*

ISSN 1350-8539

M500 233

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation.

Leibnitz's Formula for π

Sebastian Hayes

One of the peculiarities of π, the ratio of circumference of a circle to its diameter and thus a strictly geometric entity, is that it comes up in all sorts of unexpected places, thus giving rise to the belief, common amongst pure mathematicians, that Nature has a sort of basic kit of numbers, including notably π, e, i and γ that She applies here, there and everywhere. Buffon, the eighteenth-century French naturalist, worked out a formula giving the probability of a needle of length l dropped at random onto a floor ruled with parallel lines at unit intervals cutting at least one line. If l is less than a unit in length, the formula turns out to be $2l/\pi$ and this result has even been tested experimentally by a modern scientist, Kahan. Actually, in this case and very many others, there is a perfectly rational connection between the formula and the properties of circles, but I must admit that I am floored by the connection between π and the gamma function in the weird and rather beautiful result $\Gamma(1/2) = \sqrt{\pi}$.

The number π also turns up as the limit to various numerical series, a matter which in the past was of considerable importance as manufacturing methods required better and better estimates of the value of π. Today, computers have calculated the value of π to over a billion decimal places so the question of exactitude has become academic—although computers still use formulae originally discovered by pure mathematicians such as Euler or Ramanujan.

Leibnitz, co-inventor of the Calculus, produced several centuries ago, somewhat out of a hat, the remarkable series

$$\frac{\pi}{4} \;=\; 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

British mathematicians, eager to give as much credit as possible to Newton, pointed out that a Scot, Gregory, had already derived, using Newton's version of the calculus, the formula

$$\tan^{-1} x \;=\; x - \frac{1}{3}x^2 + \frac{1}{5}x^2 \ldots$$

and that you obtain Leibnitz's formula by setting $x = 1$.

However, apart from the question of priority, one might reasonably wonder why it should be necessary to bring in calculus to validate such a simple-looking series. A problem in so-called *elementary* number theory should, so I feel at any rate, make no appeal to the methods of analysis or any other 'higher' mathematics but rely uniquely on the properties of the natural numbers. I feel so strongly about this that I had at one point even thought of offering a small money reward for a strictly numerical proof of Leibnitz's famous series, but I am glad I did not do so, since I have subsequently come across one in Hilbert's excellent book, *Geometry and the Imagination.*

The complete proof is not at all easy—'elementary' proofs in Number Theory are not necessarily simple, far from it—but the general drift of the argument is straightforward enough.

Consider a circle whose centre is at the origin with radius r, a positive integer. The formula for the circle is thus $x^2 + y^2 = r^2$. We mark off lattice points to make a network of squares (or use squared paper), and take each lattice as having a side of unit length. For any given choice of circle (with $r > 1$), there will be squares which 'overlap', part of the square falling within the circumference and part falling outside the circumference and a single point counts as 'part' of a square.

We define a function $f(r)$ with r a positive integer to be the sum total of all lattice squares where the bottom left hand corner of the lattice is either inside or on the circumference of a circle radius r. (Any other criterion, such as counting a square 'when there is more than half its area inside the circle', would do so long as we stick to it, but there are good reasons for choosing this 'left hand corner' criterion, as will shortly be apparent.)

It is not clear at a glance whether the lattice area, evaluated according to our left hand corner criterion, is larger or smaller than the true area of the circle. However, as we make the lattices smaller and smaller, i.e. increase r, we expect the difference to diminish progressively. Thus $f(1) = 5$ — remember we are counting the squares where only the left hand corner point lies on the circumference. I make $f(2)$ come to 13 and $f(3)$ come to 29, while the two higher values given below are taken from Hilbert's book *Geometry and the Imagination*:

$$\begin{aligned} f(2) &= 13, \\ f(3) &= 29, \\ f(10) &= 317, \\ f(100) &= 31417. \end{aligned}$$

The absolute value of the difference between the lattice area, $f(r)$, evaluated simply by counting the relevant lattices, and the area of the circle, πr^2, is $|f(r) - \pi r^2|$. If we use $f(r)$ as a rough and ready estimate of the area of the circle and divide by r^2 we thus get an estimate of the value of π obtaining

$$\begin{aligned} \pi &\approx 13/4 = 3.125, \\ \pi &\approx 29/9 = 3.222222\ldots, \\ \pi &\approx 317/100 = 3.17, \\ \pi &\approx 31417/10000 = 3.1417. \end{aligned}$$

Now, since the diagonal of a unit square lattice is $\sqrt{2}$, all the 'borderline cases' will be included within a circular annulus bounded within by a circle of radius of $r - \sqrt{2}$ and without by a circle of radius $r + \sqrt{2}$. The area of this annulus is the difference between the larger and smaller circles, i.e.

