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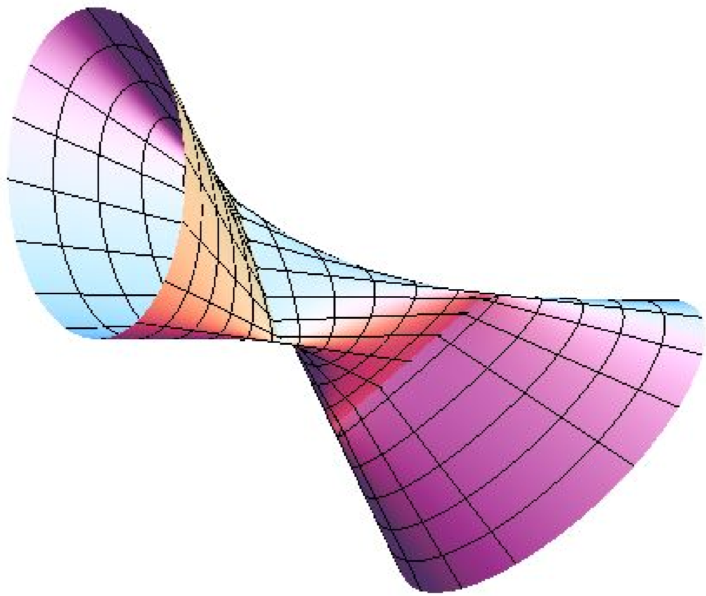
M500 234



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation.

Upon three-dimensional rotation

Dennis Morris

The rotation matrix of 2-dimensional euclidean space is

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

When $\theta = 2n\pi$, this is the identity matrix. And so it is that rotation through $2n\pi$ returns to the starting point. This is not the case in all geometric spaces; the rotation matrix of hyperbolic space is

$$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}.$$

And no value of θ other than zero will produce the identity matrix.

The 3-dimensional trigonometric functions of the $C_3L^1H^2_{(j=1,k=1)}$ geometric space are

$$\begin{aligned}
\nu A &= \frac{1}{3}\left(e^{x+y} + 2e^{-(x+y)/2}\cos\left(\frac{\sqrt{3}}{2}(x-y)\right)\right), \\
\nu B &= \frac{1}{3}\left(e^{x+y} - 2e^{-(x+y)/2}\cos\left(\frac{\sqrt{3}}{2}(x-y)+\frac{\pi}{3}\right)\right), \\
\nu C &= \frac{1}{3}\left(e^{x+y} + 2e^{-(x+y)/2}\cos\left(\frac{\sqrt{3}}{2}(x-y)+\frac{2\pi}{3}\right)\right),
\end{aligned}$$

and the rotation matrix of this space is

$$\begin{bmatrix} \nu A & \nu B & \nu C \\ \nu C & \nu A & \nu B \\ \nu B & \nu C & \nu A \end{bmatrix}.$$

This will be the identity matrix when $\nu A = 1$ and $\nu B = \nu C = 0$. That is, when

$$x = \frac{2n\pi}{\sqrt{3}}, \qquad y = -\frac{2n\pi}{\sqrt{3}}, \qquad n = 0, 1, 2, \ldots.$$

Thus, the rotation matrix is unity for many different values of $\{x, y\}$. This means that an object rotating in this space can return to its starting point by rotating through the above angle.

Fibonacci numbers

Sebastian Hayes

Recall the Fibonacci sequence F_n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377

n	15	16	17	18	19	20	21	22	23
F_n	610	987	1597	2584	4181	6765	10946	17711	28657

We shall take a look at three problems.

(1) If we consider the sequence F_n (mod 4) we obtain 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0, 1, 1, ... Show that whenever we do obtain a repeating cycle of remainders, the first pair of consecutive integers to repeat is always $(1, 1)$.

(2) Does every number divide some Fibonacci number?

(3) How long can a cycle of remainders be?

First consider the Fibonacci numbers modulo 5:

$$1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, 2, 3, 0, 3, 3, \ldots.$$

It looks as if as soon as we obtain two consecutive remainders which add to the modulus we obtain as next remainder 0 (mod 5). After this the last remainder before the 0 repeats twice and in effect multiplies the original sequence, taking into account the modulus. This becomes clearer if we use minus numbers and reduce. With modulus 5 we get

$$1, 1, 2, -2, 0, -2, -2, 1(= -4), -1, 0, -1, -1, -2, 2, 0, 2, 2, -1, 1, 0, 1, 1, \ldots.$$

