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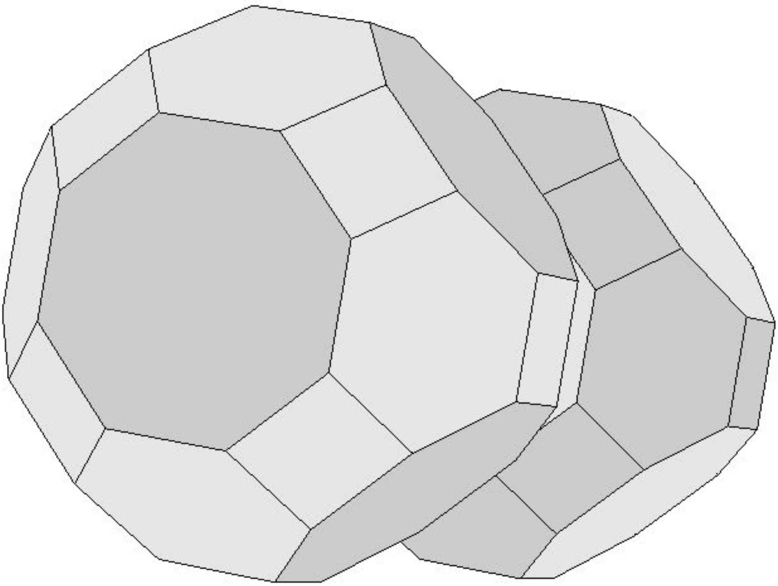
M500 232



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation.

Editor – *Tony Forbes*

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Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation.

Solution 200.1 – Well spaced

> There are n slots, numbered 1 to n, arranged in a circle. They are to be occupied by n objects, one by one, such that at all times the objects are as well spaced as possible. The first object can go anywhere. Thereafter, when an object is added to the system it must be placed such that the minimum distance to its two neighbours is as large as possible. Find a general formula for the number of ways this can be achieved.

Dave Wild

After adding m objects we can count the number of slots between the neighbouring objects. If there are g_1 gaps of size s_1, g_2 of size s_2, ..., then we will describe the current state as $g_1 * s_1,\ g_2 * s_2,\ \ldots$.

For example, if we start with 75 slots then the states after adding the first few objects are

Objects in Ring	State
1	$1 * 74$
2	$1 * 37,\ 1 * 36$
3	$1 * 36,\ 2 * 18$
4	$3 * 18,\ 1 * 17$

.

We are going to look at the case where

1. the ring already contains 2^k objects;
2. the maximum and minimum gap sizes differ by at most one;
3. the maximum gap size is greater than 3.

We will work out the numbers of ways that we can add the next 2^k objects and show that the new maximum and minimum gap size still differ by at most 1. We will consider the cases where the maximum size is odd and even separately.

Maximum gap size is odd

The initial state is $p * (2s + 1),\ q * 2s$, where $s > 1$, $p > 0$ and $p + q = 2^k$.

If we place an object in a gap of size $2s + 1$ then we can do this in one way and we are left with two gaps of size s. If we place an object in a gap of size $2s$ then we can do this in two ways and we are left with gaps of size s and $s - 1$.

Since $s > 1$, it follows that the first p objects have to be used to split each of the gaps of size $2s+1$ and the remaining q objects to split the gaps of size $2s$. Therefore there are $p!\,q!\,2^q$ ways of adding 2^k objects. The resultant state is $(2p+q)*s,\ q*(s-1)$. ♦

Maximum gap size is even

The initial state is $p*2s,\ q*(2s-1)$, where $s>1$, $p>0$ and $p+q=2^k$.

If we place an object in a gap of size $2s$ then we can do this in two ways and we are left with gaps of size s and $s-1$. If we place an object in a gap of size $2s-1$ then we can do this in one way and we are left with two gaps of size $s-1$. Since the minimum gap size is the same in both cases, as $s>1$, we can place the 2^k additional objects in the 2^k gaps in any order.

Therefore there are $(2^k)!\,2^p$ ways of adding 2^k objects. This leaves us in the state $p*s,\ (p+2q)*(s-1)$. ♦

In both cases we end up in a state where the difference between the minimum and maximum sizes is at most 1. Therefore, while the maximum gap size is greater than 3, we can use the above results to calculate the number of ways to add the next $2^0, 2^1, \ldots$ objects. When $n=75$ we have the following.

Initial Objects in Ring	Objects to be added	State after adding objects	Ways of adding objects
0	1	$1*74$	75
2^0	2^0	$1*37,\ 1*36$	$1!\cdot 2^1$
2^1	2^1	$3*18,\ 1*17$	$1!\,1!\cdot 2^1$
2^2	2^2	$3*9,\ 5*8$	$4!\,2^3$
2^3	2^3	$11*4,\ 5*3$	$3!\,5!\,2^5$
2^4	2^4	$11*2,\ 21*1$	$16!\,2^{11}$
2^5	$75-32=43$	$75*0$	?