$$\big((r + \sqrt{2})^2\pi - (r - \sqrt{2})^2\pi\big) = 4\sqrt{2}\pi r.$$

But $|f(r) - \pi r^2|$, the discrepancy between the lattice area and the area of the circle, is bound to be less than the annulus area since some lattices falling within the annulus area get counted in $f(r)$, and certainly $f(r)$ cannot be greater than the annulus area. Thus

$$|f(r) - \pi r^2| \le 4\sqrt{2}\pi r$$

which, dividing right through by r^2, gives

$$\left|\frac{f(r)}{r^2} - \pi\right| \le \frac{4\sqrt{2}\pi}{r}. \tag{i}$$

Assuming Cartesian coordinates with 0 as the centre of the circle, for any value of r there will be a certain number of points which lie on the circumference of the circle, those points (x, y) which satisfy the equation $(x^2 + y^2) = r^2$, where r is a positive integer. But we must count all the negative values of x and y as well. For example, with $r = 2$, the circumference will pass through the lattice points $(2, 0)$, $(-2, 0)$, $(0, 2)$ and $(0, -2)$ and no others.

We now introduce a new variable $n = r^2$ making the radius $\sqrt{n}$, and the equation of the circle becomes $x^2 + y^2 = n$. Although n must be an integer, we lift the restriction on r so that the radius is not necessarily an integer, e.g. $r = \sqrt{7}$, $r = \sqrt{13}$ and so on.

Now, the number of lattice points on the circumference of a circle with radius $\sqrt{n}$ is equivalent to four times the number of ways that an integer n can be expressed as the sum of two squares—four times because we allow x and y to take minus values. This is strictly a problem in Number Theory, and we have an important theorem.

> *The number of ways in which an integer can be expressed as the sum of the squares of two integers is equal to four times the excess of the number of factors of n having the form $4k+1$ over the number of factors having the form $4k+3$.*

Take $35 = 5 \cdot 7$. We have as factors of 35 : 1, 5, 7 and 35 which are respectively $1, 1, 3, 3 \pmod 4$. Since there are two of each type and $2-2=0$, there is no excess of the $(4k+1)$ type and so, if the theorem is correct, 35 cannot be represented as the sum of two squares, which is the case.

The proof of the theorem is quite complicated and will not be attempted here. What we can show at once is that

> *no prime p which is* $3 \pmod 4$ *can be represented as the sum of two (integer) squares.*

This is so because any odd number, whether it be 1 or $3 \pmod 4$, will be $1 \pmod 4$ when squared. And every even number, whether 2 or 0 (mod 4) will be $0 \pmod 4$ when squared. So if m happens to be $3 \pmod 4$ like 7 or 11, it will have no representation as the sum of two squares, i.e. the equation $a^2 + b^2 = 3 \pmod m$ is insoluble in integers.

However, if p prime is $1 \pmod 4$ it may be possible to find a representation in two squares since $(4k+1)^2 + \text{even}^2 = 1 \pmod 4$ is possible. A theorem given by Fermat, which goes some way towards establishing the principal theorem, states that

> *an odd prime p is expressible as the sum of two squares if and only if $p = 1 \pmod 4$.*

The 'if' part means that every odd prime p such as 5, 13, 17 and so on can be expressed as the sum of two squares; $13 = 3^2 + 2^2$ for example and $17 = 4^2 + 1^2$.

From our point of view, any representation such as $5 = 1^2 + 2^2$ gives us eight lattice points, four for the different ways of forming $1^2 + 2^2$ and four for the different ways of forming $2^2 + 1^2$; i.e. the lattice points with coordinates

$$(1,2),\ (1,-2),\ (-1,2),\ (-1,-2)$$

and those with coordinates

$$(2,1),\ (2,-1),\ (-2,1),\ (-2,-1).$$

But $65 = 5 \cdot 13$ has factors, 1, 5, 13 and 65 all of which are positive integers which are $1 \pmod 4$. There should, then, be four different ways of representing 65 as the sum of two squares, where the order in which we write the two squares matters. And in effect we have

$$65 \;=\; 1^2+8^2 \;=\; 8^2+1^2 \;=\; 4^2+7^2 \;=\; 7^2+4^2.$$

We end up with eight lattice points for each combination, namely

$$(1,8),\ (-1,8),\ (1,-8),\ (-1,-8), \qquad (8,1),\ (8,-1),\ (-8,1),\ (-8,-1)$$

and

$$(4,7),\ (-4,7),\ (4,-7),\ (-4,-7), \qquad (7,4),\ (7,-4),\ (-7,4),\ (-7,-4).$$

The idea now is that, by considering every number $n \le r^2$, working out how many times it can be expressed as a sum of two squares and adding the results, we will obtain $f(r)$ on multiplying by 4. Actually, this would include the origin, the point $(0,0)$, which we do not want to consider, so, excluding this, we have

$$(f(r)-1) \;=\; 4\sum \text{representations of } n \le r^2 \text{ as two squares.}$$

Now 1 has a representation since $1^2 = 1^2 + 0^2$ giving the four points $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$; $2 = 1^2 + 1^2$ has a representation giving four points, 3 none and $4 = 2^2 + 2^2$ gives four points producing twelve in all. I made $f(2) = 13$, which checks out with the above since $(f(2)-1) = 12$.