More generally, we will have a sequence of remainders

$$1, 1, 2, 3, 5, \ldots, m, -m, 0, -m, -m, -2m, -3m, \ldots, -m^2, m^2,$$
$$0, m^2, m^2, \ldots, -m^3, 0, -m^3, \ldots.$$

Eventually, at some stage we shall obtain a last remainder of 1 relative to the modulus when the entire cycle will repeat identically. If $(k, p) = 1$, the powers k, k^2, k^3, ..., k^{p-1} will be all different (mod p) and 1 will be amongst them. Thus, if we take $k = 3$, $p = 5$,

$$\begin{array}{rcrcrl} r & = & 3 & = & -2 & \pmod 5, \\ r^2 & = & 9 & = & -1 & \pmod 5, \\ r^3 & = & 27 & = & 2 & \pmod 5, \\ r^4 & = & 81 & = & 1 & \pmod 5, \\ r^5 & = & 343 & = & -2 & \pmod 5. \end{array}$$

The question is whether this is going to happen for every modulus p. We do not in fact need to look out for a consecutive pair $(m, -m)$ giving 0 as the next remainder; if *any* pair (a, b) crops up more than once, this means that the cycle is already repeating. To see this we only have to bear in mind that, just as the sum of two remainders (mod p) is unique, so is the antecedent of two remainders (mod p). Now, we can trace back the second occurrence of the pair (a, b) to the preceding pair $(b, a - b)$ and thence to $(a - 2b)$ giving $\ldots$, $a - 2b$, b, $a - b$, a, b, $\ldots$. The pattern is going to be the same as that from the *first* occurrence of (a, b), working backwards, which culminates in the original 1, 1, dots. This means that there is a part which is $\ldots$, -1, 1, 0, 1, 1, $\ldots$ and so the cycle is already repeating.

Could it not be, however, that for at least one modulus p the remainders manage to avoid repetition in much the same way as the sequences of digits in π manage to avoid a period? This is not possible since, for any modulus p, there is a maximum of p^2 pairs of consecutive numbers (x, y), where $x, y \in \{0, p-1\}$ — in fact only $p^2 - 1$ possible pairs since $(0, 0)$ is impossible in this context. If we suppose that every Fibonacci fresh number after the first eliminates a pair from the list of $p^2 - 1$ pairs, we shall soon run out of possible pairs. This is an application of the Dirichlet 'Pigeon-hole Principle' whereby, if there are more than n objects to place in n pigeon-holes, at least one pigeon-hole with have two or more objects in it.

Thus, rather surprisingly, not only does every number divide some Fibonacci number, it divides an infinite amount of them.

As for the last question, I do not know but it depends on three factors:

(1) How large the modulus is in the first place, because the larger it is, the more Fibonacci numbers will precede it.

(2) How long the sequence 1, 1, 2, 3, 5, $\ldots$ is before we reach the first 0 (mod p). This is not the same as (1) since in most cases the modulus will fall between two Fibonacci numbers (e.g. $F_5 < 7 < F_6$) and there is no way of knowing *a priori* how soon the first 0 will come.

(3) How many sequences ending in 0 we have to go through before we reach $\ldots$, 0, 1, 1, dots, when the whole cycle repeats.

Item (2) is a minimum for a given modulus if it is itself a Fibonacci number; thus, for (mod 5) we have a sequence length of $p = 5$, namely 1, 1, 2, 3, 0. But this does not necessarily mean that Fibonacci number moduli have the shortest cycles relative to their size.

Item (3) is a minimum, a single sequence ending in 0, if the penultimate digit before the first 0 is 1. Then we get 0, 1, 1, ... at once but the only modulus which is a Fibonacci number and has a cycle of one sequence only is 2.

For example the cycle for 5 is three sequences long

$$1, 1, 2, -2, 0, -2, -2, 1, -1, 0, -1, -1, -2, 2, 1, 0, 1, 1, \ldots.$$

The second best minimal number of sequences is when $-1 \pmod{p}$ precedes 0 since the next sequence will end $(-1)^2 = 1$.

It would be nice to be able to distinguish numbers which have maximum cycles, similar to fractions which have the maximum decimal period such as 1/7.

It seems on the face of it unlikely that any modulus can get near to the theoretical maximum of p^2 Fibonacci numbers for modulus p, though I am not sure how to determine the maximum in terms of p. Thus 6 as a modulus has 36 possibilities for (x, y) and with a cycle of

$$1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, 5, 5, 4, 3, 1, 4, 5, 3, 2, 5, 1, 0; 1, 1, 2, 3, 5, 2, \ldots$$

attains twenty-four of them, which seems not at all bad.