When we can no longer use the method of doubling the number of objects then, if $n>1$, we will be left in one of the following states: $p*3,\ q*2$, $p>0$, where $p+q=2^k$, or $p*2,\ q*1$, where $p+q=2^k$. We will now show how many ways there are of adding the remaining $n-2^k$ objects.

$p*3,\ q*2,\ p>0$, where $p+q=2^k$

The next p objects have to be added to the gaps of size 3. This can be done in $p!$ ways and leaves us in the state $q*2,\ 2p*1$. The remaining slots can be filled in any order so this can be done in $(2q+2p)! = (2k+1)!$ ways. So the ways of adding the final objects is $p!\,(2k+1)!$. ♦

$p * 2,\ q * 1$, where $p + q = 2^k$

The remaining slots can be filled in any order; so there are $(2p + q)! = (2k + p)!$ ways of doing this. ♦

We can now complete the table for $n = 75$ and replace the ? by 43!. So multiplying the entries in the last column of the table we see that the number of ways of adding 75 objects is $75 \cdot 2^2 13!\,4!\,5!\,16!\,43!$.

This method described allows us to find a general formula for the case where $n > 3$.

Let $b_N b_{N-1} \ldots b_1 b_0$ be the binary representation of n, and $m_k = n \bmod 2^k$. Then the number of ways of adding the n objects to the ring is

$$n\,2^p \prod_{k=1}^{N-2} f_k F,$$

where

$$p \;=\; m_{N-1} + \sum_{k=1}^{N-2} (-1)^{b_k} m_k,$$

$$f_k \;=\; \begin{cases} \left(2^k\right)! & \text{if } b_k = 0,\\ \left(m_k\right)!\,\left(2^k - m_k\right)! & \text{if } b_k = 1,\end{cases}$$

and

$$F \;=\; \begin{cases} \left(2^{N-1} + m_{N-1}\right)! & \text{if } b_{N-1} = 0,\\ \left(2^N\right)!\,\left(m_{N-1}\right)! & \text{if } b_{N-1} = 1.\end{cases}$$

If we return to our example of $n = 75$ then $N = 6$.

$k =$	6	5	4	3	2	1	0
$b_k =$	1	0	0	1	0	1	1
$m_k =$		11	11	3	3	1	
$f_k =$			16!	3! 5!	4!	1! 1!	

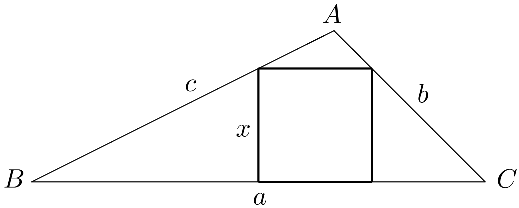
Thus $p = 11 + 11 - 3 + 3 - 1 = 21$ and $F = (32 + 11)! = 43!$. So the number of ways of adding the objects is $75 \cdot 2^{21} \cdot 3!\,4!\,5!\,16!\,43!$.

In the [autumn] of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for re-election. — Hugo Rossi

Solution 226.4 – Three squares

Let $\mathcal{T}$ be a triangle with sides a, b, c and in-circle radius r. Let x be the side of the square such that (i) one side of the square shares a common border with side a of $\mathcal{T}$, (ii) the other two vertices of the square lies on sides b and c of $\mathcal{T}$. Define y and z similarly in terms of sides b and c respectively. Show that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \;=\; \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{r}.$$



Sebastian Hayes

The solution to this problem depends on two results.

(a) If r is the radius of the in-circle of triangle ABC, $\triangle = rs = r(a+b+c)/2$.

(b) If a is the base and h_a is the height of triangle on base a, the side of the inscribed square is half the harmonic mean of the base and the height, i.e. (height × base)/(height + base). (Later we show how to inscribe a square in a triangle.)

Thus, with the diagram used, where a is the base and the angles are A, C, B proceeding clockwise:

$$\begin{aligned}
x &= \frac{ah_a}{a+h_a} = \frac{ac\sin B}{a+c\sin B},\\
y &= \frac{bh_b}{b+h_b} = \frac{ba\sin C}{b+a\sin C},\\
z &= \frac{ch_c}{c+h_c} = \frac{cb\sin A}{c+b\sin A}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{x}+\frac{1}{y}+\frac{1}{z} &= \frac{a+c\sin B}{ac\sin B}+\frac{b+a\sin C}{ab\sin C}+\frac{c+b\sin A}{bc\sin A}\\
&= \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{c\sin B}+\frac{1}{a\sin C}+\frac{1}{b\sin A}.
\end{aligned}$$

