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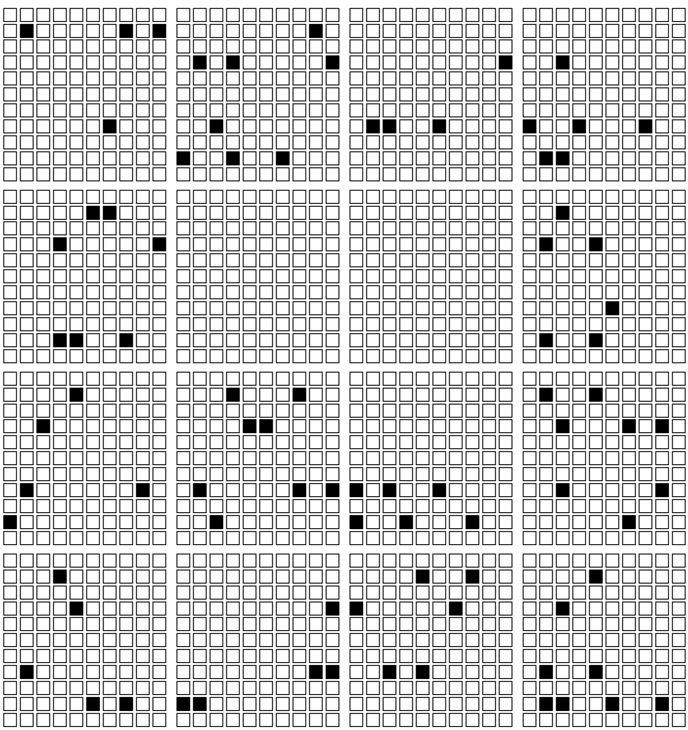
M500 236



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Arithmotriangulation

Bryan Orman

The general problem: to find a system of n points in the plane such that their mutual separations are rationals or integers.

By its very nature this problem produces complicated systems of Diophantine equations. Nevertheless it is possible to find solutions in simple cases with the aid of trigonometric formulae and elementary geometry.

We start with the arithmotriangle, which is, by definition, a triangle with rational sides (equivalently with integer sides, by simple scaling). With the usual designation of the angles A, B, C and the sides a, b, c of a triangle, it is clear that any triple (a, b, c) of integers satisfying $\max(a, b, c) < s$ will give an arithmotriangle. Here $2s = a + b + c$ is the perimeter of the triangle. So it is quite easy to produce arithmotriangles. We note here that an arithmotriangle is Heronian if its area is an integer. In order to construct arithmoquadrilaterals based on four points we need a classification of arithmotriangles. The two diagonals have to be rational, so the approach we will adopt requires two arithmotriangles to have a diagonal as one of their sides, with the other diagonal itself producing two further arithmotriangles. Thus we seek four matching arithmotriangles.

Some basic trigonometric results are needed:

the cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$,

the area: $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$,

the sine rule, involving Δ: $2\Delta = bc\sin A = ca\sin B = ab\sin C$,

the sine rule, involving R: $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$, $abc = 4R\Delta$, where R is the radius of the circumscribed circle.

From the cosine rule we observe that $\cos A$, $\cos B$ and $\cos C$ will all be rational and, from the remaining formulae, we have that the products and ratios of any two quantities from $\sin A$, $\sin B$, $\sin C$, Δ and R are all rational.

An example. Take $a = 7$, $b = 8$ and $c = 9$. Then the cosine rule gives $\cos A = 2/3$, $\cos B = 11/21$ and $\cos C = 2/7$. As $s = \frac{1}{2}(7 + 8 + 9) = 12$, it follows that $\Delta = \sqrt{12(12-7)(12-8)(12-9)} = 12\sqrt{5}$ and $R = abc/(4\Delta) = 21\sqrt{5}/10$. Finally, from the sine rule, $\sin A = a/(2R) = \sqrt{5}/3$, $\sin B = 8\sqrt{5}/21$, $\sin C = 3\sqrt{5}/7$.

The appearance of $\sqrt{5}$ in the above quantities suggests that arithmotriangles could be classified by the number $\sqrt{T}$, with T being the product of distinct primes. We will call T the order of the arithmotriangle.

As a further example, take $a = 13$, $b = 14$, $c = 15$. Then $\cos A = 3/5$, $\cos B = 33/65$, $\cos C = 5/13$, $s = 21$. But now $\Delta = 84$, and the triangle is Heronian. The other quantities are $R = 65/8$, $\sin A = 4/5$, $\sin B = 56/65$, $\sin C = 12/13$, and all the quantities are rational so that in this case we have $T = 1$. So this Heronian arithmotriangle has order $T = 1$, as is the case for all Heronian arithmotriangles.

Our next task is to parametrize an arithmotriangle of order T. Any angle a having a rational cosine will have both $\sin\alpha$ and $\tan\frac{1}{2}\alpha$ characterized by the same T. This follows from the identity $\sin\alpha = (1+\cos\alpha)(\tan\frac{1}{2}\alpha)$. It is convenient to introduce the rational q/p and define $\tan\frac{1}{2}\alpha = q\sqrt{T}/p$ so that, by the half-angle formulae

$$\cos\alpha = \frac{p^2 - q^2T}{p^2 + q^2T}, \ \ \sin\alpha = \frac{2pq\sqrt{T}}{p^2 + q^2T}, \ \ \tan\alpha = \frac{2pq\sqrt{T}}{p^2 - q^2T}.$$

Furthermore the sum (or difference) of two arithmetical angles of order T is also arithmetical of order T. This is easily demonstrated since, given

$$\tan\frac{1}{2}\theta_1 \ = \ \frac{q_1}{p_1}\sqrt{T}, \quad \tan\frac{1}{2}\theta_2 \ = \ \frac{q_2}{p_2}\sqrt{T},$$

it follows that

$$\tan\left(\frac{1}{2}\theta_1 + \frac{1}{2}\theta_2\right) \ = \ \frac{(p_1q_2 + p_2q_1)\sqrt{T}}{p_1p_2 - q_1q_2T}.$$

Right-angled Heronian triangle ($T = 1$)

Since we have a right-angle, $A = \frac{1}{2}\pi$ and it is sufficient to set $\tan\frac{1}{2}B = q/p$ with $p > q$. Straightforward calculations give $a = p^2 + q^2$, $b = 2pq$, $c = p^2 - q^2$, recognized as the general Pythagorean triple, with $\Delta = pq(p^2 - q^2)$.