Problem 234.1 – Two ellipses

Tony Forbes

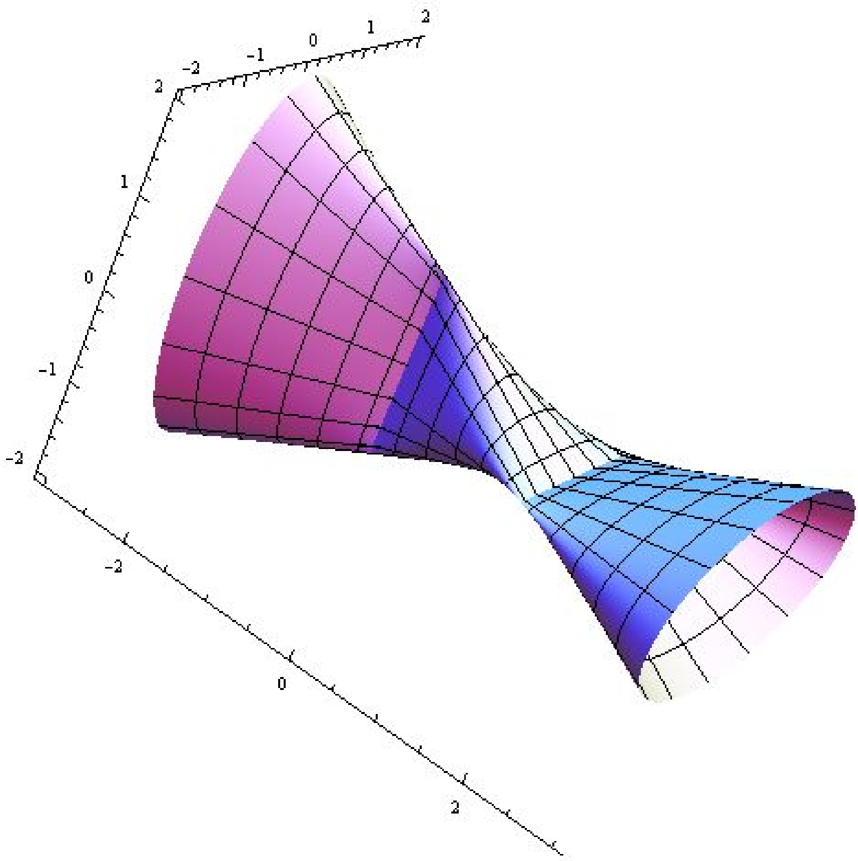
Let a, b and c be positive numbers with $b > a$. Let E_1 be an ellipse with axes $2a$ and $2b$ and situated in the plane $x = -c$ with its centre at $(-c, 0, 0)$ and its $2b$-axis vertical. Let E_2 be the same ellipse but centred at $(c, 0, 0)$ and with its $2a$-axis vertical. Thus

$$\begin{aligned} E_1(t) &= (-c, a\cos t, b\sin t), \\ E_2(t) &= (c, b\cos t, a\sin t), \qquad 0 \le t < 2\pi. \end{aligned}$$

Now join each point of E_1 to the diametrically opposite point of E_2. In other words, join $(-c, a\cos t, b\sin t)$ to $(c, -b\cos t, -a\sin t)$. The resulting surface, bounded by the ellipses, has two singularities in the form of straight line segments L_1 and L_2, say, with L_i parallel to the major axis of E_i and centred on a point between the origin and the centre of E_i.

Thanks to Dick Boardman, who showed me what to do, it is not too difficult to illustrate this interesting surface convincingly. In the picture, below, you can clearly see the ellipses E_1 and E_2 as well as the line segments L_1 and L_2.

Now for the problem. What is the volume enclosed by the surface between the two line segments?



Borromean rings and things

Chris Pile

The Borromean rings are usually shown in two dimensions as three circular rings linked together but no two rings are linked. This is not possible in three dimensions without distortion as the rings have constant diameter. A three dimensional model can be made with three mutually perpendicular ellipses or rectangles, so that the smaller diameter of each passes through the larger diameter of another, as in the photo of a chunky wooden model (I). Using flat polygonal shapes a three dimensional model can be made with three triangles (II, front). With a view to maximizing the width of the 'ring', and an appeal to symmetry, I made the altitude of the internal aperture equal to the external width between the mid-points of the sides. That is, side length 100 mm, internal altitude 50 mm, givng the width of ring $= (50\sqrt{3} - 50)/3 \approx 12.2\,\text{mm}$.

Unfortunately this does not work as I intended because the altitudes and mid-points for the second triangle do not align so some adjustment has to be made. It works well enough to give a plausible model but it would be good to know the exact configuration / orientation for a symmetrical arrangement.