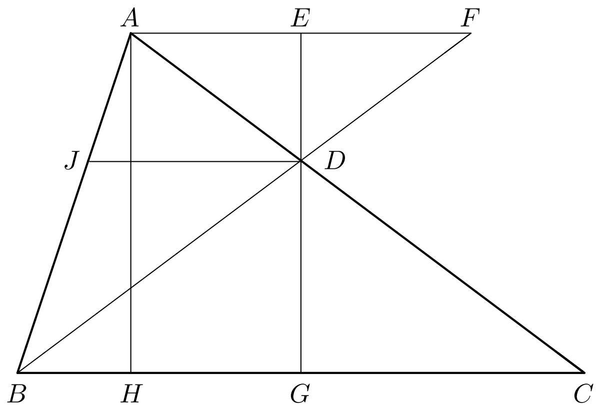
But

$$\begin{aligned}\frac{1}{c\sin B}+\frac{1}{a\sin C}+\frac{1}{b\sin A} &= \frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\\ &= \frac{a}{ah_a}+\frac{b}{bh_b}+\frac{c}{ch_c}\\ &= \frac{a}{2\,\triangle}+\frac{b}{2\,\triangle}+\frac{c}{2\,\triangle}\\ &= \frac{a+b+c}{2\,\triangle} = \frac{s}{\triangle} = \frac{1}{r}.\end{aligned}$$

This shows that the sum of the reciprocals of the three heights is $1/r$.

To inscribe a square in a triangle

The solution suggests a method for constructing an inscribed square in a triangle all of whose angles are acute.



If the triangle is BAC with height AH, we draw $AF = AH$ parallel to BC. Then $AH = EG = AF$ and $ED/DG = AF/BC$ since ED and DG are the heights of similar triangles. Hence

$$\frac{EG}{DG} = \frac{AF+BC}{BC}$$

and therefore

$$DG = \frac{BC\cdot EG}{AF+BC} = \frac{BC\cdot AH}{AH+BC} = \frac{\text{height}\times\text{base}}{\text{height}+\text{base}}$$

since $AH = EG = AF$, the height of the triangle.

Also, if we draw a line through D meeting BA at J, then $JD/BC = ED/AH$ by similar triangles. Hence

$$JD = \frac{BC \cdot ED}{AF} = BC \cdot \frac{ED}{AF}$$

since $AH = AF$. But $ED/AF = DG/BC$ by similar triangles. Therefore

$$JD = DG = \frac{\text{height} \times \text{base}}{\text{height} + \text{base}}.$$

We recall that, for two quantities x and y,

$$\text{A.M.} = \frac{x+y}{2}, \quad \text{G.M.} = \sqrt{xy}, \quad \text{H.M.} = \frac{2xy}{x+y} = \frac{(\text{G.M.})^2}{\text{A.M.}}.$$

Now, H.M. $\le$ G.M. $\le$ A.M., with equality only occurring when $x = y$. Thus

$$\begin{aligned} DG \cdot JD &= \text{inscribed square} \\ &= \left(\frac{\text{height} \times \text{base}}{\text{height} + \text{base}}\right)^2 \le \frac{\text{height} \times \text{base}}{4} = \frac{\text{area } \triangle}{2}. \end{aligned}$$

The area of the inscribed square is thus always less than half the area of the triangle except when the triangle is itself inscribed within a square, when the area of the inscribed square is exactly half that of the triangle and a quarter that of the larger square.

Problem 232.1 – $\pi + e$

Tony Forbes

Not long ago I made a decision to spend the rest of my life believing that $\pi + e$ is rational. Not unreasonable behaviour—unless of course someone can prove that I am wrong. Perhaps I am hoping that something useful and interesting will occur to me. Meanwhile, what we are asking you to do for this problem is decide the whether the following numbers are rational or irrational, on the assumption that $\pi + e$ is rational.

$\pi - e$,

πe, π/e, $\pi^2 e^2$, π^2/e^2, $\pi^3 e^3$, $\pi^3 e^3$, etc.,

π^e, e^π,

$\pi^2 + e^2$, $\pi^2 - e^2$, $\pi^3 + e^3$, $\pi^3 - e^3$, $\pi^3 + e^3$, $\pi^3 - e^3$, etc.

Can you find any new rational and irrational numbers?

The Generalized Principle of Relativity

Dennis Morris

Einstein postulated the now very much verified principle that physical reality is invariant under change of velocity. Change of velocity is rotation in space-time. Space-time is the geometric space that is found in the finite group C_2 (one simply exponentiates the Cayley table). It is common in particle physics to require that physical reality is invariant under transformations in the Lie groups U(1) and SU(2) (and some others). Transformations in these two groups are just rotations in euclidean space and quaternion space respectively. These are the geometric spaces found in the groups C_4 and $\mathbb{H}$. There are 18 infinite families of simple finite groups and 26 simple sporadic groups. I hereby postulate the generalized principle of relativity.

