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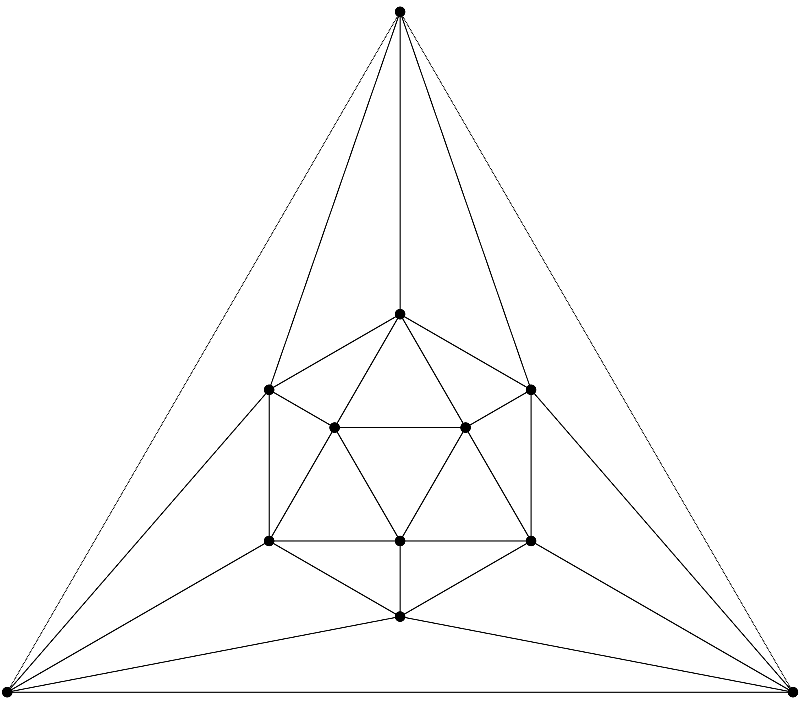

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The arithmetization of quadratic irrationals

Finding rational approximations to $\sqrt{N}$

Bryan Orman

Problem statement: *To obtain a rational approximation to the square root of a positive, square-free integer N, to a prescribed accuracy, the calculations to be performed without the aid of a calculator. The procedure needs to be efficient and relatively easy to apply.*

There are several methods that could be employed to address the above problem and these will be considered in turn, from the most demanding to the most accessible. All the methods have special merits and they are related, as will be seen. Reference will be made to just one book, David Burton's *Elementary Number Theory*, Allyn & Bacon, 1980, but other books on number theory cover much of the material needed for background reading.

Continued fractions (Burton, pages 313 ff)

The continued fraction representation of $\sqrt{N}$ gives rise to convergents $C_k = p_k/q_k$ and these furnish the best approximations to $\sqrt{N}$ in that every other rational number with the same or smaller denominator differs from $\sqrt{N}$ by a greater amount. Furthermore, the accuracy of the approximation is given by the inequality $|\sqrt{N} - p_k/q_k| < 1/(q_k q_{k+1})$. So, to apply this result to a given integer N, we need to produce the infinite continued fraction representation of $\sqrt{N}$, and from it the convergents, C_k. The standard notation for the continued fraction is

$$\sqrt{N} \;=\; a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}},$$

or

$$\sqrt{N} \;=\; [a_0; \overline{a_1, a_2, a_3, \dots, a_3, a_2, a_1, 2a_0}].$$

The bar over the block of integers indicates that this block is repeated over and over. The number of terms in the block is called the period of the continued fraction. Note the symmetry exhibited in the block. The convergents are given by the finite continued fraction, obtained by truncating the infinite continued fraction, that is, $C_k = p_k/q_k = [a_0; a_1, a_2, a_3, \dots, a_k]$. In practice the convergents are determined recursively from the linear second-order system:

$$\begin{aligned} p_0 &= a_0, & q_0 &= 1, \\ p_1 &= a_1 a_0 + 1, & q_1 &= a_1, \\ p_k &= a_k p_{k-1} + p_{k-2}, & q_k &= a_k q_{k-1} + q_{k-2}. \end{aligned}$$