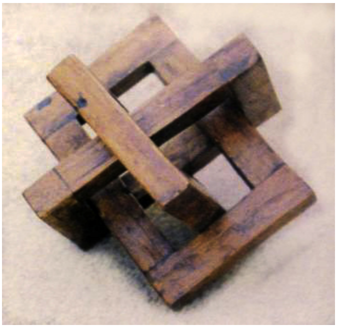
With square 'rings' a three-dimensional model can be made with the square sides fitting inside the diagonals. Square side 100 mm = internal diagonal, giving a ring width of $(141.2 - 100)/2\sqrt{2} \approx 14.6\,\text{mm}$.

The three squares have one square inside diagonally and one outside diagonally. Again it was almost possible to have the squares inclined at 45° and the internal corners at the mid-points of the sides. To make an attractive model I added two more squares so that each square had two other squares inside diagonally and two outside diagonally, with no two squares linked together; see (II, right) and (III). The model has twenty corners, suggesting that a polyhedral model with twenty vertices could be made. Although an accurate drawing showed that the constraints imposed were not possible (!), I assumed that the vertices were arranged as ten equilateral triangles, ten squares and two pentagons and made the appropriate number of 3, 4, and 5 sided pyramid 'cups' with the same edge length and 45° face angle, cemented them together and cut away the edges of the diametral octagons to leave the five diametral squares (II, left) arranged in the same configuration as the skeletal square model. Although it again looks plausible it had to be distorted somewhat. However, it appears that a 'solid' polyhedron is possible with 22 regular polygonal faces, 10 triangles, 10 squares and two pentagons (II, rear). Is there a name for it?

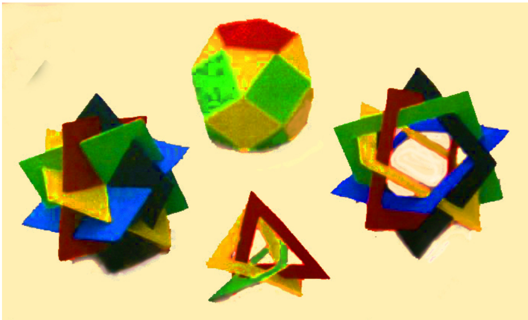
There is further scope with pentagonal 'rings' as shown in the photo. A further three rings can be added to make a more symmetric model.

Five 'Borromean' squares

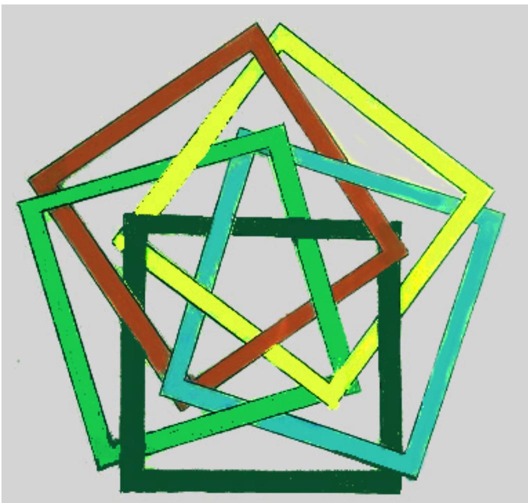
The schematic (III) shows that no two squares are linked together yet all five are linked in five sets of Borromean triples. Starting at the bottom and going clockwise, call the squares black, green, red, yellow and blue. Then the Borromean triples are {red, yellow, blue}, {yellow, blue, black}, {blue, black, green}, {black, green, red} and {green, red, yellow}. The other five triples are not linked: {red, yellow, black}, {yellow, black, green}, {blue, green, red}, {green, yellow, blue} and {red, blue, black}.



(I)



(II)



(III)

Matrix operations on a number grid

Comments on a question that arose in the January 2010 Winter Weekend

Dennis Morris

Consider the following 9×9 number grid.

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81

Draw a T shape anywhere upon this grid that is three boxes wide and three boxes deep like so:

1	2	3
	11	
	20	

and define the *T-total* to be the total of all the numbers in the T; in this case, the T-total is $1 + 2 + 3 + 11 + 20 = 37$. Define the number at the bottom of the T to be the *T-number*; in this case, the T-number is 20. The T-total and the T-number are two ordered numbers and are thus a vector. Translation to the right by one column of this vector is

$$\begin{bmatrix} \text{T-total} \\ \text{T-number} \end{bmatrix} = \begin{bmatrix} T \\ N \end{bmatrix} \rightarrow \begin{bmatrix} T+5 \\ N+1 \end{bmatrix}.$$