Actually, rather than work out the excess for each number n individually, it is much more convenient to add up the number of factors of all numbers of the form $4k+1$ and then subtract the number of factors of all numbers of the form $4k+3$. In the first list we have $1, 5, 9, \ldots, 4k+1 \le r^2$ and in the second, $3, 7, 11, \ldots, 4k+3 \le r^2$. Each of these numbers must appear in the total for its class as many times as there are multiples of it that are at most r^2. Clearly 1 will obviously appear r^2 times, but 5 will only appear $[r^2/5]$ times, where the square brackets indicate the nearest integer $\le r^2/5$.

Finally, since we are not removing or adding anything, we can subtract the first term in the $4k+3$ category from the first term in the $4k+1$ category, the second term from the second and so on. We end up with the open-ended series, depending on the choice of r

$$\begin{aligned}(f(r)-1) \;&=\; 4\sum \text{representations of } n \le r^2 \text{ as two squares} \\ &=\; 4\left([r^2] - \left[\frac{r^2}{3}\right] + \left[\frac{r^2}{5}\right] - \left[\frac{r^2}{7}\right] + \ldots\right). \qquad \text{(ii)}\end{aligned}$$

Now the 'least integer' series $[r^2] - [r^2/3] + [r^2/5] - [r^2/7] + \ldots$, unlike the series $r^2 - r^2/3 + r^2/5 - r^2/7 + \ldots$, is not an infinite series since it terminates as soon as we reach the point where $r^2/(4k+1) < 1$ making all subsequent terms 0. We assume for simplicity that r is odd and of the type $4k+1$ so that $r-1$ is a multiple of 4. Since all the terms with $4k+1$ as denominator are positive, we can split the series into two, and then add up the pairs, where the first member of a pair is taken from the + and the second from the − series. The '+ series' contains $[r^2/r]$ and the final non-zero term is $[r^2/r^2]$.

$$\left[\frac{r^2}{1}\right] + \left[\frac{r^2}{5}\right] + \left[\frac{r^2}{9}\right] + \cdots + \left[\frac{r^2}{r}\right] + \left[\frac{r^2}{r+4}\right] + \cdots + \left[\frac{r^2}{r^2}\right],$$

$$0 + \left[\frac{r^2}{3}\right] + \left[\frac{r^2}{7}\right] + \left[\frac{r^2}{11}\right] + \cdots + \left[\frac{r^2}{r-2}\right] + \left[\frac{r^2}{r+2}\right] + \cdots + \left[\frac{r^2}{r^2-2}\right].$$

If we cut off the series at $[r^2/r]$ the error involved, namely the rest of the original series, will be less than r, or αr where α is some proper fraction, i.e.

$$\left[\frac{r^2}{r+4}\right] - \left[\frac{r^2}{r+2}\right] + \cdots + \left[\frac{r^2}{r^2}\right] - \left[\frac{r^2}{r^2-2}\right] \;<\; r.$$

To see this, we write all terms after $[r^2/r]$ as $[r^2/(r+k)]$, where k is even and ranges from 2 to $r^2 - r$, since $r + r^2 - r$ is the denominator of the final non-zero term. The absolute values of all these terms are less than $[r^2/r] = r$ and they come in pairs which alternate in sign. Also, all terms where $2(r+k) > r^2$ or $k > (r^2/2 - r) = r(r-2)/2$ will make $[r^2/(r+k)] = 1$. The first such term comes when $k = r(r-2)/2 + 1/2$ (since k is even) i.e. when $k = (r-1)^2/2$. From this point on all pairs will sum to zero so we can ignore them and only need consider the pairs between $[r^2/r]$ and ending $[r^2/(r + (r-1)^2/2)]$. There will be $(r-1)^2/8$ such pairs with a maximum difference of 1 in each case, and so the sum total of the error cannot exceed $(r-1)^2/8 < r$ since $(r-1)^2 < 8r$ for $r \geq 2$.

An example may make this more intelligible. Take $r = 9$, which is a number of the form $4k+1$. Then $[9^2/9] = 9$ and all terms from then on have their absolute values < 9 while the final last term is $[9^2/9^2] = [9^2/(9+72)]$. The last term where $[9^2/(9+k)] \geq 2$ comes when $k = 30$ and we can neglect all pairs where k has values > 32 (we make the last value $k = 32$ to make up the pair); k even thus ranges from 2 to 32:

$$2,\ 6,\ 10,\ \ldots,\ 30, \qquad 4,\ 8,\ 12,\ \ldots,\ 32.$$

The maximum absolute amount possible will thus be $32/4 = 8$ (in this case -8) and $8 < 9 = r$. A similar argument can be used to establish the case where r is odd and of the form $4k + 3$ and any even value of r will be sandwiched between the two cases. We thus have, returning to (ii),

$$\frac{f(r)-1}{4} = [r^2] - \left[\frac{r^2}{3}\right] + \left[\frac{r^2}{5}\right] - \left[\frac{r^2}{7}\right] + \left[\frac{r^2}{9}\right] \pm \left[\frac{r^2}{r}\right] \pm \alpha r,$$

where $0 \le \alpha < 1$.