Arithmotriangle with angle $A = \frac{1}{3}\pi$ ($T = 3$)

With $T = 3$ and $\tan\frac{1}{2}A = 1/\sqrt{3} = \sqrt{3}/3$ we set $\tan\frac{1}{2}B = \sqrt{3}\,v/u$. Then

$$\tan\frac{1}{2}C \ = \ \cot\left(\frac{1}{2}A + \frac{1}{2}B\right) \ = \ \frac{u-v}{u+3v}\sqrt{3}.$$

Next the sines of the angles can be calculated:

$$\sin A \ = \ \frac{1}{2}\sqrt{3}, \quad \sin B \ = \ \frac{2uv\sqrt{3}}{u^2+3v^2}, \quad \sin C \ = \ \frac{(u-v)(u+3v)\sqrt{3}}{2(u^2+3v^2)}.$$

The sine rule gives

$$\begin{aligned} a &= 2R\sin A = \frac{R\sqrt{3}}{u^2+3v^2}(u^2+3v^2), \\ b &= 2R\sin B = \frac{R\sqrt{3}}{u^2+3v^2}4uv, \\ c &= 2R\sin C = \frac{R\sqrt{3}}{u^2+3v^2}(u-v)(u+3v), \end{aligned}$$

where we have extracted a common term that leaves integers on the right-hand side. Set $R = \frac{1}{3}(u^2+3v^2)\sqrt{3}$, and $a = u^2+3v^2$, $b = 4uv$, $c = (u-v)(u+3v)$.

Examples: $u = 2$, $v = 1$ produces $a = 7$, $b = 8$, $c = 5$ and $u = 4$, $v = 1$ produces $a = 19$, $b = 16$, $c = 21$.

Arithmotriangle with angle $A = \frac{2}{3}\pi$ ($T = 3$ again)

In this case $\tan\frac{1}{2}A = \sqrt{3}$ and, setting $\tan\frac{1}{2}B = \sqrt{3}\,v/u$ as before, we have

$$\tan\frac{1}{2}C = \cot\left(\frac{1}{2}A + \frac{1}{2}B\right) = \frac{u-3v}{u+v}\frac{1}{\sqrt{3}}.$$

Performing the same calculation as for the arithmotriangle with angle $\frac{1}{3}\pi$ we find $R = \frac{1}{3}(u^2+3v^2)\sqrt{3}$ and $a = u^2+3v^2$, $b = 4uv$, $c = (u+v)(u-3v)$.

An example: $u = 4$, $v = 1$ produces $a = 19$, $b = 16$, $c = 5$.

A quadrilateral can now be constructed using $(19, 16, 5)$ with angle $\frac{2}{3}\pi$ and $(19, 16, 21)$ with angle $\frac{1}{3}\pi$, and both these triangles have the same order, $T = 3$. If the common side 19 is used as one diagonal of the quadrilateral then we need to check that the other diagonal is rational, otherwise the quadrilateral will not be an arithmoquadrilateral. To this end, we note that the quadrilateral is cyclic since a pair of opposite angles sum to π (so that's why we introduced $T = 3$ arithmotriangles!). Ptolemy's theorem, that the sum of the products of its two pairs of opposite sides is equal to the product of its diagonals, gives the other diagonal as $26 \cdot 16/19$. So we have constructed a cyclic arithmoquadrilateral. See Figure 1.

Any equilateral triangle is a $T = 3$ arithmotriangle and so a cyclic arithmoquadrilateral can be constructed as in Figure 2. And the general $T = 3$ one is simply as in Figure 3.

So far our investigation has employed only $T = 3$ arithmotriangles to produce cyclic arithmoquadrilaterals but now we look at the general arithmotriangle, of order T.