This is accomplished by the matrix:

$$\begin{bmatrix} 1 & \dfrac{5}{N} \\ \dfrac{1}{T} & 1 \end{bmatrix} \begin{bmatrix} T \\ N \end{bmatrix} = \begin{bmatrix} T+5 \\ N+1 \end{bmatrix}.$$

The adjoint (inverse multiplied by the determinant) of this matrix is

$$\begin{bmatrix} 1 & \dfrac{-5}{N} \\ \dfrac{-1}{T} & 1 \end{bmatrix} \begin{bmatrix} T \\ N \end{bmatrix} = \begin{bmatrix} T-5 \\ N-1 \end{bmatrix}.$$

We might expect that a translation to the right followed by a translation to the left would be the identity, but in fact

$$\begin{bmatrix} 1 & \dfrac{-5}{N} \\ \dfrac{-1}{T} & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{5}{N} \\ \dfrac{1}{T} & 1 \end{bmatrix} = \begin{bmatrix} 1-\dfrac{5}{NT} & 0 \\ 0 & 1-\dfrac{5}{NT} \end{bmatrix}.$$

We can see that this approaches the identity as the T-total and the T-number approach infinity. Similarly, we might expect that a translation to the right followed by another translation to the right would be the same as a translation by two columns to the right. However,

$$\begin{bmatrix} 1 & \dfrac{5}{N} \\ \dfrac{1}{T} & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{5}{N} \\ \dfrac{1}{T} & 1 \end{bmatrix} \begin{bmatrix} T \\ N \end{bmatrix} = \begin{bmatrix} T+10+\dfrac{5}{N} \\ N+2+\dfrac{5}{T} \end{bmatrix},$$

which again approaches what we would expect as the T-total and the T-number approach infinity. The determinant of the single translation to the right approaches unity as the T-total and the T-number approach infinity. This means that the adjoint approaches the inverse as the T-total and the T-number approach infinity.

Matrices do not 'work properly' over the number chart because it is a non-uniform space. This means we can measure how non-uniform a space is by measuring how much matrices do not 'work properly' over it.

Problem 234.2 – Series

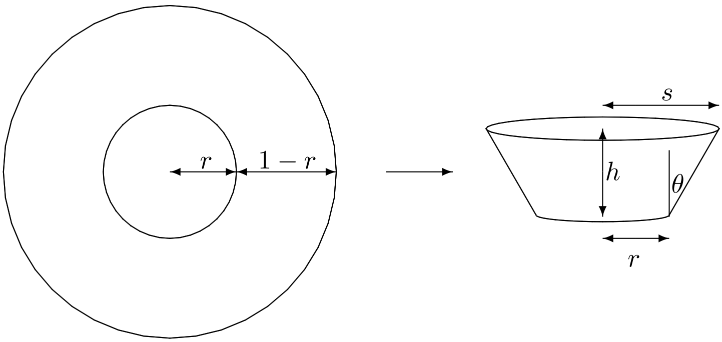
Show that

$$1-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}+\frac{1}{25}-\dots = \frac{\pi(\sqrt{2}+1)}{8},$$

an interesting relation between π and $\sqrt{2}$.

Solution 230.5 – Cup-cake holder

You have a circular piece of tin foil of radius 1. Use it to make a cup-cake holder of maximum possible volume. For simplicity, assume that the pleating of the foil to make the sides is done at an infinitesimal level; so you can compute the volume of the finished cup-cake holder by the usual formula for a truncated cone.



Dick Boardman

The metallic foil is pleated into a cup or truncated cone so that the base has radius r, the slope side is $1-r$ and the upper radius is s. Let θ be the semi-angle of the cone. Then the height of the cone is $h=(1-r)\cos\theta$ and $s=r+(1-r)\sin\theta$. We use the standard formula for the volume of a truncated cone: $v=\pi h(r^2+rs+s^2)/3$.

Plugging into v the expressions for h and s gives

$$v(r,\theta) \;=\; \frac{\pi}{3}(1-r)(\cos\theta)\big(3r^2+3(1-r)r\sin\theta+(1-r)^2\sin^2\theta\big).$$

As usual, we differentiate this expression with respect to r and θ and equate to zero. After some simplification we obtain these two rather complicated expressions:

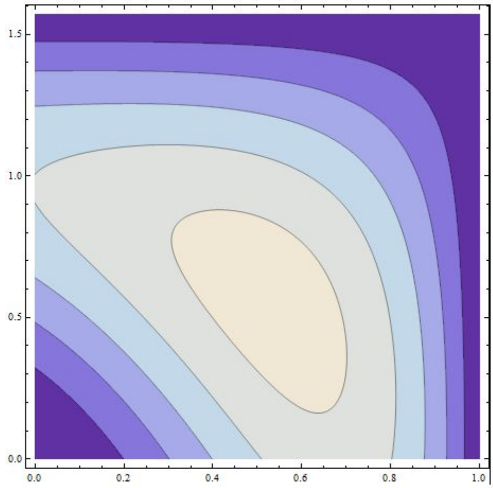
$$\begin{aligned}\frac{\partial v}{\partial r} &= \pi(\cos\theta)\big((2-3r)r+(1-4r+3r^2)\sin\theta-(r-1)^2\sin^2\theta\big),\\ \frac{\partial v}{\partial \theta} &= \frac{\pi}{6}(r-1)\big((1-2r-5r^2)\sin\theta\\ &\quad + 3(r-1)(\cos 2\theta)(-2r+(r-1)\sin\theta)\big).\end{aligned}$$

We want to find r and θ such that $\partial v/\partial r=\partial v/\partial\theta=0$. Solving these two equations is very messy. It is easier to use numerical methods. After doing the computation we see that the relevant root occurs at

$$r \approx 0.53564, \qquad \theta \approx 0.560008 \approx 32.0856^\circ$$

to give a maximum volume of approximately 0.542991.

The contour plot of $v(r, \theta)$ shows the maximum volume in more or less the correct place.



Solved in a similar manner by **Steve Moon** and **Tamsin Forbes**.

Tony Forbes

Being curious, I could not help wanting to see what the exact solution of $\partial v/\partial r = \partial v/\partial\theta = 0$ would look like. The two derivatives don't look very friendly and it seems that any attempt to obtain a solution by hand is going to be extremely tedious. Fortunately I have MATHEMATICA to do most of the work. It's so simple—after setting up v as a function of r and t (for θ) you type in `Solve[D[v,r] == D[v,t] == 0, {r,t}]`, wait for something to appear and then decide which of the 15 solutions presented is the one you really want.

The expressions delivered by MATHEMATICA for the parameters r and θ are actually not too bad, and with further simplification (by hand) I managed to reduce each of them to one-liners. It serves little useful purpose to go through the lengthy process of deriving the expressions; so to save space I give just the final results. Let

$$p = \frac{11+9\sqrt{-107}}{2} \quad \text{and} \quad q = \frac{11-9\sqrt{-107}}{2}.$$

Then

$$r = \frac{11}{63} + \frac{1}{1638}\left(\left(-5+3\sqrt{-107}\right)q^{2/3} + \left(109-3\sqrt{-107}\right)q^{1/3}\right),$$

$$\theta = \arccos\sqrt{\frac{13+q^{1/3}+p^{1/3}}{27}},$$

and you can verify that they reproduce the values calculated by numerical methods. The corresponding exact value of $v(r,\theta)$ is a little more complicated. Let

$$f = 14 - p^{1/3} - q^{1/3}$$

and

$$g = 1352 - \left(109-3\sqrt{-107}\right)q^{1/3} - \left(-5+3\sqrt{-107}\right)q^{2/3},$$

both of which are real and have approximate values 7.618485390 and 760.6216695 respectively. Then

$$v(r,\theta) = \frac{g\pi\sqrt{81-3f}\left(fg^2+9g\sqrt{3f}(1638-g)+81(1638-g)^2\right)}{14742^3}, \qquad (*)$$

a real number with value 0.54299099140894696683 to about 20 decimal places. The denominator is $2\cdot 3^4\cdot 7\cdot 13$ cubed.

It is interesting to see that p and q are related by $pq = 13^3$. Furthermore, the other two expressions which involve $\sqrt{-107}$ are related to p by

$$\left(-5+3\sqrt{-107}\right)\left(109-3\sqrt{-107}\right) = 38p.$$

I am intrigued by the appearance of the square root of -107 in a simple problem involving cooking materials. I wonder if cake-makers can offer an explanation. I also wonder if the catering industry is aware of the solution to this problem and in particular the value of $(*)$, an important universal number, which one might wish to refer to as the *cup-cake* constant.

Letters to the Editor

Centuries, squares and balls

Dear Tony,

Thank you for M500 229 with the first 16 'century patterns' of primes on the front cover. It gives me the incentive to investigate further but, as mentioned previously, my knowledge of prime number distribution is not as great although it provides me with some entertainment and amusement. I am not yet addicted or dedicated to the search! I enclose my lastest attempts to answer the questions posed in the magazine. I suspect you have the answers already and I would welcome your expert comments.

With regard to Pythagorean Squares [M500 **289**, page 11, this was intended as a postscript to Problem 225 (Pythagorean triangles). That is why it has nothing to do with buried treasure!