To lift the square brackets, we note that the error in each term is less than 1 and that there will be, for r odd, $(r+1)/2$ terms if we cut the series off at $[r^2/r]$. The total possible error is thus $< (r+1)/2 < r$ for $r \ge 2$ and can be written as $\pm\beta r$, where $\beta < 1$. We can thus write

$$\frac{f(r)-1}{4} = r^2 - \frac{r^2}{3} + \frac{r^2}{5} - \frac{r^2}{7} + \frac{r^2}{9} \pm \alpha r \pm \beta r.$$

Dividing right through by r^2 we obtain

$$\frac{1}{4}\left(\frac{f(r)}{r^2} - \frac{1}{r^2}\right) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \pm \frac{\alpha}{r} \pm \frac{\beta}{r},$$

which has limit as $r \to \infty$, $f(r)/(4r^2) = 1 - 1/3 + 1/5 - 1/7 + \ldots$.

Finally, we note that the discrepancy between the area of the circle and the lattice representation is $|f(r)/r^2 - \pi| \le 4\sqrt{2}\pi/r$ with limit 0 as $r \to \infty$, giving us the desired limit $1 - 1/3 + 1/5 - 1/7 + \cdots = \pi/4$.

As I see things, π is not a unique number in the way in which 5 or 27/6 are unique numbers, π is basically a label given to a whole family of convergent numerical series—there are any number of 'infinite' series which are 'limit equivalent' to $4\,(1 - 1/3 + 1/5 - 1/7 + \cdots \pm 1/(2r+1))$, for example the one we can elicit from the remarkable continued fraction

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ldots}}}}.$$

What did the mathematician say after a heavy lunch?

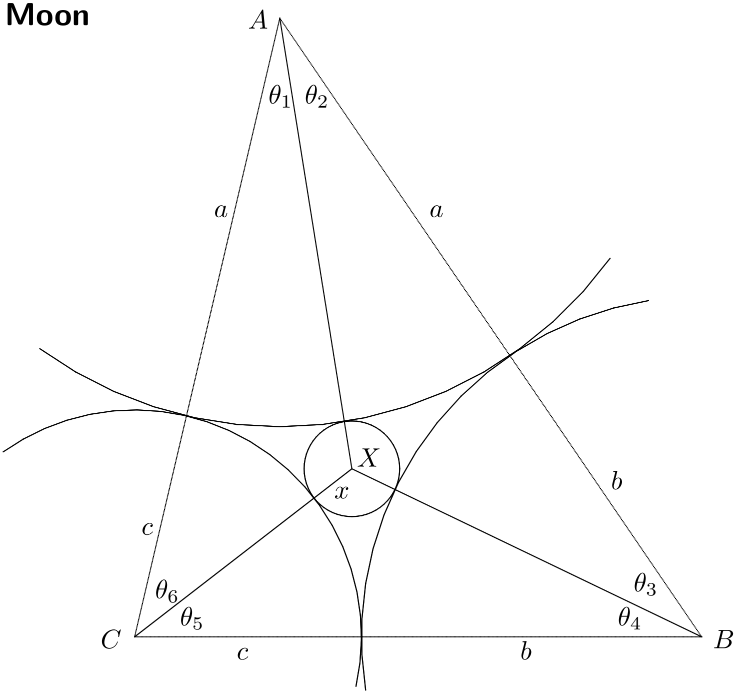
"$\sqrt{-1/64}$."

Solution 226.5 – Three circles

Three circles touch each other externally and have radii a, b and c. A fourth circle of radius x touches the other three externally. Show that

$$\sqrt{\frac{a+b+x}{c}}+\sqrt{\frac{b+c+x}{a}}+\sqrt{\frac{c+a+x}{b}} = \sqrt{\frac{a+b+c}{x}}.$$

Steve Moon



First use the cosine rule with angles θ_1, θ_2 and $\theta_1+\theta_2$:

$$\cos\theta_1 = \frac{(a+c)^2+(a+x)^2-(c+x)^2}{2(a+c)(a+x)} = 1-\frac{2cx}{(a+c)(a+x)}.$$

