forbes, How to solve cubics, M500, number **202**, I offer here various alternative methods for solving cubic equations.

Cube roots of 1 We have

$$x^3 - 1 = (x - 1)(x^2 + x + 1) = 0.$$

Therefore

$$\begin{aligned} x &= 1 \\ \text{or } x &= \omega = \frac{1}{2}(-1 + \sqrt{3}i) \approx -0.5 + 0.86602i \\ \text{or } x &= \omega^2 = \frac{1}{2}(-1 - \sqrt{3}i) \approx -0.5 - 0.86602i. \end{aligned}$$

Also $1 + \omega + \omega^2 = 0$ and $\omega - \omega^2 = \sqrt{3}i$. Alternatively,

$$x = e^{i\theta} = \cos \theta + i \sin \theta, \quad \theta = 0, 120^\circ, 240^\circ.$$

Note that the cube roots of -1 are -1 , $-\omega = 1 + \omega^2$ and $-\omega^2 = 1 + \omega$.

The general cubic Start with

$$z^3 + Az^2 + Bz + C = 0;$$

then $z = x - A/3$ gives $x^3 - Ax^2 + \frac{1}{3}A^2x - \frac{1}{27}A^3 + Ax^2 - \frac{2}{3}A^2x + \frac{1}{9}A^3 + Bx - \frac{1}{3}AB + C = 0$ and the general cubic becomes

$$y = x^3 - 3ax = b, \tag{1}$$

where $-3a = -\frac{1}{3}A^2 + B$ and $b = -\frac{2}{27}A^3 + \frac{1}{3}AB - C$. We shall assume henceforth that the coefficients a , b and c are real.

Solution by trial and error Every cubic with real coefficients has at least one real solution, which can be found by trial and error, perhaps with the help of a graph. The other two solutions can then be found by factorization. However, trial and error is not very satisfactory from a pure-mathematical point of view.

Classification of roots Differentiating, we have $dy/dx = 3x^2 - 3a$, which can be zero only if $a \geq 0$, in which case the cubic has a maximum at $x = -\sqrt{a}$, $y = 2a\sqrt{a}$ and a minimum at $x = \sqrt{a}$, $y = -2a\sqrt{a}$. We consider two alternatives.

Case I: There is one real root and there are two imaginary roots. Either $a \leq 0$ or $|b| > 2a\sqrt{a}$. But there are three zero roots if $a = b = 0$.

Case II: There are three real roots if $a > 0$ and $-2a\sqrt{a} \leq b \leq 2a\sqrt{a}$. Two roots are equal if $|b| = 2a\sqrt{a}$.

Solution in case I (Cardano's method) Put $x = z_1 + z_2$, giving

$$z_1^3 + z_2^3 + 3z_1z_2(z_1 + z_2) - 3a(z_1 + z_2) = b.$$

Let

$$z_1z_2 = a. \quad (2)$$

Then

$$z_1^3 + z_2^3 = b. \quad (3)$$

Since $z_1^3z_2^3 = a^3$, z_1^3 and z_2^3 are roots of $(z^3)^2 - b(z^3) + a^3 = 0$. Therefore $z_1^3 = \frac{1}{2}(b + \sqrt{b^2 - 4a^3})$ and $z_2^3 = \frac{1}{2}(b - \sqrt{b^2 - 4a^3})$ are real since $b^2 > 4a^3$. Write

$$s = \sqrt[3]{\frac{1}{2}(b + \sqrt{b^2 - 4a^3})} \quad \text{and} \quad t = \sqrt[3]{\frac{1}{2}(b - \sqrt{b^2 - 4a^3})},$$

taking real values of the cube roots. Then the full solutions satisfying (2) are $(z_1, z_2) = (s, t)$, $\omega(s, \omega^2t)$ and $\omega^2(s, \omega t)$. The final solutions of (1) are then $x = s + t$, $\omega s + \omega^2t$ and $\omega^2s + \omega t$.

Solution in case II Although the roots are real, Cardano's method is unsuitable because $b^2 < 4a^3$; so s and t become cube roots of complex numbers.