The generalized principle of relativity is that physical reality is invariant under rotation in the geometric spaces of every simple group. The aim is to

1. correlate the Lie groups with the finite simple groups
2. discover the nature of rotation in each simple sporadic group and each family of simple finite groups
3. write down a lagrangian that is invariant under rotations in all simple finite groups
4. derive the field equation(s) from that lagrangian; the speculation is that the field equation(s) will be the unified field theory of physics.

About 100 years work methinks.

Problem 232.2 – Angles

Suppose $A + B + C = 45°$. Show that

$$\begin{aligned}(\cos A + \sin A)(\cos B + \sin B)(\cos C + \sin C)\\ = 2(\cos A \cos B \cos C + \sin A \sin B \sin C).\end{aligned}$$

Problem 232.3 – Three degrees

Devise a ruler-and-compasses geometric construction for

$$\frac{1}{16}\left(\left(\sqrt{6}+\sqrt{2}\right)\left(\sqrt{5}-1\right)-2\left(\sqrt{3}-1\right)\left(\sqrt{5+\sqrt{5}}\right)\right).$$

Solution 226.8 – 999 nines

What are the last nine digits of the number whose value is an exponential tower of 999 nines?

$$9^{9^{9^{\cdot^{\cdot^{\cdot^{9^{9^9}}}}}}}$$

Tony Forbes

Since nobody has sent a solution to this interesting problem I thought I might like to have a go.

To save constructing large and unwieldy towers of nines we first define some notation. Let $T_k(r)$ denote a tower of k nines with r at the top,

$$T_k(r) \;=\; 9^{9^{9^{\cdot^{\cdot^{\cdot^{9^{9^{r}}}}}}}} \qquad (k \text{ nines}),$$

and for brevity let $T_k = T_k(1)$. Denote as usual by $\phi(n)$ the number of positive integers less than or equal to n and co-prime to n. This is Euler's ϕ function. Recall that if we have a congruence such as $x^y \equiv z \pmod n$ with $\gcd(x, n) = 1$, then we can simplify it by reducing the exponent modulo $\phi(n)$ thus: $x^{y \bmod \phi(n)} \equiv z \pmod n$.

Our task is to compute $T_{999} \bmod 1000000000$. Using the ϕ function as indicated this is equivalent to

$$9^{T_{998} \bmod \phi(1000000000)} \;\equiv\; T_1(T_{998} \bmod 400000000) \pmod{1000000000}.$$

Moving up another nine and using the fact that $\phi(400000000) = 160000000$, this is equivalent to $T_2(T_{997} \bmod 160000000) \pmod{1000000000}$.

I hope the pattern is becoming revealed. Each time we move up the tower of nines we apply the ϕ function to the exponent. Applying ϕ repeatedly, starting with 1000000000, gives

400000000, 160000000, 64000000, 25600000, 10240000,
4096000, 1638400, 655360, 262144, 131072, 65536, 32768,
16384, 8192, 4096, 2048, 1024, 512, 256, 128, 64, 32,
16, 8, 4, 2 and finally 1.

Counting these iterations very carefully, we see that our task is to compute

$$T_{27}(T_{972} \bmod 1) \;\equiv\; T_{27}(0) \;\equiv\; T_{26}(9^0 \bmod 2) \;\equiv\; T_{26}$$

with everything modulo 1000000000.

We continue to work backwards, descending the tower of nines and reversing the iterations of ϕ:

$$\begin{aligned}
T_{26} &\equiv T_{25}(9 \bmod 4) \equiv T_{25} \equiv T_{24}(9 \bmod 8) \equiv T_{24} \\
&\equiv T_{23}(9 \bmod 16) \equiv T_{22}(9^9 \bmod 32) \equiv T_{21}(9^9 \bmod 64) \\
&\equiv T_{20}(9^9 \bmod 128) \equiv T_{19}(9^{73} \bmod 256) \\
&\equiv T_{18}(9^{73} \bmod 512) \equiv T_{17}(9^{329} \bmod 1024) \\
&\equiv T_{16}(9^{841} \bmod 2048) \equiv T_{15}(9^{841} \bmod 4096) \\
&\equiv T_{14}(9^{841} \bmod 8192) \equiv T_{13}(9^{4937} \bmod 16384) \\
&\equiv T_{12}(9^{4937} \bmod 32768) \equiv T_{11}(9^{21321} \bmod 65536) \\
&\equiv T_{10}(9^{21321} \bmod 131072) \equiv T_{9}(9^{21321} \bmod 262144) \\
&\equiv T_{8}(9^{21321} \bmod 655360) \equiv T_{7}(9^{545609} \bmod 1638400) \\
&\equiv T_{6}(9^{873289} \bmod 4096000) \equiv T_{5}(9^{873289} \bmod 10240000) \\
&\equiv T_{4}(9^{9065289} \bmod 25600000) \equiv T_{3}(9^{3945289} \bmod 64000000) \\
&\equiv T_{2}(9^{16745289} \bmod 160000000) \equiv T_{1}(9^{112745289} \bmod 400000000) \\
&\equiv 9^{192745289} \equiv 392745289 \pmod{1000000000};
\end{aligned}$$

and that is the final answer, 392745289, assuming I have counted correctly.