Similarly

$$\cos\theta_2 = 1-\frac{2bx}{(a+b)(a+x)}$$

and

$$\cos(\theta_1+\theta_2) = 1-\frac{2bc}{(a+b)(a+c)}.$$

Hence

$$\sin\theta_1 = \frac{2\sqrt{acx(a+c+x)}}{(a+c)(a+x)}$$

and

$$\sin\theta_2 = \frac{2\sqrt{abx(a+b+x)}}{(a+b)(a+x)}.$$

We now have expressions for all terms in the identity

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2,$$

or

$$\sin\theta_1\sin\theta_2 = \cos\theta_1\cos\theta_2 - \cos(\theta_1+\theta_2). \qquad (1)$$

The left-hand side is

$$\sin\theta_1\sin\theta_2 = \frac{4abcx}{(a+b)(a+c)(a+x)^2}\sqrt{\frac{a+c+x}{b}}\sqrt{\frac{a+b+x}{c}} \qquad (2)$$

and on the right of (1) we have

$$\begin{aligned}\cos\theta_1\cos\theta_2 - \cos(\theta_1+\theta_2) &= \left(1-\frac{2cx}{(a+c)(a+x)}\right)\left(1-\frac{2bx}{(a+b)(a+x)}\right)\\ &\quad - \left(1-\frac{2bc}{(a+b)(a+c)}\right)\\ &= \frac{2abcx}{(a+b)(a+c)(a+x)^2}\left(\frac{a}{x}+\frac{x}{a}-\frac{a}{c}-\frac{x}{c}-\frac{a}{b}-\frac{x}{b}\right). \qquad (3)\end{aligned}$$

Hence, cancelling the common factors on each side of (2) and (3), we get

$$2\sqrt{\frac{a+c+x}{b}}\sqrt{\frac{a+b+x}{c}} = \frac{a}{x}+\frac{x}{a}-\frac{a}{c}-\frac{x}{c}-\frac{a}{b}-\frac{x}{b}. \qquad (4)$$

Then repeat the process using the cosine rule on θ_3, θ_4, $\theta_3+\theta_4$ and then θ_5, θ_6, $\theta_5+\theta_6$:

$$2\sqrt{\frac{b+a+x}{c}}\sqrt{\frac{b+c+x}{a}} = \frac{b}{x}+\frac{x}{b}-\frac{b}{a}-\frac{x}{a}-\frac{b}{c}-\frac{x}{c} \qquad (5)$$

and

$$2\sqrt{\frac{c+b+x}{a}}\sqrt{\frac{c+a+x}{b}} = \frac{c}{x}+\frac{x}{c}-\frac{c}{b}-\frac{x}{b}-\frac{c}{a}-\frac{x}{a}. \qquad (6)$$

Now add (4), (5) and (6); some cancellation and grouping of terms on the right gives

$$2\sqrt{\frac{a+c+x}{b}}\sqrt{\frac{a+b+x}{c}}+2\sqrt{\frac{b+a+x}{c}}\sqrt{\frac{b+c+x}{a}}$$

$$+2\sqrt{\frac{c+b+x}{a}}\sqrt{\frac{c+a+x}{b}} = \frac{a+b+c}{x}-\frac{b+c+x}{a}-\frac{a+c+x}{b}-\frac{a+b+x}{c},$$

which after rearranging becomes

$$\frac{b+c+x}{a}+\frac{a+c+x}{b}+\frac{a+b+x}{c}+2\left(\sqrt{\frac{a+c+x}{b}}\sqrt{\frac{a+b+x}{c}}\right.$$

$$\left.+\sqrt{\frac{b+a+x}{c}}\sqrt{\frac{b+c+x}{a}}+\sqrt{\frac{c+b+x}{a}}\sqrt{\frac{c+a+x}{b}}\right) = \frac{a+b+c}{x},$$

or

$$\left(\sqrt{\frac{b+c+x}{a}}+\sqrt{\frac{a+c+x}{b}}+\sqrt{\frac{a+b+x}{c}}\right)^2 = \frac{a+b+c}{x},$$

as required.

Problem 233.1 – Hill

A cannon of mass M fires a shot of mass m to hit a target at distance a. At distance b in the line of fire there is a hill of height h. Assuming that the shell just clears the hill before going on to strike the target, prove that the gun must have been aiming at an angle of

$$\arctan\left(\frac{M}{M+m}\cdot\frac{ah}{b(a-b)}\right)$$

to the horizontal. Assume also that this takes place in a vacuum on a planet where gravity always acts vertically downwards and that the gun, the base of the hill from which its height is measured and the target all lie in the same horizontal plane.

If that was too easy, obtain a formula which works for a spherical planet of radius R, say. (Ideally the planet should have an atmosphere, but we would still nevertheless be interested if you want to neglect air resistance.)

Differential calculus in C_3 space

Dennis Morris

The nature of the 3-dimensional C_3 geometric space is such that it does not have a 2-dimensional subspace. This is because the 3-dimensional complex numbers do not have a 2-dimensional subalgebra, and this derives from the fact that the order 3 cyclic group C_3 does not have an order 2 subgroup. One of the consequences of this is that we cannot partially differentiate in the C_3 algebra by holding one variable constant because doing so is to assume the existence of a 2-dimensional plane in the 3-dimensional space.