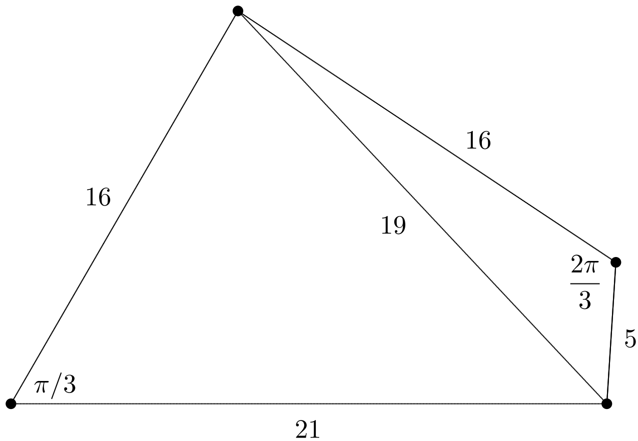


Figure 1

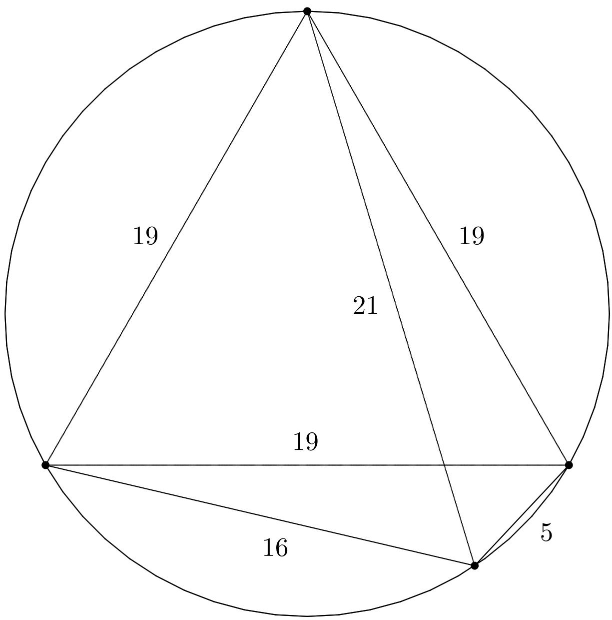


Figure 2

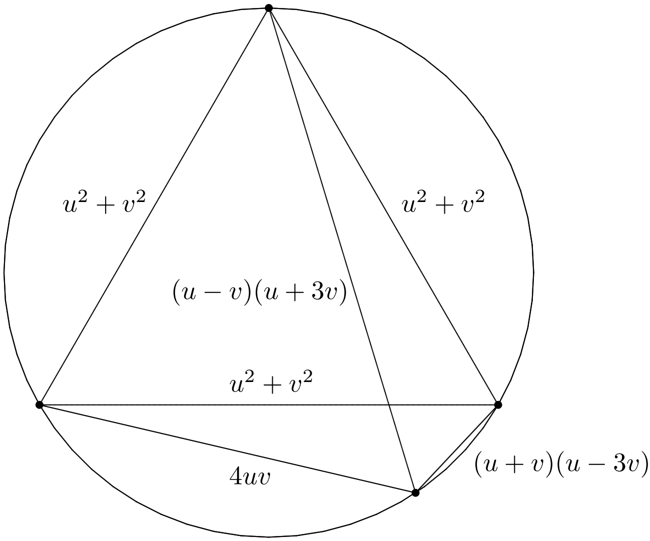


Figure 3

Arithmotriangle of order T

Setting $\tan\frac{1}{2}B = q/(s\sqrt{T})$ and $\tan\frac{1}{2}C = s\sqrt{T}/p$ then, by the usual calculations,

$$\tan\frac{1}{2}A = \frac{s(p-q)\sqrt{T}}{pq+s^2T}.$$

Furthermore

$$\begin{aligned}
\sin A &= \frac{2s(p-q)(pq+s^2T)\sqrt{T}}{(p^2+s^2T)(q^2+s^2T)},\\
\sin B &= \frac{2qs\sqrt{T}}{q^2+s^2T},\\
\sin C &= \frac{2ps\sqrt{T}}{p^2+s^2T}.
\end{aligned}$$

The sine rule produces

$$a = (p-q)(pq+s^2T), \quad b = q(p^2+s^2T), \quad c = p(q^2+s^2T)$$

with

$$R = \frac{(p^2+s^2T)(q^2+s^2T)}{4s\sqrt{T}}$$

and

$$\Delta = pqs(p-q)(pq+s^2T)\sqrt{T}.$$

The cyclic arithmopolygon of order T

It should be clear from the construction of our example of an order-3 cyclic arithmoquadrilateral that all the angles subtended by a vertex on the chord joining any pair of vertices are arithmetical of order 3. This means that the general cyclic arithmopolygon with arithmetical angles of order T can be investigated with, for example, the arithmoquadrilateral as a particular case. To this end we consider n successive points $A_1, A_2, \ldots, A_n$ on a circle of radius R, with an origin O between A_n and A_1, and a tangential base line OB at O from which all relevant angles are measured. Let θ_i, $i = 1, 2, \ldots, n$, be the angle between OB and OA_i. See Figure 4.

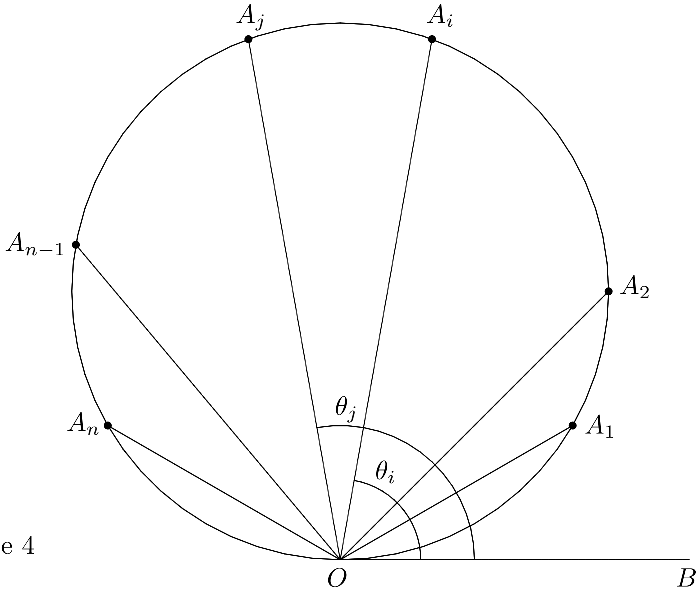


Figure 4

With $\tan \frac{1}{2}\theta_i = s\sqrt{T}/p_i$, and $p_1 > p_2 > \cdots > p_n > 0$, then a_{ij}, the length of the side from A_i to A_j, $i > j$, is equal to $2R\sin(\theta_j - \theta_i)$. Finally

$$a_{ij} \;=\; \frac{|p_i - p_j|(p_i p_j + Ts^2)}{(p_i^2 + Ts^2)(p_j^2 + Ts^2)},$$

with $R = 1/(4s\sqrt{T})$.

Example. Consider the cyclic arithmoquadrilateral with $T = 2$, $s = 2$, and $p_1 = 10$, $p_2 = 5$, $p_3 = 2$, $p_4 = 1$. The mutual separations of the four points are $a_{12} = 145$, $a_{23} = 243$, $a_{34} = 165$, $a_{14} = 297$, $a_{13} = 308$,

$a_{24} = 312$, where the lengths have been scaled up by the factor $33 \cdot 54$ to produce integer values. The radius of the circumscribed circle is then $R = 33 \cdot 27\sqrt{2}/8$ (Figure 5).