Problem 232.4 – Gradients

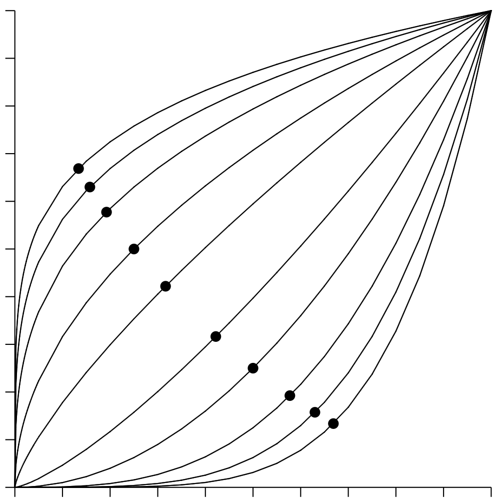
Robin Whitty

For each real number α in the range $(0, \infty)$, draw the graph of the function

$$x \mapsto x^\alpha,$$

$0 \le x \le 1$, and mark the point where the gradient is 1.

What is the function traced out by these points (assuming that a suitable choice is made when $\alpha = 1$)?



Solution 226.6 – Two bombs

> There is a collection of bombs, all of identical construction. Your task is to determine the minimum height from which a bomb must be dropped for the detonation mechanism to work. Great accuracy is not necessary. Measurement to the nearest 10 feet is all that is required. You have at your disposal a very tall building whose floors are spaced ten feet apart.
>
> If you are given just one bomb to test, all you can do is this, starting at $n = 1$. Drop the bomb from floor n and see what happens. If it explodes, report '$10n$ feet'. If not, retrieve the bomb and repeat the test from floor $n + 1$. You may assume that a bomb which survives being dropped will not sustain any damage, and therefore a future test will be valid. On the other hand, once the bomb explodes it cannot be used again.
>
> Now suppose instead you are given *two* test bombs. How can you improve your strategy?

Tony Forbes

Nobody has sent anything on this, so I thought I would have a try, even though I do not really have a satisfactory answer. If there is no known upper bound to the detonation height, the problem seems to be more difficult. Also we need to be reasonably clear about the way we measure a strategy. So let us take the easier option: we assume that there does exist a known upper bound, and we look for the strategy that minimizes the number of tests in the worst possible case.

Actually it is better to solve a slightly different problem, and we might as well dive straight into the general case. Given a number of test bombs, b, and a number of tests, t, we ask ourselves the question, "What is the maximum height $H = H(b, t)$ such that we can guarantee to determine the bomb's detonation height, assuming it is at most H?"

If $b = 1$, we carry out the procedure described in the statement of the problem. Thus $H(1, t) = t + 1$. Moreover, if $t < b$ then the $(t + 1)$th bomb is redundant and $H(b, t) = H(t, t)$. In the remaining case, with $b > 1$ bombs and $t \geq b$ tests, our plan is as follows. For the first test we go to floor $H(b - 1, t - 1)$ and drop the bomb. If it explodes, then we know that the detonation height is at most $H(b - 1, t - 1)$, and, having used one bomb and one test, the problem is indeed reduced to $b - 1$ bombs and $t - 1$ tests. On the other hand, if the bomb survives, we move the ground up to floor $H(b - 1, t - 1)$ and continue the process with b bombs and $t-1$ tests, the relevant height value being $H(b,t-1)$. Thus $H(b,t) = H(b-1,t-1) + H(b,t-1)$. Putting all this together, we have

$$\begin{aligned} H(1,t) &= t+1, \\ H(b,t) &= \begin{cases} H(t,t) & \text{if } t < b, \\ H(b-1,t-1) + H(b,t-1) & \text{if } t \geq b. \end{cases} \end{aligned}$$