The solution to Problem 229.4 (Balls) is easier done than said (i.e. it is easier to solve in practice than to explain the strategy). See page 15.

Yours sincerely,

Chris Pile

Formulae

Tony,

Perhaps at my venerable age (my daughters tell me that I am not 'old', just 'retro'—I'm not sure whether this is a compliment) I should know the answer to this question, but I don't, and so I ask for pearls of wisdom from the Great and the Good who read M500:

Is it possible to prove whether a formula exists?

I can usually cope with the 'Quadratic Formula', and I am vaguely aware that there are formulae for finding the roots of cubics and quartics. And lurking in a dark recess of my brain is a seditious voice telling me that a formula for finding the roots of quintics and higher order polynomials does not exist. Is this because the appropriate formula has not yet been found, or because someone has actually proved that it cannot exist?

It is my conjecture that the answer to my question is, in general, 'no'. This is based on the observation that many people have devoted large parts of their life looking for a general formula to find the nth prime number. If it were possible to prove that such a formula can not exist, then presumably most of them would have gone out and got a life instead.

Tony [Huntington]

Where does mathematics come from?

I haven't read the book [George Lakoff & Rafael Núñez, *Where Mathematics Comes From*, reviewed in M500 **231** by Sebastian Hayes]; so can't really comment in detail but I personally think that some aspects of mathematics are independent of human existence. I take this to be something akin to the 'Platonist view'.

It is undoubtedly true that everything we perceive comes via our senses and is processed by our brains but I don't think this is the same as saying that without human beings there would be no mathematics.

There are creatures, e.g. wolves or whales, which cooperate and communicate and it is not impossible to imagine a super creature, which, via natural selection, could find an advantage in being able to count 'there are more of us than there are in the competing pack' and measure 'my opponent is bigger than me'. Such a creature could still be around when human beings have become extinct. If it evolved further so as to be able to add, subtract, multiply and divide then it would find that there were prime numbers and that there are more prime numbers than you could count. This is a theorem in Greek mathematics.

This is not to say that all mathematics has an independent existence. Where you draw the line I leave to wiser minds than mine.

Regards,

R. M. Boardman

Problem 234.3 – Fixed point

If you ever owned a calculator of the type that was popular a decade or two ago you will surely be familiar with α, the fixed point of the cosine function, the unique real number that solves the equation

$$\cos\alpha \;=\; \alpha.$$

I (TF) am told by **Robin Whitty** that this is called *Dottie's number* and if you still have one of the aforementioned calculators, this is how to compute it. Enter any number and then hit the `cos` key infinitely many times. You should see the number 0.739085133215160641655, or something like it, appear in the display. The attraction is universal; you really can start from any number you like, and you will always end up with α.

Now for the problem. *Prove that α is transcendental.*

Solution 229.4 – Balls

There are bn balls, n each of b different colours. They are arranged in a line in b blocks of n.

$$\odot\odot\odot\odot\odot\odot\otimes\otimes\otimes\otimes\otimes\otimes\oplus\oplus\oplus\oplus\oplus\oplus$$

A *move* is to take a ball from the line, place it somewhere else in the line and close up the gap.

$$\odot\odot\otimes\odot\odot\odot\odot\otimes\otimes\otimes\otimes\otimes\oplus\oplus\oplus\oplus\oplus\oplus$$

What is the minimum number of moves necessary to create a line with no two adjacent balls having the same colour? Prove that the answer is $\geq \frac{1}{2}b(n-1)$.

Chris Pile

In a line of n balls of the same colour there are $n-1$ 'interballs'. If there are b coloured blocks of n balls, there are $b(n-1)$ interballs between adjacent balls of the same colour. An optimum move is to take one ball from one coloured block (reducing the number of interballs by 1) and insert it between two balls of another block to separate off one ball (reducing the number of adjacent balls of the same colour by 2. Hence the minimum number of moves is $b(n-1)$ unless n is even and b is odd. A convenient strategy is to start from the first block on the left and make a move to the next block on the right, making the bth move from the last block back to the first. Where n is even and b is odd there will be one pair remaining, but there will also be a run of b different colours. Move the middle ball to split the pair for the last move. This strategy will also work if the blocks of balls are arranged in a circle

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Problem 234.4 – Tetrahedron

Three sides of a tetrahedron form an equilateral triangle of side a. The other three sides have length 1. Show that the diameter of the sphere that circumscribes the tetrahedron is

$$\frac{\sqrt{3}}{\sqrt{3-a^2}}.$$

What happens if, as before, three sides have length 1 and three sides have length $a \neq 1$, but no face of the tetrahedron is equilateral?