Putting $x = r \cos \theta$, (1) becomes $r^3 \cos^3 \theta - 3ar \cos \theta = b$. Therefore $4 \cos^3 \theta - \frac{12a}{r^2} \cos \theta = \frac{4b}{r^3}$. Now let

$$\frac{12a}{r^2} = 3, \quad r = 2\sqrt{a}. \quad (4)$$

Then

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta = \frac{b}{2a\sqrt{a}}. \quad (5)$$

Therefore $3\theta = \cos^{-1} b/(2a\sqrt{a}) + n360^\circ$ for any integer n . The solutions are $x = 2\sqrt{a} \cos \theta$ for $\theta = \phi, \phi + 120^\circ$ and $\phi + 240^\circ$, where $\phi = \frac{1}{3} \cos^{-1} b/(2a\sqrt{a})$. Note that $r = -2\sqrt{a}$ gives the same solutions by replacing θ with $180^\circ - \theta$. It is remarkable that for this method to be valid, (4) and (5) require $a > 0$ and $|b/(2a\sqrt{a})| < 1$, which are precisely the criteria for case II.

Letters to the Editor

Quartics

Dear Tony,

I was extremely interested to study your solution of quartics in M500 **223**. I have never seen such a solution in a large amount of mathematical literature, although it is often stated that Galois proved that it is impossible to solve an equation of higher degree by analytical methods.

To investigate your comment that there appear to be twelve solutions instead of four, I used your method to solve a large number of quartics, and in every case the three values of u gave the *same* four values of x . I believe this is always the case, but I have not been able to prove it.

Yours sincerely,

Norman Graham

Tony Forbes replies. For convenience I outline the method here. Let $x^4 + bx^2 + cx + d = 0$ be the quartic after losing the ax^3 term in the usual manner. Rewrite it as

$$(x^2 + u)^2 = (2u - b)x^2 - cx + u^2 - d, \quad (1)$$

and choose u to be a root of the cubic $8u^3 - 4bu^2 - 8du + 4bd - b^2$. Then (1) is equivalent to

$$(x^2 + u) = \pm \sqrt{2u - b} \left(x - \frac{c}{2(2u - b)} \right),$$

which yields four roots for a given value of u .

I am reminded of an incident at a summer fair on Ham Common many years ago. As I approached the book stall I overheard one of the elderly ladies griping about the unmerchantability of the material that was donated. "Who on earth is going to buy this?" she asked her companion. Well, *this* was in fact a second-hand copy of *Guided Weapon Control Systems* by P. Garnell & D. J. East, and about a millisecond later it became my property in exchange for 20 pence. A fortuitous acquisition, for it turned out to be the very book in which I discovered the quartic solution outlined above. It's in Chapter 6, Autopilot Design, where the authors describe it merely as the standard method of solving such an equation.

See elsewhere in the magazine for an article by Norman explaining various alternative methods of solving cubics.

Inspecting the column, laces and a sequence

Tony,

I have been a member of the M500 Society for over twenty years but I have never written to the Society before. I usually conclude that either a problem is far too difficult for me to attempt, or that if I have done it then it must have been so easy that everyone else will have done it as well! I am prompted to write, however, by the fact that you gave two solutions to Problem 224.3 – Inspecting the column. One of the solutions—the one with a negative square root in it—implies negative distances and so is wrong (unless the column marches backwards, which is not something which can reasonably be inferred from the question). The acceptable solution—with a positive square root in it—is the sum of the lengths of two sides of a right-angled triangle, which makes me wonder if the problem can be reformulated in geometrical terms. I am still working on this!

Problem 227.5 – Laces seems obvious. The time taken walking on fixed ground will be the same regardless of where you tie your laces, and part of the walk on the travelator will be common to both scenarios. The time taken to tie your laces will be also common to both. If you step on the travelator before you tie your laces you will travel to a point X on the ground while doing so. If, however, you tie your laces before you step on the travelator you will then have to travel to point X and this will take time that you would not have had to spend if you had stepped on the travelator before tying your laces.

Finally, here is one for you. [See Problem 230.1 – Sequence, below.] I don't know the answer!

Best wishes,

Roger Dennis

Laces

Without loss of generality (Oh, how I love that phrase!) we may assume that I tie my shoe laces for one minute: either just before or just after boarding the travelator. Let us also assume that I am accompanied by a friend who shares my walking pace.

In the first case the time lost (precisely one minute) is equivalent to the distance my friend achieves walking along the moving travelator for that minute. In the second case the time lost is equivalent only to the distance he achieves walking for that minute.

Thus the better place to tie the laces is on the travelator.

Ian Adamson

Hadwiger's Conjecture

What is it, and why was it conjectured?