Consider the function $z = xy$ (in $\mathbb{R}^3$). We can choose the 2-dimensional (x, z) plane by holding y constant and differentiate within that (x, z) plane. Doing so from first principles gives

$$\begin{aligned} z + \delta z &= (x + \delta x)y \;=\; xy + y \cdot \delta x, \\ \delta z &= y \cdot \delta x, \\ \frac{\partial z}{\partial x} &= y. \end{aligned}$$

However, in C_3 space, we cannot hold y constant but must allow that it varies as x and z vary. We therefore have

$$\begin{aligned} z + \delta z &= (x + \delta x)(y + \delta y) \;=\; xy + x \cdot \delta y + y \cdot \delta x + \delta x \delta y, \\ \delta z &= x \cdot \delta y + y \cdot \delta x, \\ \frac{\partial z}{\partial x} &= y + \frac{\partial y}{\partial x}, \end{aligned} \tag{1}$$

where we have taken $\delta x \delta y$ to be infinitesimal. If we integrate both sides of (1) with respect to x, we get back to the original equation, and so the differentiation is reversible.

There is an alternative view that we differentiate at an infinitesimal point and that the difference between the 3-dimensional infinitesimal point and a 2-dimensional infinitesimal point is infinitesimal and that, even in $\mathbb{R}^3$, we do not differentiate in a 2-dimensional plane when we hold y constant it is misguided teachers drawing misguided diagrams on innocent blackboards that belies this fact.

Problem 233.2 – Three secs

Show that

$$\sec^4 \frac{\pi}{7} + \sec^4 \frac{2\pi}{7} + \sec^4 \frac{3\pi}{7} \;=\; 416.$$

Solution 230.4 – Tanks

> There are an odd number of tanks in a field. At the appointed instant each tank fires a shell at its nearest neighbour. Prove that if all the distances are distinct, then there is at least one tank which escapes being shot at. It is easy to show that the distinct distances part is essential. It is also easy to see that oddness is essential; otherwise, for instance, we could have pairs of nearest neighbours separated from each other by great distances.

Ian Adamson

Suppose that there are n tanks and also suppose to the contrary that each tank is shot at exactly once. Let D be an $n \times n$ matrix (d_{ij}), where d_{ij} is the distance from tank i to tank j and set $d_{ii} = \infty$.

(1) Circle the minimal distance in each row and note that $d_{ij} = d_{ji}$.

(2) Now let A be an $n \times n$ matrix (a_{ij}) satisfying (i) $a_{ij} \in \{0, 1\}$; and (ii) $a_{i,j} = 1$ if and only if d_{ij} is circled in D, so that there is exactly one non-zero element in each row.

(3) Since each of n tanks is shot at whenever n shots are fired, there is exactly one non-zero element in each column.

(4) From (1) we have $a_{ij} = a_{ji}$.

It is easy to see that criteria (2) to (4) are satisfied only if n is even.

This works even if the tanks are in an m-dimensional continuum, where m is any positive whole number, although $m > 1$ avoids cross-fire.

Steve Moon

We need only concern ourselves with the set of distances which are the shortest distance from a tank i to its nearest neighbour j, d_{ij}

Let there be $2n + 1$ tanks, since it is an odd number. All distances are distinct. Hence the set of distances d_{ij}, where $1 \le i, j \le 2n + 1$ and $i \ne j$, has a least member, say j_{12}. So tanks 1 and 2 fire at each other.

If for any tank i, $i \ge 3$, its nearest neighbour was either tank 1 or 2, it will fire at that tank. So either tank 1 or 2 will be fired at twice. Since a tank can only fire once, there must be a tank that is not shot at, and we are done.

If that is not the case, we can ignore tanks 1 and 2, and look at the set of $2n-1$ remaining tanks. Repeating the exercise on the set d_{ij} for these tanks, two will fire at each other and either one other tank will fire at them, or we can eliminate them and look at the remaining $2n-3$ tanks.

Continuing in this way, since $2n-3$, $2n-5$, etc. are all odd, we will eventually be left with one tank that will fire at either of the last two 'nearest neighbours' and hence escape unscathed—if we get that far.

Solution 229.2 – Tank

> I am driving a tank and I have to make a circular tour of various military bases along a given route. I can arrange to have my tank transported to a starting point of my choice. Initially my (fuel) tank is empty, but distributed along the route there is sufficient fuel to complete my tour. Show that I can choose my starting point so that I can complete the whole journey and return to the waiting tank-transporter without running out of fuel.

We have had one or two contributions solving this problem but not in the simplistic way we were looking for. Assuming it is of interest we give our solution here.