t	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$	$b=6$	$b=7$	$b=8$
1	2	2	2	2	2	2	2	2
2	3	4	4	4	4	4	4	4
3	4	7	8	8	8	8	8	8
4	5	11	15	16	16	16	16	16
5	6	16	26	31	32	32	32	32
6	7	22	42	57	63	64	64	64
7	8	29	64	99	120	127	128	128
8	9	37	93	163	219	247	255	256
9	10	46	130	256	382	466	502	511
10	11	56	176	386	638	848	968	1013
11	12	67	232	562	1024	1486	1816	1981
12	13	79	299	794	1586	2510	3302	3797
13	14	92	378	1093	2380	4096	5812	7099
14	15	106	470	1471	3473	6476	9908	12911
15	16	121	576	1941	4944	9949	16384	22819
16	17	137	697	2517	6885	14893	26333	39203
17	18	154	834	3214	9402	21778	41226	65536
18	19	172	988	4048	12616	31180	63004	106762
19	20	191	1160	5036	16664	43796	94184	169766
20	21	211	1351	6196	21700	60460	137980	263950

To answer the original problem we look up the maximum height in the column headed $b = 2$, or use the formula

$$H(2,t) = \frac{1}{2}\left(t^2 + t + 2\right)$$

to determine t, the (maximum) number of tests required.

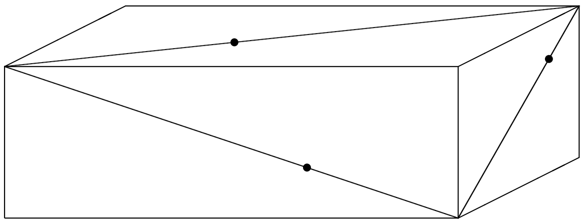
Upside down? Show that the quadratic $ax^2 + bx + c = 0$ has solutions

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}.$$

[Sent by **Emil Vaughan**.]

Problem 232.5 – Three points on a cuboid

Take a cuboid (a solid object which looks like a cube except that its length, width and height are not necessarily the same). Mark a point on each of three mutually orthogonal faces. The problem we want you to solve is to show how to construct the lines on each of the three faces at the intersections of the plane that passes through the three points.



In the diagram you can see the easy case, where each point happens to lie on the appropriate face diagonal. I must admit that I (TF) really wanted to use a more typical example to illustrate the problem. Unfortunately I do not know how to do the general case.

Thanks to **John Faben** for showing me this interesting problem.

Mathematics in the kitchen – VII

Tony Forbes

Here is a nice little problem communicated to me by **Robin Whitty** and which might be of use to readers, especially those who cook food.

Robin wants to suspend a large, heavy kitchen cupboard on a wall. So he drills two holes in the back panel, near the top-left and the top-right corners, and continuing into the wall. Through each hole he inserts a suitable fixing bolt and the cupboard is hung. But as the years go by gradual disruption of the brickwork in the wall by minor earthquakes causes the fixings to work loose, and eventually the cupboard falls down.

Fortunately Robin can have his life again. This time he tiles the lower part of the kitchen wall and then lets the bottom edge of the rear of the cupboard rest on the top of the last row of tiles. Now the tiles take a share of the load. So should Robin be surprised to see the cupboard break away from the wall considerably sooner than before?

Warning: Do not try this at home. Hanging cupboards is dangerous work that is best left to competent builders.

Problem 232.6 – Thrackles

In graph theory one spends a lot of time drawing diagrams of graphs. The vertices are represented by distinct points in the plane $\mathbb{R}^2$. An edge that joins vertices (points) a and b is drawn as a *curve* that starts at a, ends at b but otherwise does not pass through any vertex; that is, a continuous one-to-one mapping $C : [0,1] \to \mathbb{R}^2$ such that $C(0) = a$, $C(1) = b$, and if v is any vertex point then $C(x) \neq v$ for all $x \in (0,1)$.

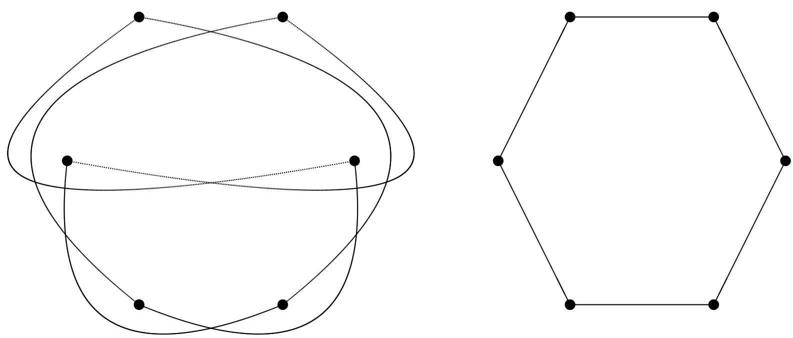
If, additionally, any two edges have disjoint interiors (i.e. allowed to meet only at vertices), the graph is called *planar*. Obviously this is a nice property, and if a graph does admit a planar representation, there is some merit in trying to find one.