Ian Adamson

We all know what the Four Colour Map Conjecture is. And we all know it was proved thirty-odd years ago. And that the proof needed the *equivalent* of over seven weeks' (continuous) computer time. This 1977 proof by Appel and Haken has been heavily criticized not so much because of their use of a computer but because it consisted in effect of 1,482 different possibilities (later reduced to 633). One writer said it read more like a dictionary.

The history of this conjecture, its proof and its many false 'proofs' have been written about extensively. Read *Four Colours Suffice* by Robin Wilson.

Let us suppose that it hasn't been proved. How would we go about it? We would probably start with a few drawings and conclude (correctly but without proof) that we can't draw five mutually adjacent regions although we can draw four. Let's move from maps (with regions) to **graphs** (with vertices) by putting a **vertex** (point) in each region and joining it to those vertices in adjacent regions. And we can't join five vertices to each other in the plane.

Now that we've mentioned graphs: What *is* a graph? A simple graph is a (usually finite) set of vertices and a set (which may be empty) of edges, where an **edge** is an unordered pair of distinct vertices. That's the only definition (there are no axioms) and from just that, there's an enormous body of theory and many, many unsolved conjectures. There are about 400 such conjectures concerning just the colouring of vertices. Graph Theory was created by Euler *after* he disposed of the famous Königsberg Bridge Problem, which he did in 1736. The Map Conjecture was one of the three famous unsolved problems in Mathematics until 1977. The other two were Fermat's Last Theorem (proved in the 1990s) and the Riemann Conjecture (still neither proved nor disproved).

Back to the Four Colour Theorem. The fact that five vertices can't be joined to each other does not prove it. Consider a pentagon A, B, C, D, E where each vertex is joined to F (inside the pentagon). We call such graph a **wheel**. Clearly this graph requires 4 colours but it's not **complete**, that is where all vertices mutually joined. But our fuzzy attempt does in fact hit the nail on the head. Almost. Since the complete graph on 4 vertices is, as it were, the 'essence' of 4-chromaticity. And a complete graph on 5 vertices is the 'essence' of 5-chromaticity but such a graph can't be drawn in the plane; it's not **planar**. Let us agree to call a complete graph on r vertices a K_r .

Now we must get a bit technical. We must introduce the concept of a graph **contraction**. (Or, if you prefer, an **elementary connected homomorphism**.) But we'll stick with contraction. Suppose we take the wheel $A, B, C, D, E; F$: it's obviously not a K_4 . Replace the edge $[AB]$ by vertex (AB) so (AB) joins all the vertices previously joined to both the original A and B . We'll have a smaller wheel $(AB), C, D, E; F$. That's all a contraction is. And we may successively edge-contract and here replace edge $[CD]$ by vertex (CD) and obtain $(AB), (CD), E; F$: a K_4 . Formed by successively (in this case only twice) contracting the original graph. Successive contractions of a graph yield what is called a **(graph) minor**.

What's the point to all this? It has been shown (by Kuratowski and Wagner) that if a graph has K_5 as a minor then it cannot be planar. What about Hadwiger's Conjecture? It states that a graph which is r -chromatic has K_r as a minor. So a 5-chromatic graph must have K_5 as a minor (according to Hadwiger) and so can't be planar (according to Kuratowski and Wagner). So, together with Kuratowski–Wagner and Hadwiger, we could prove the Four Colour Theorem. 'Could' because Hadwiger's 1943 Conjecture hasn't yet been proved in general.

Now we know what Hadwiger's Conjecture is and (probably) why he conjectured it. It shows how the *essence* of r -chromaticity is linked *formally* to a K_r .

But as I said the Conjecture hasn't yet been proved—or disproved. And it's not easy. So far we've been waiting over 60 years. One complication (of many complications!) is that in obtaining a minor by a succession of contractions we might end up with say a K_3 even if the graph *has* K_4 as a minor! (The statement 'the graph has K_r as a minor' should be replaced by 'there exists *some* sequence of graph-contractions that includes K_r .) Return to the wheel $A, B, C, D, E; F$ and were we first to contract the edge $[AF]$ then we could never finish up with K_4 as a minor. We might even finish with K_2 . Try.

Later if *you* think you've proved Hadwiger's Conjecture then you probably haven't.

But you just might!