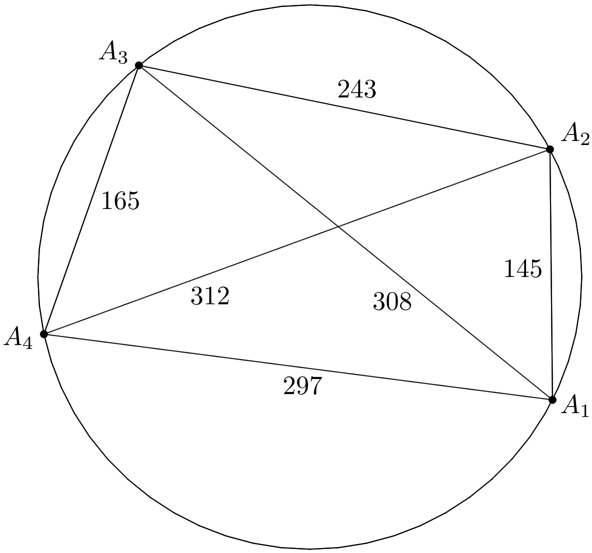


Figure 5

With the addition of a fifth point A_5, between A_2 and A_3 with $p_5 = 3$, a cyclic arithmopentagon is obtained with mutual separations $a_{12} = 2465$, $a_{13} = 5236$, $a_{14} = 5049$, $a_{15} = 4389$, $a_{23} = 4131$, $a_{24} = 5304$, $a_{25} = 2484$, $a_{34} = 2805$, $a_{35} = 2079$, $a_{45} = 4356$. The radius of the circumscribed circle is now $R = 17 \cdot 33 \cdot 27\sqrt{2}/8$. The additional point requires a further scaling of the system by a factor of 17 (Figure 6).

To confirm that the calculations have been performed correctly, it is sufficient to examine, say for example, the angles subtended by the vertices A_2, A_5 and A_3 on the chord A_1A_4. The cosine rule gives the angle as $\tan^{-1} 2\sqrt{2}$, in keeping with the order $T = 2$ for this system.

The general construction of cyclic arithmopolygons has been achieved. To go further and produce just arithmopolygons requires the techniques of inversion with respect to a vertex of a cyclic arithmopolygon, the vertex being employed as the pole of the transformation. Angles are preserved and lengths remain rational. However this approach, although quite fruitful, leads to fairly involved algebra, even for arithmoquadrilaterals, so enough is enough!

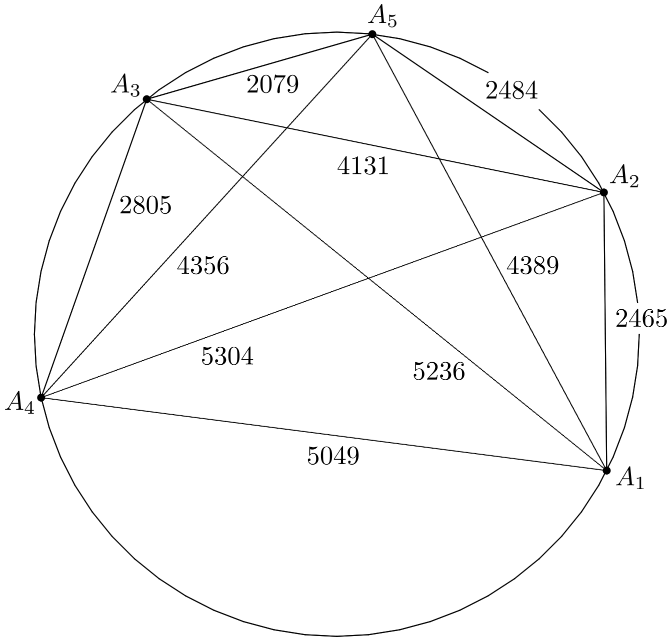


Figure 6

Problem 236.1 – Relationships

Tony Forbes

'With humans being what they are, if you had a drama centring on three women and three men, there's no end to the amount of platonic and romantic entanglements that could ensue.' So says Caitlin Moran, writing in *The Times*, 6 March 2010 (thanks to Robin Marks for sending me a copy) about ITV's *Married Single Other*.

The article continues, 'Well obviously there is—simple mathematics tells us that it's 36.' A few days later a *Times* correspondent asserted that the correct figure is not 36 but 15, her reason being that human relationships are symmetric and not reflexive.

Well, I can see 15 comes from six-choose-two, but I have to disagree with it. Before I attempt to explain my own answer (which I believe to be nearer 755) I thought it would be a good idea to collect M500 readers' thoughts on the matter. So here is the problem for you to solve: *How many distinct sets of relationships can you have involving six people.* Remember that there are two types of entanglements, platonic and romantic.

Problem 236.2 – Series

Tommy Moorhouse

It is well known that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; \frac{\pi^2}{6}.$$

In contrast, the sums

$$\sum_{n=1}^{\infty} \frac{1}{n(n+N)}$$

with N an integer greater than zero, are all rational. Consider

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

This sum converges and is equal to 1. If we let

$$S(M) \;=\; \sum_{n=1}^{M} \frac{1}{n(n+1)},$$

we find that $S(9) = 0.9, S(99) = 0.99$. Show that this pattern holds for all the sums of the form $S(9999\ldots9)$. Find a closed expression for $S(M)$. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+N)} \;=\; \frac{1}{N}\sum_{k=1}^{N} \frac{1}{k}.$$

Problem 236.3 – Periodic function

A function $f : \mathbb{R} \to \mathbb{R}$ is periodic with period 2π and is differentiable any number of times. Must $f(x)$ be a polynomial in $\cos x$ and $\sin x$? What if instead $f : \mathbb{C} \to \mathbb{C}$ is analytic?