However, for this problem we wish to consider a kind of extreme opposite to planarity. A *thrackle* is a representation of a graph where any pair of edges meet exactly once, either at a common vertex or, if there is no common vertex, at a crossing-point somewhere in their interiors. By a crossing-point we mean that edge E approaches edge F, say, meets it, and then continues on the other side of the F. We do not allow E to meet F tangentially.

1. Show that the cycle graphs C_2, C_3, C_5, C_6, ... have thrackle representations but that C_4 does not.

2. Investigate **Conway's conjecture**: a graph with more edges than vertices has no thrackle representation.

And for illustration, we show C_6 as a thrackle on the left and in its familiar planar representation on the right.



Women in mathematics

Eddie Kent

The film *Fermat's Room* concerns a small group of mathematicians who are drawn together to work on a special project. So that they should remain anonymous they are given name-tags to wear: Fermat for the host, Hilbert, Galois, Oliva and Pascal for the others. In my review [M500 **230**] of the film I mentioned that Oliva who, we were told, was a female mathematician was not known to me. I therefore tried to look her up, but could find no trace. This has now changed; possibly because of the film. Maria Vintro has put a note on the Net about a site—women-philosophers.com—which has some details of the life and work of Oliva Sabuco.

The lady was born in Alcaraz, Spain, and baptized on 2 December 1562. As she was taught by her father and brother, as well as by some family friends, it can be assumed that she had an ability that might have benefited from formal schooling. Most of her learning was in medicine, because that was her father's and brother's profession, but it was still far removed from what a nicely brought up girl should do. The family lived very close to a Dominican-run convent and since Oliva willed part of her house to the nuns we can assume that this was to thank them for their help, possibly with latin and philosophy.

In December 1580 she married Acacio de Buedo. It might be that Acacio knew Don Simon Abril, humanist, native of Alcaraz, and writer of works on grammar, logic, mathematics, human nature and law. It is said that Don Simon acted as mentor and possibly tutor to Oliva. In any case his publisher brought out Oliva's *New Philosophy of Human Nature, unknown and not reached by the ancients which improves human health and life.* In 1586 Oliva had petitioned the king for permission to publish her work which appeared the next year.

When he saw it begin to make money her father claimed authorship of the book. He gave his son power of attorney to publish it in Portugal, and put an 'anathema' on his daughter when she protested; this caused the Inquisition to prick up its ears, with the possibility of another witch to burn. In the end they failed to kill her but later editions of her book appeared duly expurgated by the Inquisition.

That's about the end of it. She lived, gave birth and died in due time, and was clearly never destitute; one assumes she had friends. Although the first edition of *New Philosophy* clearly carries Oliva's name, and records show that in February 1587 Oliva Sabuco payed the taxes due for publication, nevertheless some twentieth-century and later scholars still dispute its authorship. It is today attributed by the Biblioteca Nacional to 'Miguel Sabuco (antes Oliva)'; antes means 'before'.

At first sight there doesn't appear to be much mathematics in all this, though one should read her book to be certain, but perhaps Oliva is the nearest thing to an internationally known woman mathematician Spain has got.

What's the next number?

Tony Forbes

Recall that sequence at the bottom of M500 **210** page 28 and, later, at the top of M500 **213** page 22.

1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, 31131211131221, 13211311123113112211, 11131221133112132113212221, ...

If we look instead at the lengths of these numbers, we obtain a new sequence, d_n, whose first few terms are

1, 2, 2, 4, 6, 6, 8, 10, 14, 20, 26, 34, 46, 62, 78, 102, 134, 176, 226, 302, 408, 528, 678, 904, 1182, 1540, 2012, 2606, 3410, 4462, 5808, 7586, 9898, 12884, 16774, 21890, 28528, 37158, 48410, 63138, 82350, 107312, ...

Also in M500 **213** I observed that the ratio of consecutive terms of d_n is about 1.3.

I was surprised to discover that this 1.3 has been computed exactly. It is known as *Conway's constant* and I am sure you, like me, will be astonished to learn that it is the unique positive real root of the polynomial

$$\begin{aligned}
&x^{71} - x^{69} - 2x^{68} - x^{67} + 2x^{66} + 2x^{65} + x^{64} - x^{63} - x^{62} - x^{61} - x^{60}\\
&- x^{59} + 2x^{58} + 5x^{57} + 3x^{56} - 2x^{55} - 10x^{54} - 3x^{53} - 2x^{52} + 6x^{51}\\
&+ 6x^{50} + x^{49} + 9x^{48} - 3x^{47} - 7x^{46} - 8x^{45} - 8x^{44} + 10x^{43} + 6x^{42}\\
&+ 8x^{41} - 5x^{40} - 12x^{39} + 7x^{38} - 7x^{37} + 7x^{36} + x^{35} - 3x^{34} + 10x^{33}\\
&+ x^{32} - 6x^{31} - 2x^{30} - 10x^{29} - 3x^{28} + 2x^{27} + 9x^{26} - 3x^{25} + 14x^{24}\\
&- 8x^{23} - 7x^{21} + 9x^{20} + 3x^{19} - 4x^{18} - 10x^{17} - 7x^{16} + 12x^{15}\\
&+ 7x^{14} + 2x^{13} - 12x^{12} - 4x^{11} - 2x^{10} + 5x^{9} + x^{7} - 7x^{6} + 7x^{5}\\
&- 4x^{4} + 12x^{3} - 6x^{2} + 3x - 6.
\end{aligned}$$