Problem 234.5 – Graham's number

What are the last 20 digits of Graham's number?

Graham's number appears in R. L. Graham & B. L. Rothschild, Ramsey's theorem for n-parameter sets (*Trans. Amer. Math. Soc.* **159** (1971)) as an upper bound for a problem in Ramsey theory. For details of the problem, see Eddie Kent's article in M500 **215** (or look it up in *Wikipedia*). The number itself is G_{64}, where G_n is defined recursively by

$$\begin{aligned} G_1 &= 3 \uparrow\uparrow\uparrow\uparrow 3, \\ G_n &= 3 \uparrow^{G_{n-1}} 3. \end{aligned}$$

We are using Donald Knuth's 'upward arrow' notation, a device he invented to extend the sequence of arithmetic operators: addition, multiplication, exponentiation, ...

We shall adopt the convention of writing a string of n upward arrows as $\uparrow^n$. Also let us agree to do computations involving arrows *from right to left*, as with repeated exponentiation. For example, the expression $a \uparrow^3 b \uparrow^4 c \uparrow^5 d$ is shorthand for $a \uparrow\uparrow\uparrow (b \uparrow\uparrow\uparrow\uparrow (c \uparrow\uparrow\uparrow\uparrow\uparrow d))$.

With that out of the way, we can now define $x \uparrow^n y$ for positive integers x and y by

$$\begin{aligned} x \uparrow^{-1} y &= x + y, \\ x \uparrow^n y &= x \uparrow^{n-1} x \uparrow^{n-1} \ldots \uparrow^{n-1} x, \qquad n = 0, 1, 2, \ldots, \end{aligned}$$

where the right hand side of the second expression when written out in full contains exactly y copies of x.

Roughly speaking, we are taking the previous operation and inserting $y - 1$ copies of it in the spaces between y copies of x. For instance, we start with $x \uparrow^{-1} 3$, which is defined as x added to 3, $x + 3$, and to get the next level, $x \uparrow^0 3$, we apply addition to three copies of x, or multiply x by 3: $x + x + x = x \cdot 3$. Then $x \uparrow 3$ is multiplication applied to three copies of x: $x \uparrow 3 = x \cdot x \cdot x = x^3$, and to get $x \uparrow\uparrow 3$ we apply exponentiation to three copies of x: $x \uparrow\uparrow x = x \uparrow x \uparrow x = x^{x^x}$.

If you put $x = y = 2$, the computations are quite easy:

$$2 \uparrow^{-1} 2 = 2 \uparrow^0 2 = 2 \uparrow 2 = 2 \uparrow\uparrow 2 = 2 \uparrow\uparrow\uparrow 2 = \ldots = 4.$$

However, to compute Graham's number, we have $3 \uparrow 3 = 3^3 = 27$, $3 \uparrow\uparrow 3 = 3^{3^3} = 7625597484987$, and that seems to be about as far as you can get with ordinary decimal notation.

Problem 234.6 – Simplification

Simplify

$$\frac{\tan\theta\sin\theta}{1-\cos\theta}.$$

Thanks to **Emil Vaughan** for this. Looks easy but I think the answer we are looking for might surprise you.

Problem 234.7 – Directed triangles

Draw a directed graph as follows. Take n points, numbered 0, 1, ..., $n-1$, and place them in order clockwise (or anticlockwise if you prefer) around the circumference of a circle. For $i = 0, 1, \ldots, n-1$ and $j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$, join point i to point $(i+j) \bmod n$ by an arrow unless an arrow going in the opposite direction is there already. A *directed triangle* is where you have three points a, b, c and arrows joining them thus: $a \longrightarrow b \longrightarrow c \longrightarrow a$. (If this doesn't make sense, try drawing a diagram.)

How many directed triangles do you get?

M500 Mathematics Revision Weekend 2010

The thirty-sixth **M500 Society Mathematics Revision Weekend** will be held at

Aston University, **Birmingham**

over

Friday $10^{\textbf{th}}$ – **Sunday** $12^{\textbf{th}}$ **September 2010**.

The cost, including accommodation (with en suite facilities) and all meals from bed and breakfast Friday night to lunch Sunday is £250 (in Aston's Lakeside flats) or £298 (Aston Business School), The cost for non-residents is £120 (includes Saturday and Sunday lunch). M500 members get a discount of £10. For full details and an application form, see the Society's web site at www.m500.org.uk, or send a stamped, addressed envelope to

Jeremy Humphries, M500 Weekend 2010.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most mathematics-based OU courses, subject to sufficient numbers.

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