Problem 236.4 – Real function

Suppose A and θ are real. Show that

$$(A + iA\tan\theta)^{\log(A\sec\theta) - i\theta}$$

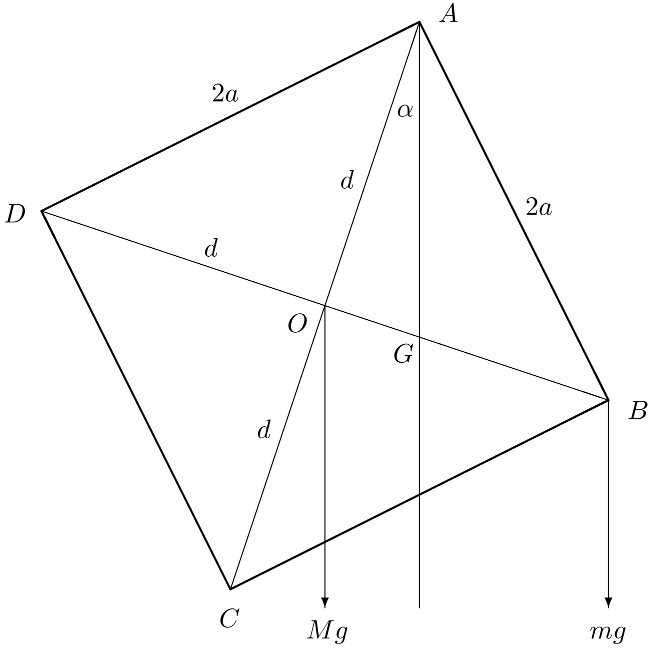
is also real.

A simple statics problem

Bryan Orman

A uniform square lamina $ABCD$ of mass M is freely supported at A, such that AC is vertical. A mass m is attached at the vertex B and the lamina is allowed to rotate to a new equilibrium position. Show that the angle through which AC has rotated is $\arctan \dfrac{m}{m+M}$.

Let the sides of the lamina have length $2a$ and its diagonals $2d$. Since the lamina is uniform its centre of mass is at O, and is at G when the mass m is attached.



Taking torques about G we have $OG = \dfrac{m}{M+m} d$ and $GB = \dfrac{M}{M+m} d$ and these can be written as $OG = \lambda d$ and $GB = \mu d$, with $\lambda = \dfrac{m}{M+m}$ and $\lambda + \mu = 1$. If the angle OAG is α, the angle through which the lamina has rotated, then $\tan\alpha = OG/OA = \lambda$. Finally we have the result that $\alpha = \arctan \dfrac{m}{m+M}$.

Now consider the right angled triangle OAB. Let P and Q be the feet of the perpendiculars from G and O on to the side AB. Now $\tan\beta = GP/AP$. Since

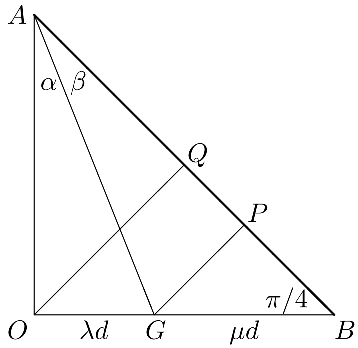
$$AP \;=\; AQ + QP \;=\; a + OG\cos\frac{\pi}{4} \;=\; a + \frac{\lambda d}{\sqrt{2}} \;=\; (1+\lambda)a$$

and $GP = PB = (1-\lambda)a$, it follows that $\tan\beta \;=\; \dfrac{1-\lambda}{1+\lambda}$. As $\alpha + \beta = \dfrac{\pi}{4}$, we have

$$\arctan\lambda + \arctan\frac{1-\lambda}{1+\lambda} \;=\; \frac{\pi}{4}.$$

This identity is interesting, since, by using the double-angle formula $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ with $\tan\theta = 1/5$, we have $\tan 2\theta = 5/12$ and $\tan 4\theta = 120/119$. These give $\arctan 120/119 = 4\arctan 1/5$. With $\lambda = 120/119$ in the identity we end up with the well-known Machin's formula:

$$\frac{\pi}{4} \;=\; 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$



Problem 236.5 – Harmonic quotients

Let

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \;=\; \frac{a(n)}{b(n)},$$

the nth harmonic number expressed as a fraction in its lowest terms (so that $\gcd(a(n), b(n)) = 1$). Let $s(n) = b(n)/b(n-1)$.

For which n do we have $s(n) < 1$. In particular, show that $s(2{\cdot}3^n) = 1/3$.

Thanks to Victor Moll for the idea behind this problem.

A differential equation on the integers

Tommy Moorhouse

Preliminaries The set of functions from the positive whole numbers with values in the integers has some interesting properties. One of these is the fact that the product of two such functions f and g given by

$$f * g(n) \; = \; \sum_{jk=n} f(j)g(k)$$

is commutative, associative and distributive over addition. That is, it is a product with properties very much like ordinary multiplication. When considering differential equations satisfied by ordinary functions we often make use of the following property, Leibniz's rule:

$$\frac{d}{dx}(fg) \; = \; \left(\frac{d}{dx}f\right)g + f\frac{d}{dx}g.$$

If we want this rule to extend to our integer functions with the $*$ product we find that, denoting differentiation by ∇, so that the function obtained by differentiating f is ∇f, the only possibilities are

$$\nabla f(n) \; = \; \kappa(n)f(n),$$

where κ satisfies $\kappa(mn) = \kappa(m) + \kappa(n)$. We will use the specific function $\kappa(p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}) = k_1p_1 + k_2p_2 + \cdots + k_rp_r$. Here the ps are all primes and the function maps an integer to the sum of its prime factors properly counted.

An aside Some readers will be familiar with the 'finite difference' derivative $\nabla f(n) = f(n) - f(n-1)$. Although this derivative can be applied to integer-valued functions there is a sense in which it is not natural—it does not satisfy Leibniz's rule when the product is given by $*$. It does arise more naturally in a related context to be explored in a future article.

Differential equations It is now interesting to explore the solutions of 'differential equations' in the above setting. The simplest such equation is $\nabla f = z$, where $z(n) = 0$ for all n (in other words $\nabla f(n) = 0$ for all n.) Since $\kappa(1) = 0, \kappa(n) > 0$ we must have $f(n) = 0$ for $n > 0$ with $f(1)$ undetermined. Then f is just a multiple of I, the identity function. This is analogous to the constant functions in ordinary calculus having vanishing derivatives.