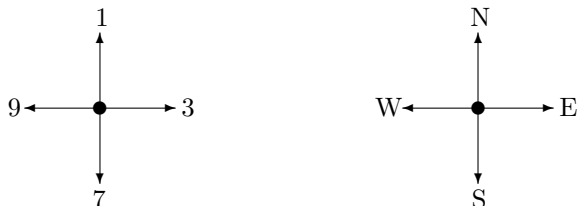
Wilson Stothers

We are very sorry to hear that Dr Wilson Stothers, tutor in OU mathematics, died in July after a long fight with cancer. Wilson was a considerable asset to the M500 Society at the Revision Weekends over many years. He was a great tutor, and a great character too. Our sympathy goes to his widow, Andrea.

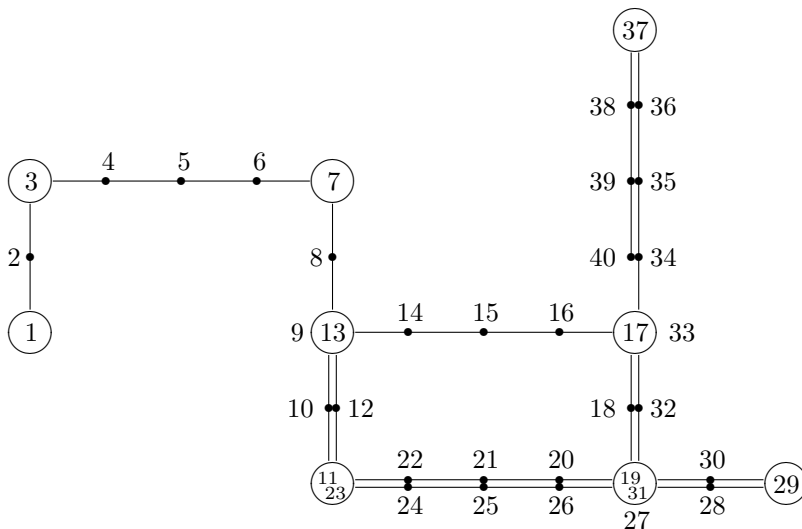
Another prime pathway

Chris Pile

Further to ‘Professor Pile’s prime pathway’ [M500 229, 14–15] for primes ending in 1, 3, 7 and 9. Since 3 and 9 are on opposite sides of a clock face horizontally and 1 and 7 are (almost) vertically opposite, they could be considered as compass points.



Using consecutive integers as your route map, take strides North when encountering a prime ending in 1, East when encountering a prime ending in 3, etc. Where do you end up? Here are the first few steps.



The trend is NE. But how far West and how far South do you go? (In my limited ramblings I have reached 36W and 46S.)

Problem 230.1 – Sequence

Roger Dennis

Find a general formula for $u(n)$, defined by

$$u(1) = 1, \quad u(n) = u(n-1) + \frac{1}{u(n-1)} \quad \text{for } n = 2, 3, \dots$$

To save you the trouble we have computed the first few terms:

$$1, 2, \frac{5}{2}, \frac{29}{10}, \frac{941}{290}, \frac{969581}{272890}, \frac{1014556267661}{264588959090}, \frac{1099331737522548368039021}{268440386798659418988490} \dots$$

Problem 230.2 – Sum

Sebastian Hayes

In how many ways can you sum n different non-negative integers which add to p ? The same numbers in a different order count as different selections.

For example, suppose we have $n = 2$, $p = 3$. We are looking for different ways in which $a + b = 3$. There are four ways, namely $3 + 0 = 2 + 1 = 1 + 2 = 0 + 3 = 3$.

A general formula is required.

Problem 230.3 – Magic

Sebastian Hayes

A number N is ‘magic’ if any number which ends with N is divisible by N . (Base 10 is assumed.) Example: any number ending in 2 is even and thus divisible by 2.

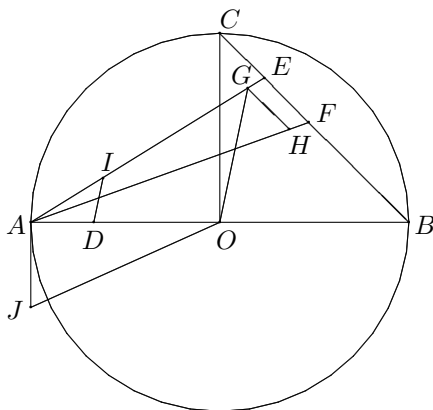
Problem: (a) How many magic numbers are there $\leq 10^n$? (b) Give a sufficient and necessary condition for a positive integer to be magic.