Thus $\lim_{n\to\infty} d_n/d_{n-1}$ exists and is equal to 1.30357726903429639125 . . . . Needless to say, I haven't a clue as to where that 71th degree polynomial came from. It is mentioned in *The Book of Numbers* by John Conway.

Pronunciation

Eddie Kent

A question on a recent University Challenge was 'What is the smallest number that is the sum of two cubes in two different ways?' One contestant (no doubt a mathematics student) buzzed immediately with the correct answer, but I wondered, how fair a problem is that. You would not learn the answer routinely in any normal course—it is not general enough. In fact you could know the answer with such little effort in one of only two ways. Either you are Srinivasa Aiyangar Ramanujan, FRS, or you have read *A Mathematician's Apology* by G. H. Hardy. It is therefore a truly unfair question since to be on such a programme you might expect to have some knowledge from many areas, including mathematics. But not the anecdotage! On the other hand if you are a student of mathematics and do not know the answer I would suggest you get organized pretty quickly.

I would have forgotten all about this except that I listened to *Start the Week* (Radio 4) and heard Nitin Sawhney interviewed. He was about to perform at Sadlers Wells, and in the conversation he mentioned 'the Indian mathematician Ramanujan.' He pronounced the name with the accent on the third syllable. I, in common with most Western speakers I've heard, have always put it on the second syllable. I shall now assume that an Indian, even a dancer and musician, is likely to have it right.

TF writes — I have to confess that I do get irritated when I hear mathematicians using the third-syllable-stressed option to produce a ghastly Cockney/Essex sound, 'RamanOOjn'. Anyone who has read Robert Kanigel's book, *The Man Who Knew Infinity* will know that Kent and most Westerners have the right pronunciation.

Curiously, I am reliably informed that the correct pronunciation of my name is with two syllables, 'Forbees' rather than 'Forbs'—but that's another story.

Problem 232.7 – Zero

Show that

$$\cos\tfrac{1}{3}\pi + \frac{\cos\frac{2}{3}\pi}{2} + \frac{\cos\frac{3}{3}\pi}{3} + \frac{\cos\frac{4}{3}\pi}{4} + \ldots \;=\; 0.$$

How would Euler have pronounced 'Euclid'?

Twenty-five years ago

Here is something interesting by **Gareth Harries**, lifted from M500 **90** (January 1985, which would have been exactly 25 years ago if the parity of the M500 months hadn't changed at some time during the interval from then till now, February 2010) and which in turn was inspired by an article of Tony Gardiner in *The Mathematical Gazette.*

A finite sequence of integers $(N_0, N_1, N_2, \ldots, N_k)$ called *self-descriptive* if N_j is the number of js that occur in the sequence. For an example with $k = 6$, the sequence (3,2,1,1,0,0,0), more informally written as 3211000, is self-descriptive. Indeed, there are three zeros, two ones, one two, one three and none each of 4, 5 and 6.

Some things for you to do.

(i) Find all self-descriptive sequences with $k < 6$ (these are not easy).

(ii) Compute $\displaystyle\sum_{j=0}^{k} N_j$.

(iii) Find a sequence for each of $k = 6, 7, 8, 9$ and 10. (Hint: work backwards from N_k).

(iv) Find a general form of one sequence for each $k \geq 7$. Are these the only self-descriptive sequences?

(v) What happens if you start with N_1? What about N_r, $r < k$?

And I see that on the back cover of **90** there are three interesting problems, which will conveniently fill this page.

(90.1) This is also due to **Gareth Harries**. At a conference meeting the delegates were all given place settings equally spaced around a table. At the pre-conference hospitality session the delegates all became fatigued, so they all sat down randomly at their table. To avoid disturbing them, the table was rotated until the maximum number of delegates were seated in their correct positions. Is it always possible to place more than one delegate in the correct place?

(90.2) Investigate the probability that in any group of P people, there are D consecutive days of the year in which at least B of the people have their birthdays. The case $D = 1$ is well known.

(90.3) A region of the plane has two axes of symmetry. What other symmetries must it have?

Contents M500 232 – February 2010