In ordinary calculus the equation

$$\frac{df}{dx} = \lambda f$$

may be used to define the exponential functions $Ce^{\lambda x}$, with constant C, following the idea that $de^{\lambda x}/dx = \lambda e^{\lambda x}$. We can take the differential equation to define a set of functions of x, one for each value of λ, by

$$e^{\lambda}(x) = e^{\lambda x}.$$

The functions $E[m]$ We consider solutions of the analogous equation for integer functions

$$\nabla f = mf,$$

where m is a positive integer. You can check that, since

$$\nabla f(n) = \kappa(n) f(n),$$

f must be non-zero only for n such that $\kappa(n) = m$, and at these arguments $f(n)$ is not determined. This means that the differential equation has many distinct solutions (corresponding, perhaps, to the choice of C above). For later convenience we choose one solution, namely $f(n) = n$ if $\kappa(n) = m$, $f(n) = 0$ otherwise, and call it $E[m]$. It is important to note that $E[m]$ is a function, not a number. Each $E[m]$ acts on the natural numbers to give an integer $E[m](n)$.

It is interesting to see to what extent the function $E[m]$ shares the familiar properties of the exponential function. For example, we can check that

$$E[m] * E[n] = E[m+n]$$

as follows:

$$\begin{aligned} \nabla(E[m] * E[n]) &= (\nabla E[m]) * E[n] + E[m] * \nabla E[n] && (1) \\ &= (m+n) E[m] * E[n]. && (2) \end{aligned}$$

Therefore $E[m] * E[n]$ satisfies the definition of $E[m+n]$ and, by our choice above, is uniquely determined.

The function $E[0]$ satisfies $\nabla E[0] = z$, where $z(n) = 0$ for all n, from which we deduce that $E[0](n)$ vanishes except, possibly, at $n = 1$, where its value is undetermined. We choose $E[0](1) = 1$.

Analogous to the property $\log(e^x) = x$ we see that

$$\kappa(E[m](n)) \;=\; m$$

for all n in $\kappa^{-1}(m)$. The analogue of $e^{\log(x)} = x$ is not as straightforward to interpret. It would give

$$E[\kappa(m)](n) \;=\; n$$

if $\kappa(n) \;=\; \kappa(m)$, but this does not allow us to uniquely specify the result. Although $E[\kappa(m)](n) = m$ is one of the possibilities other integers, namely all those in $\kappa^{-1}(m)$, are candidates.

You may like to check whether any of the other properties of the exponential function make sense in this setting, or try to solve other differential equations on the integers.

Letter

Leibniz's formula for π

Dear Tony,

I would like to make the following comments on the historical introduction to Sebastian Hayes's paper 'Leibniz's Formula for π' in the April 2010 issue of M500.

In the first place neither Newton nor Leibniz invented calculus, a clear statement of the calculus relationship between the infinite series for sines and cosines is to be found in the works of Madhava of Sangamagrama in 1425. The infinite series for arctangents is based on Newton's reversion of infinite series described in the *epistola posterior*, 1676, intended for Leibniz. In 1671 James Gregory had applied Newton's reversion of series technique to the infinite series for tangents to arrive at the infinite series for arctangents, leading to the so-called discovery of Leibniz. The reversion of an infinite series has nothing to do with calculus.

I would agree with Sebastian Hayes's query about the $\Gamma(0.5) = \sqrt{\pi}$; this I think was obtained from Stirling's formula reconciling factorials with powers. This formula was not intended for low amounts, nevertheless someone seems to have ignored this limitation to produce an elegant but nonsensical result.

Peter L. Griffiths

Panic strikes again

Ralph Hancock

Once more Dr Urban Panic summoned me to Antibes. It was with a familiar sinking feeling that I approached his sumptuous villa, where some gardeners were cleaning graffiti from the boundary wall, including a puzzling drawing of a winged pig that seemed to be relieving itself on the heads of angry people below.

Inside there was now a small, square ploughed field beside the house, surrounded by a tall wire fence. Although the Mediterranean sky was cloudless, the furrows did not look sunlit. I watched a gull glide over the fence—and abruptly vanish.

With customary effusiveness the doctor welcomed me into his laboratory, which contained some odd-looking devices, one of them apparently a cross between a combine harvester and a bathyscaphe. "You remember that jam sandwich I did last year?" (It had taken weeks to get the taste of his iron jam out of my mouth.) "Well, I've really made progress on that. The thing was just inverted in the fourth dimension—no more topologically significant than turning over a sheet of paper. But now we're well on the way to a real four-dimensional sandwich."

"Four-dimensional bread?" I said. "What are you growing in that field?"

"Clever, isn't it? I had to dig down 50 metres to fit in the field. Funny thing is, we found that ordinary bread wheat grows quite well in 4D. It's a hexaploid, you see—three times the usual number of chromosomes. We used hexaploid brassicas for the trials, so you can have mustard on your sandwich."

"And butter?"

"Ah, butter is *naturally* 4D. You make it out of cream, which is an emulsion of fat droplets in water. Churn it a bit, and suddenly it becomes an emulsion of water droplets in fat. Obviously, you couldn't do that in 3D."

He showed me a white cube. "Now, you'll be wondering how we slice the bread. Ever heard of the ham sandwich theorem? In n-dimensional space a sandwich made of n objects can be cut exactly in half by an $(n-1)$-dimensional knife. Just as an infinitely thin pancake, 2D, can be bisected by a wire, 1D. So here's our 3D knife." He produced a cubical tin box the same size as the bread, and dropped both into a machine.

"We can't put a handle on the knife, so we have to fire it through the bread." He pressed a button and there was a sharp bang. Two white cubes, seemingly the same size as the original, dropped out of the machine.

"Now we butter it." He reinserted both cubes and pressed another button. This time one cube emerged. It had a yellow line of butter halfway down each side. Worryingly, all the lines joined end to end around the edges, no matter which way you looked at it. It made my head hurt.