Let the tank do an imaginary journey starting at some arbitrary point on the route. Plot a graph of fuel in the tank against distance travelled. This graph starts at zero but must finish at a non-negative value (although at points in between it might go negative—which is why the journey has to be imaginary). Now choose as the start of the real tour the point on the graph where the fuel level is at its minimum.

Problem 233.3 – Six tans

Let $k = \pi/13$. Show that

$$\tan k \tan 2k \tan 3k \tan 4k \tan 5k \tan 6k = \sqrt{13}.$$

Problem 233.4 – Three tans

Show that the cubic $k^3 - 21k^2 + 35k - 7 = 0$ has roots

$$\tan^2 \frac{\pi}{7}, \quad \tan^2 \frac{2\pi}{7} \quad \text{and} \quad \tan^2 \frac{3\pi}{7}.$$

Solution 228.1 – Odd expression

Show that the expression

$$\left\lfloor \left(3+\sqrt{5}\right)^n \right\rfloor$$

is odd for non-negative integer n.

Tony Forbes

The case $n = 0$ is left to the reader. Suppose henceforth that $n \geq 1$.

Let

$$\alpha \;=\; \frac{\sqrt{5}+1}{2} \;\approx\; 1.61803398874989484820,$$

the golden ratio, and let

$$\beta \;=\; \frac{1}{\alpha} \;=\; \frac{\sqrt{5}-1}{2} \;\approx\; 0.61803398874989484820.$$

Consider the sequence defined by $s_n = \alpha^{2n} + \beta^{2n}$, $n = 1, 2, 3, \ldots$, the first few terms of which are given by

$$3, 7, 18, 47, 123, 322, 843, 2207, 5778, 15127, 39603, 103682, 271443, \ldots.$$

Obviously s_n is going to be an integer. One way of proving this is to observe that it would follow by induction from the fact that

$$s_{n+2} \;=\; 3s_{n+1} - s_n. \tag{$*$}$$

Since $\alpha^2 = 1 + \alpha$ and $\beta^2 = 1 - \beta$, we have

$$\alpha^4 \;=\; \alpha^2 + 2\alpha + 1 \;=\; 3\alpha^2 - 1, \quad \text{and} \quad \beta^4 \;=\; \beta^2 - 2\beta + 1 \;=\; 3\beta^2 - 1.$$

Hence

$$\begin{aligned} s_{n+2} &= \alpha^{2n+4} + \beta^{2n+4} \;=\; \alpha^{2n}\left(3\alpha^2 - 1\right) + \beta^{2n}\left(3\beta^2 - 1\right) \\ &= 3(\alpha^{2n+2} + \beta^{2n+2}) - \alpha^{2n} - \beta^{2n} \;=\; 3s_{n+1} - s_n \end{aligned}$$

and thus $(*)$ is proved.

Having established that s_n is an integer, it follows that

$$2^n s_n \;=\; \left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n$$

is an *even* integer. Furthermore, $0 < \left(3-\sqrt{5}\right)^n < 1$. Thus $\left\lfloor \left(3+\sqrt{5}\right)^n \right\rfloor$ is the even integer $2^n s_n$ minus something which is positive and smaller than 1. Hence $\left\lfloor \left(3+\sqrt{5}\right)^n \right\rfloor$ is odd.

Also solved (in an essentially similar manner) by **Sebastian Hayes**.

Solution 211.5 – Product

Prove this interesting relation between π^2 and e^2:

$$\frac{\left(\frac{\pi^2}{4}+1\right)\left(\frac{\pi^2}{4}+\frac{1}{9}\right)\left(\frac{\pi^2}{4}+\frac{1}{25}\right)\cdots}{\left(\frac{\pi^2}{4}+\frac{1}{4}\right)\left(\frac{\pi^2}{4}+\frac{1}{16}\right)\left(\frac{\pi^2}{4}+\frac{1}{36}\right)\cdots} = \frac{e^2+1}{e^2-1}.$$

Steve Moon

First note that the right-hand side is $(\cosh 1)/(\sinh 1) = \coth 1$.

For the left-hand side, by Weierstrass's factorization theorem, using the fact that sin and cos are entire functions, we have

$$\sin \pi w = \pi w \prod_{n=1}^{\infty}\left(1-\frac{w^2}{n^2}\right) \quad \text{and} \quad \cos \pi w = \prod_{n=1}^{\infty}\left(1-\frac{4w^2}{(2n-1)^2}\right),$$