Problem 230.4 – Tanks

There are an odd number of tanks in a field. At the appointed instant each tank fires a shell at its nearest neighbour. Prove that if all the distances are distinct, then there is at least one tank which escapes being shot at. It is easy to show that the distinct distances part is essential.

Solution 226.7 – Squaring the circle

In the diagram, AOB is a diameter of the circle with centre O . The radius of the circle is $|OA| = 1$. Also CO is perpendicular to AB , $|AD| = |CE| = |EF| = \frac{1}{3}$, $|AH| = |AE|$, GH is parallel to EF , DI is parallel to OG , AJ is perpendicular to AB and $|AJ| = |AI|$. Show how to construct the diagram with ruler and compasses only. What is $3\sqrt{|OJ|}$?



Hugh Luxmoore–Peake

As I was in the middle of reading Thomas Heath's glorious translation of Euclid's Thirteen Books, I was in just the right frame of mind to tackle Problem 226.7. And the puzzle comes from the finest stable, though I thought Ramanujan was more of a numbers man than a geometer.

To construct the diagram:

- (1) Draw a line of sufficient length through O .
- (2) Mark off A and B three units from O (one on each side of O), and D one unit from A towards O .
- (3) Draw the circle centre O through A and B .
- (4) Draw a perpendicular to AB through O , determining C its intersection with the circle.
- (5) Join points C and B .
- (6) Mark off E and F one and two units respectively from C along the line CB .
- (7) Join points A and E , and A and F .
- (8) Using the compass at centre A and radius AE , mark off H such that AH is equal to AE .
- (9) Draw a line through H parallel to CB , and mark off G where it intersects the line AE .
- (10) Join points O and G .
- (11) Draw a line through D parallel to OG , and mark off I where it intersects the line AG .

It's fair to assume, I think, that steps (4), (9) and (11) are trivial, and do not need to be spelled out.

For simplicity's sake, set $a = |OJ|$, $b = |AI|$, $c = |AG|$, $d = |AE|$, $e = |AF|$, and radius $|OA| = 3$ rather than one. We now want the value of \sqrt{a} . Working backwards,

by Pythagoras in $\triangle AJO$,

$$a^2 = 9 + b^2;$$

from similar \triangle s AID and AGO ,

$$b = c/3;$$

from similar \triangle s AGH and AEF ,

$$c = d^2/e;$$

from $\triangle AEB$ (cosine rule),

$$d^2 = 6^2 + (3\sqrt{2} - 1)^2 - 12(3\sqrt{2} - 1)/\sqrt{2};$$

and from $\triangle AFB$ (cosine rule),

$$e^2 = 6^2 + (3\sqrt{2} - 2)^2 - 12(3\sqrt{2} - 2)/\sqrt{2}.$$

Working forwards, we obtain $d^2 = 19$, $e^2 = 22$, whence $a = \sqrt{2143/198}$ and $\sqrt{a} = \sqrt{\sqrt{2143/198}}$.

Ramanujan, having got this far without putting pen to paper, would proceed to find a far more elegant expression for $a \dots$

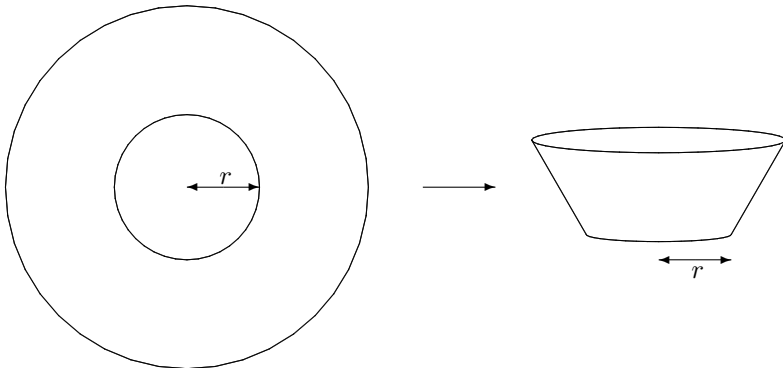
Problem 230.5 – Cup-cake holder

Tamsin Forbes

You have a circular piece of tin foil of radius 1. Use it to make a cup-cake holder of maximum possible volume.