I said, "You talked about a *ham* sandwich."

"Hmm, we're still working on our four-dimensional pigs. We got as far as hexaploids, but the mutation was lethal." He passed me some distressing photographs. "But I'm sure we'll crack it in time."

"Well," I said, "at least you've got as far as bread and butter." I picked up the cube.

"Don't eat that!" shouted the doctor. "It'll disrupt the very fabric of your being. We fed one to one of our pigs, and it literally exploded. They found part of its head in Nice, and a leg in Cannes."

"Then how do propose to market your sandwiches if people can't eat them?"

"Not a problem. Think of noodles. People buy them all the time and no one can eat *them*."

Leibniz's rules for a certain function ring

Tommy Moorhouse

Consider the integer-valued functions from the non-negative integers (i.e. $0, 1, 2, \dots$) with pointwise addition $(f+g)(n) = f(n)+g(n)$ and the product

$$f \circ g(n) \equiv \sum_{j+k=n} f(j)g(k),$$

where the sum extends over pairs of non-negative integers summing to n. Define $N(m) = m$ for integer m, and define a pointwise product $f \cdot g$ by

$$f \cdot g(n) \;=\; f(n)g(n).$$

See if you can show that $\nabla f \equiv N \cdot f$ satisfies Leibniz's rule

$$\nabla(f \circ g) \;=\; (\nabla f) \circ g + f \circ (\nabla g),$$

i.e.

$$N \cdot (f \circ g) \;=\; (N \cdot f) \circ g + f \circ (N \cdot g).$$

Can you describe the solutions to the differential equation $\nabla f = 0$?

Professor Pile's prime pathway revisited

Chris Pile

Returning from my visit to the professor [M500 **229** 14–15] I attempted to answer some of the queries raised. I started by looking for a century with no primes around the 4000th, as he had suggested, and I discovered that the 4133rd century has only one prime, 413353. I thought that primes had a gregarious nature so I was surprised to see this isolated example. I envisaged trudging along the pathway over the blue ridge mountains on the trail of the (most) lonesome prime!

The 4921st century has two primes (492103 and 492113), two adjacent orange [black] tiles, and then 113 yellow [white] tiles before the next orange tile (492227). Over a hundred yellow tiles but still no complete century block. The professor must have misunderstood!

With fewer primes in each block it is easier to check for patterns. The 5404th century with four primes is symmetrical about the '5' line, as is the 6327th century with six primes and the 6828th with two primes.

The 7018th century, with four primes, is the same when turned through 180 degrees. The professor has admitted there were times when he didn't know whether he was coming or going!

Checking the centuries with two primes, I have found two patterns the same: the 6194th century and the 7454th century. The pattern extends for -2 and $+1$ decades into adjoining centuries. The 7837th century has 17 primes while the 7839th century has only two.

There are 40 potential prime positions in each century (giving a maximum of 2^{40} possible patterns) although it appears that only about half the positions can be occupied in one block. Of these 40 positions, there are at least 12 (and at most 14) non-primes divisible by 3, at least five (and at most six) non-primes divisible by 7. Allowing for multiples of 21, at least 15 of the 40 potential positions are occupied by numbers divisible by 3 or 7, leaving 25 potential primes. Removing multiples of 11 brings the maximum possible number of primes down to 23. Apart from the first two century blocks, I had not expected to see any more than 17 primes. Then I read M500 **226** and saw Wroblewski's 26-digit prime 18-tuplet, which has 18 primes in the century block.

More questions arise.

(1) Where is the first prime-free century?

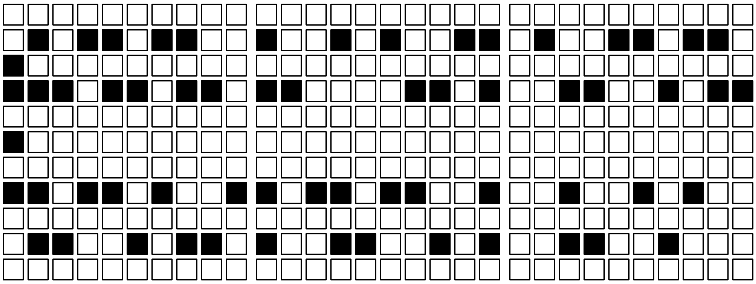
(2) How many different patterns are there?

(3) How many patterns occur a finite number of times?

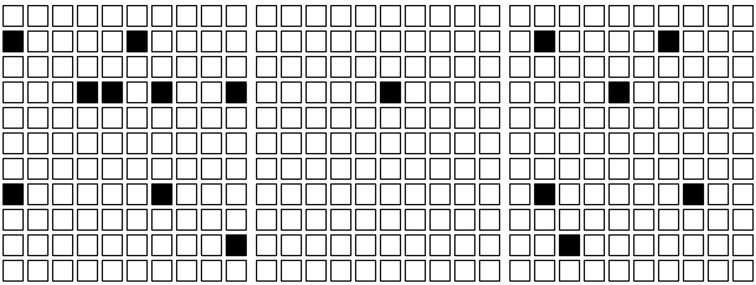
(4) Is the number of 18-prime centuries finite?

(5) What is the average number of primes per century?