$w \in \mathbb{C}$. Replace w by iz/π. Then

$$\sin iz = i \sinh z = iz \prod_{n=1}^{\infty}\left(1+\frac{z^2}{\pi^2 n^2}\right);$$

hence

$$\sinh z = z \prod_{n=1}^{\infty}\left(1+\frac{z^2}{\pi^2 n^2}\right),$$

and similarly

$$\cosh z = \prod_{n=1}^{\infty}\left(1+\frac{4z^2}{\pi^2 (2n-1)^2}\right).$$

Therefore

$$\coth z = \frac{\prod_{n=1}^{\infty}\left(1+\frac{4z^2}{\pi^2(2n-1)^2}\right)}{z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{\pi^2 n^2}\right)} = \frac{\prod_{n=1}^{\infty}\left(1+\frac{4z^2}{\pi^2(2n-1)^2}\right)}{z\prod_{n=1}^{\infty}\left(1+\frac{4z^2}{\pi^2 (2n)^2}\right)},$$

to get the denominator in the right form. Now we put $z = 1$ to obtain

$$\coth 1 = \frac{\prod_{n=1}^{\infty}\left(1+\frac{4}{\pi^2(2n-1)^2}\right)}{\prod_{n=1}^{\infty}\left(1+\frac{4}{\pi^2 (2n)^2}\right)} = \frac{\prod_{n=1}^{\infty}\left(\frac{\pi^2}{4}+\frac{1}{(2n-1)^2}\right)}{\prod_{n=1}^{\infty}\left(\frac{\pi^2}{4}+\frac{1}{(2n)^2}\right)},$$

as required.

Solution 215.2 – Three angles

$$\text{If } A+B+C=180°, \text{ show that } \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix} = 0.$$

Steve Moon

Subtracting the first column from the third (and using $1 = \sin^2\theta + \cos^2\theta$) we see that the determinant becomes

$$\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix} = \begin{vmatrix} \sin^2 A & \cot A & \cos^2 A \\ \sin^2 B & \cot B & \cos^2 B \\ \sin^2 C & \cot C & \cos^2 C \end{vmatrix}.$$

Then expand down the middle column:

$$\begin{aligned} & -\cot A\,(\sin^2 B\cos^2 C - \sin^2 C\cos^2 B) \\ & +\cot B\,(\sin^2 A\cos^2 C - \sin^2 C\cos^2 A) \\ & -\cot C\,(\sin^2 A\cos^2 B - \sin^2 B\cos^2 A) \end{aligned}$$

$$\begin{aligned} = \;& \cot A\ \sin(C-B)\sin(C+B) \;+\; \cot B\ \sin(A-C)\sin(A+C) \\ & + \;\cot C\ \sin(B-A)\sin(B+A). \end{aligned}$$

Since $A+B+C = 180°$, we have $\sin(A+B) = \sin C$, $\sin(B+C) = \sin A$ and $\sin(C+A) = \sin B$. Hence the determinant simplifies to

$$\cos A\sin(C-B) + \cos B\sin(A-C) + \cos C\sin(B-A).$$

When this is expanded we obtain

$$\begin{aligned} & \cos A\,(\sin C\cos B - \cos C\sin B) \;+\; \cos B\,(\sin A\cos C - \cos A\sin C) \\ & \quad + \;\cos C\,(\sin B\cos A - \cos B\sin A) \end{aligned}$$

and in this last expression everything cancels to leave zero.

Problem 233.5 – Croquet

A croquet hoop made of wire of diameter 1 has an opening of width w and is set into the (x, y) plane with the opening occupying the interval from $(-w/2, 0)$ to $(w/2, 0)$. A croquet ball has diameter $d < w$. What is the set of points from which the ball, when struck in a non-spin-inducing manner, will eventually go through the hoop, possibly after bouncing off its uprights a number of times. (See L. Carroll, *Alice's Adventures in Wonderland* for further details about croquet.)

Problem 233.6 – The quartic and the golden mean

A quartic polynomial has two points of inflection, with x coordinates p and q. Show that straight line which passes through the points of inflection meets the quartic again at two points with x coordinates

$$\frac{\sqrt{5}+1}{2}\,p \;-\; \frac{\sqrt{5}\;-\;1}{2}\,q \quad \text{and} \quad \frac{\sqrt{5}+1}{2}\,q \;-\; \frac{\sqrt{5}\;-\;1}{2}\,p.$$

Thanks to **Robin Whitty** for communicating to me (TF) this interesting and surprising appearance of the golden ratio.

Problem 233.7 – Cyclic quadrilateral

Robin Whitty

A convex quadrilateral of sides a, b, c and d is inscribed in a circle of radius 1. What is d in terms of a, b and c?

M500 Mathematics Revision Weekend 2010

The thirty-sixth **M500 Society Mathematics Revision Weekend** will be held at

Aston University, **Birmingham**

over

Friday 10**th** – **Sunday** 12**th** **September 2010**.

The cost, including accommodation (with en suite facilities) and all meals from bed and breakfast Friday night to lunch Sunday is £250 (in Aston's Lakeside flats) or £298 (Aston Business School), The cost for non-residents is £120 (includes Saturday and Sunday lunch). M500 members get a discount of £10. For full details and an application form, see the Society's web site at www.m500.org.uk, or send a stamped, addressed envelope to

Jeremy Humphries, M500 Weekend 2010.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most mathematics-based OU courses, subject to sufficient numbers.

Contents — M500 233 – April 2010