For simplicity, you may assume that the pleating of the tin foil to make the sides is done at an infinitesimal level; so you can compute the volume of the finished cup-cake holder by the usual formula for a truncated cone. And for definiteness, denote by r the radius of the part of the tin foil that forms the bottom of the cup-cake holder (leaving $1 - r$ for the side). Obviously $0 \leq r < 1$.

What is the maximum volume?



Fermat's Room

Eddie Kent

I went to the pictures the other day. *Fermat's Room*. Spanish, low-budget, and supposedly concerned with mathematics. The first thing you hear and see is 'Do you know what a prime number is? If you don't, leave now'. We are introduced to a young man who claims to have proved Goldbach's conjecture, but on the eve of his presentation of the proof his room was trashed and all his papers destroyed. From there on the action of the film mostly takes place in a room furnished like Fermat's. I can not believe that the books were authentic but we never get to see the titles so what the hell.

A problem is set, either in a newspaper or by email (I wasn't paying attention), to find the order in the sequence 8, 5, 4, 9, 1, 7, 6 (there might have been fewer numbers); solvers could send in their solutions and those who got it right would be invited to a select gathering 'to work on the greatest enigma.' Only four people manage it and they are instructed to drive to a remote spot, wearing badges they are given, leaving mobile phones behind, and not to tell anyone their real names. The badges identify them as Hilbert, Galois, Oliva and Pascal. (Oliva was a woman; I can't find anything about her.)

The four meet by a lake, then find a boat. In the boat is a personal digital assistant (PDA) which tells them to row to an island, and they end up in Fermat's room. They meet their 'host', labelled Fermat, who invites them to eat and drink. Then he is called away on a mobile which he alone carries and the others find they are locked in. The PDA sets them a problem, something like the two hourglasses taking different times and they have to show how to produce yet a third time. They are given one minute to find an answer but because they are all shouting at one another they miss the deadline.

After one minute the room starts contracting. They quickly key in an answer and the walls stop. They find a receipt for four presses and realize that all four walls are set to press in on them. (How?) Another problem arrives, and then another, till I lost interest. They're not up to M500 standard, though one I liked was three boxes, red balls and blue balls, one box has all red, one all blue and one mixed. All the boxes are labelled, but wrongly. How many balls do you have to take out to be able to correct the labelling? The answer of course is one—why? And to the problem of the shepherd, the goat the wolf and the cabbage and a restricted boat, the engineer asks "Why would a shepherd have a wolf with him?"

And so it goes on, and they are losing out all the time because although they are leading mathematicians (Goldbach's conjecture, pshaw!) it takes

them too long to find the answers, when they also have to spend time shouting at one another, attempting to stop the walls, working out why they're here, discovering they are not all strangers, finding a way out, etc. They do pretty well in the end, once they get organized and stop hitting one another, and they discover a lot about themselves in the process, so really the maths is irrelevant—the film is a straightforward but very complicated horror-crime mystery, and entertaining enough. I just wish I knew how the walls worked and why they felt it necessary to tell me that Galois died at 22 (he was in fact 21).

The film is in colour, mostly in close-up, and very loud. There was one other adult in the auditorium with me, and dozens of uniformed teenagers, so I assume the adult was their teacher. But I thought, the maths wasn't too onerous, anyone should be able to cope with it pretty well. And then a couple of days later I was listening to Radio 4, a programme about Fred Perry, a tennis player. And I hardly understood a word. That gave me some inkling of how the ordinary intelligent person could see mathematics, as gobbledegook. A sobering thought. I have left the sequence and the balls problems to entertain the interested reader.

The Schur real zeros theorem

Tony Forbes

As you will discover, after consulting Robin Whitty's THEOREM OF THE DAY web site at <http://www.theoremoftheday.org/Theorems.html>, Issai Schur proved the following.

Theorem *Let $f(x) = a_0 + a_1x + \dots + a_mx^m$ and $g(x) = b_0 + b_1x + \dots + b_nx^n$ be polynomials with real coefficients. Suppose that f and g have only real zeros and that all the zeros of g have the same sign. Then the polynomial*

$$f(x) \odot g(x) = \sum_{k=0}^{\min(m,n)} k! a_k b_k x^k$$

has only real zeros. Moreover, if a_0 and b_0 are both non-zero, then these zeros are distinct.