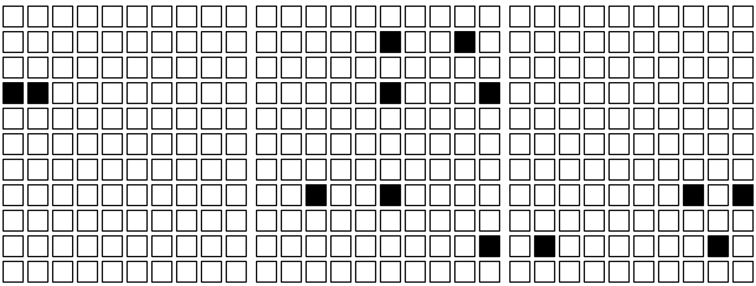
[To remind you of what we are talking about, or if you haven't seen M500 **229**, here is what the beginning of the 'prime pathway' looks like. Century-blocks 1–3. — TF]



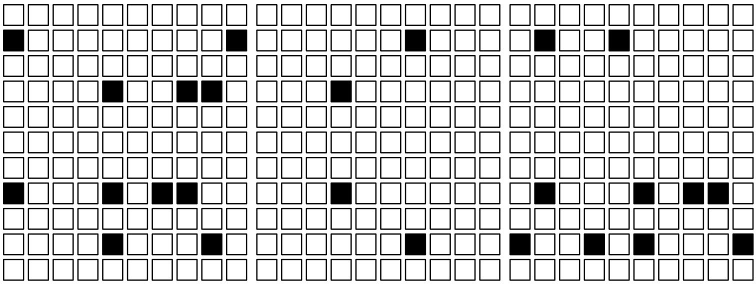
A lonesome prime, 413353. Blocks 4132–4134.



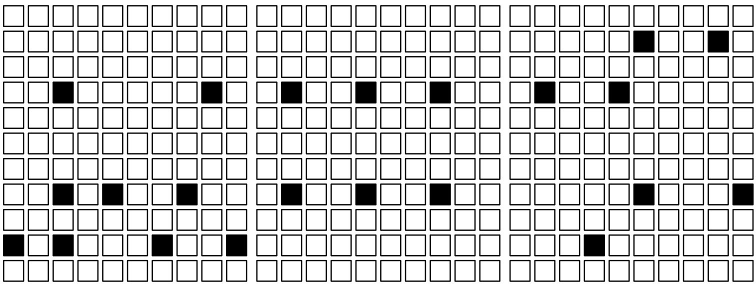
A blank century slightly displaced. Blocks 4921–4923.



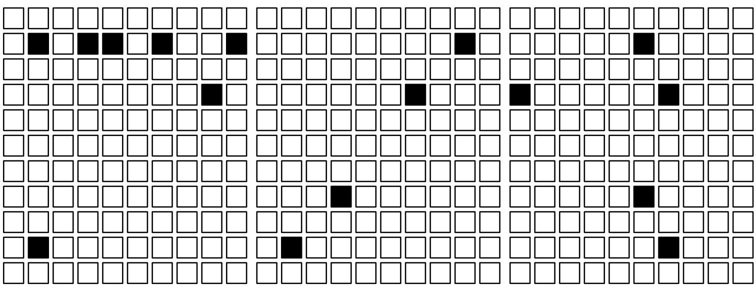
Symmetrical patterns about the '5' line. Blocks 5403–5405 . . .



. . . and 6326–6328.



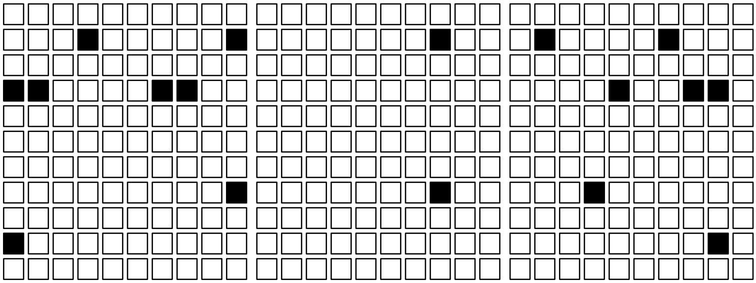
A 180-degree reversible century. Blocks 7017–7019.

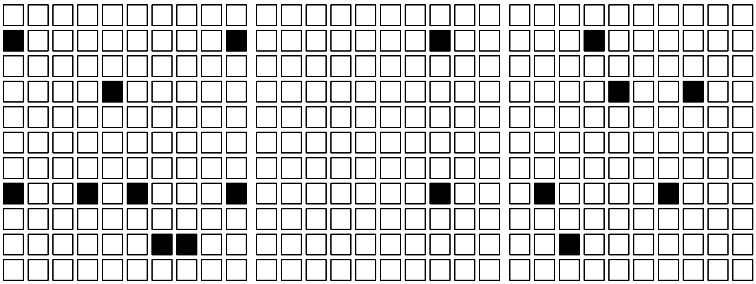


[To make use of this space at the bottom of the page before continuing the pathway, it is worth pointing out that if you manage to obtain a definitive answer to question (4) above, there might be a substantial prize to claim. And an interesting related problem occurred to me.

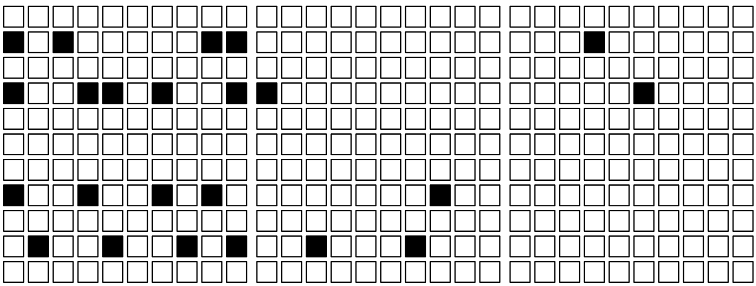
(4a) Must 18 primes in a century block include at least one occurrence of twin primes? — TF]

A repeated pattern of two primes in a century. Blocks 6193–6195 and 7453–7455.





A dense century containing 17 primes followed by two sparse centuries. Blocks 7837–7839.



Problem 236.6 – Products

For integer n, compute

$$\prod_{i=2}^{n}\prod_{j=1}^{i-1}\sin\frac{j\pi}{i}\quad\text{and}\quad\prod_{i=2}^{n}\prod_{j=1}^{i-1}\cos\frac{j\pi}{i}.$$

M500 Winter Weekend 2011

100% Proof

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Florence Boot Hall, **Nottingham University**

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http://www.m500.org.uk/winter/booking.pdf

If you have no access to the internet, send a stamped addressed envelope to

Diana Maxwell.

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