After illustrating the theorem with an example, Robin then goes on to suggest as a nice exercise that if f and g are both quadratics, then the factor $k!$ can be omitted.

And now we are going on to suggest as a further nice exercise to prove that the same applies if f and g are both *cubics*.

Solution 225.3 – GCD

Compute $\gcd(n! + 1, (n + 1)!)$, where n is a positive integer.

Basil Thompson

Write $(n + 1)!$ as $(n + 1)n!$.

(1) Consider $n! + 1$ and $n!$. There are no common factors; all divisors of $n!$ leave remainder 1. That is, no g.c.d.

(2) Consider $n! + 1$ and $n + 1$. If $n + 1$ is composite, it will divide into $n!$ but not $n! + 1$. That is, no g.c.d.

On the other hand, if $n + 1$ is prime, it will not divide into $n!$. Indeed, $n! + 1$ will be a prime number or a new composite number made up of new primes not included in $n!$. Compare Euclid's proof that there is an infinite number of primes. If $n! + 1$ is a new prime, there will be no g.c.d. with $n + 1$. But:

n	$n + 1$	$n! + 1$	g.c.d.
1	2	2	2
2	3	3	3
4	5	25	5
6	7	721	7
10	11	3628801	11
12	13	479001601	13
16	17	20922789888001	17
18	19	6402373705728001	19
22	23	1124000727777607680001	23
28	29	304888344611713860501504000001	29
30	31	265252859812191058636308480000001	31

From this small sample it is tempting to conjecture that the only g.c.d.s are $n + 1$ when $n + 1$ is prime.

The proof is completed using Wilson's theorem [see, for example, Theorem 80 in G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*], which says that if p is prime, then $(p - 1)! \equiv -1 \pmod{p}$. In other words, when $n + 1$ is prime, $n + 1$ divides $n! + 1$ and hence the g.c.d. is $n + 1$.

The start of a journey to Persia resembles an algebraical equation: it may or it may not come out

— Robert Byron, *The Road to Oxiana*

M500 Winter Weekend 2010

A Weekend of Socializing with Mathematics

Join with fellow enthusiasts for a weekend of mathematical fun and a look at some unusual, interesting and recreational problems. The **twenty-ninth M500 Society Winter Weekend** will be held at

Florence Boot Hall, Nottingham University

from Friday evening to Sunday afternoon

8th–10th January 2010.

The overall theme will be INVESTIGATIONS, and it will be run by Mel Starkings and Angela Allsopp. During the process Angela will be doing a lot of sliding and Mel will spend some time looking at sausages (!!).

Cost: £190 to M500 members, £195 to non-members. You can obtain a booking form from the M500 site:

<http://www.m500.org.uk/winter/booking.pdf>.

If you have no access to the internet, send a stamped addressed envelope to

Diana Maxwell

There will be the usual extras. On Friday Rob Rolfe is running a pub quiz with Valuable Prizes, and for the sing-song on Saturday night we urge you to bring your favourite musical instrument—or at least your voice! Hope to see you there.

Corrigendum. The erratum at the bottom of page 13 of M500 **229** should read

$$(L_g + L_{g'})(1) = L_g(1) + L_{g'}(1) = 0 + 0 = 0,$$

not $(L_g + L_{g'})(1) = L_g(0) + L_{g'}(0) \dots$ Editor's fault. Apologies.

Solution 226.4 – Three squares		
Norman Graham	1	
Reversible Numbers		
Dennis Morris	2	
Cubic equations		
Norman Graham	4	
Letters to the Editor		
Quartics	Norman Graham	6
Inspecting the column, ...	Roger Dennis	7
Laces	Ian Adamson	7
Hadwiger’s Conjecture		
Ian Adamson	8	
Another prime pathway		
Chris Pile	10	
Problem 230.1 – Sequence		
Roger Dennis	11	
Problem 230.2 – Sum		
Sebastian Hayes	11	
Problem 230.3 – Magic		
Sebastian Hayes	11	
Problem 230.4 – Tanks	11
Solution 226.7 – Squaring the circle		
Hugh Luxmoore–Peake	12	
Problem 230.5 – Cup-cake holder		
Tamsin Forbes	13	
<i>Fermat’s Room</i>		
Eddie Kent	14	
The Schur real zeros theorem		
Tony Forbes	15	
Solution 225.3 – GCD		
Basil Thompson	16	
M500 Winter Weekend 2010	17