An example will illustrate the effort involved with this approach. Consider $N = 33$. The continued fraction representation requires the integer part of $\sqrt{N}$ to be extracted: this is a_0. The remainder is written as a reciprocal, with the process repeated. Thus

$$\sqrt{33} \;=\; 5 + (\sqrt{33} - 5);$$

so $a_0 = 5$. Next,

$$\sqrt{33} - 5 \;=\; \frac{(\sqrt{33}-5)(\sqrt{33}+5)}{\sqrt{33}+5} \;=\; \frac{8}{\sqrt{33}+5} \;=\; \frac{1}{\dfrac{\sqrt{33}+5}{8}}$$

and $(\sqrt{33}+5)/8 = 1 + (\sqrt{33}-3)/8$; so $a_1 = 1$. Then

$$\frac{\sqrt{33}-3}{8} = \frac{1}{2 + \dfrac{\sqrt{33}-3}{3}};$$

so $a_2 = 2$. Thus $\sqrt{33} = [5, 1, 2, 1, 10]$, with period 4, giving $a_0 = 5$, $a_1 = 1$, $a_2 = 2$, $a_3 = 1$, $a_4 = 10$, $a_5 = 1$, etc. Applying the recurrence relations for the ps and qs,

$$\begin{array}{lll} p_0 = 5 & q_0 = 1, & \text{giving } C_0 = 5, \\ p_1 = 6 & q_1 = 1, & \text{giving } C_1 = 6, \\ p_2 = 17 & q_2 = 3, & \text{giving } C_2 = 17/3, \end{array}$$

$$C_3 = \frac{23}{4},\ C_4 = \frac{247}{43},\ C_5 = \frac{270}{47},\ C_6 = \frac{787}{137},\ C_7 = \frac{1057}{184},\ C_8 = \frac{11357}{1977} \ldots .$$

Checking the accuracy, we expect $C_5 = 270/47$ to be a rational approximation to $\sqrt{33}$ with an error of less than

$$1/q_5 q_6 \;=\; 1/(47 \cdot 137) \;\approx\; 0.00016.$$

Indeed $|\sqrt{33} - 270/47| \approx 0.00012$.

So this method, based on the continued fraction representation of $\sqrt{N}$, will produce rational approximations with predictable accuracy. However, it is extremely inefficient in that all the convergents need to be generated before the required accuracy is achieved.

Pell–Fermat equations (Burton, pages 329 ff)

The appropriate Pell–Fermat equation for our purpose is $x^2 - Ny^2 = 1$, and we seek the pairs of positive integers (x_k, y_k), $k \geq 1$, satisfying this equation for a given N. We note that $(x_0, y_0) = (1, 0)$ is a trivial solution and so we call (x_1, y_1) the fundamental solution. One way of generating solutions is to substitute successively $y = 1, 2, 3, \ldots$ into the expression $1 + Ny^2$ until perfect squares are obtained.

For our example we find that $1 + 33 \cdot 4^2 = 23^2$ and $1 + 33 \cdot 184^2 = 1057^2$, so that $(x_1, y_1) = (23, 4)$ and $(x_2, y_2) = (1057, 184)$. These are just the convergents C_3 and C_7 obtained previously. This ties in with a theorem that states that 'All positive solutions of $x^2 - Ny^2 = 1$ are given by C_{kn-1} if the period of the continued fraction representation of $\sqrt{N}$ is even, or by C_{2kn-1} if the period is odd'. For $N = 33$ we have $n = 4$ so we expect to get C_3, C_7, $\ldots$, but the period will not be known if this method is employed!

This method generates a subset of the convergents of $\sqrt{N}$ and does save some effort, a good thing, but it does require the perfect square calculation, over and over again. For $N = 33$ the solutions are found for $y = 4$ and $y = 184$ which shows that this part of the procedure is time consuming. Fortunately this can be avoided since only the fundamental solution is needed to generate all the solutions. If $x_1^2 - Ny_1^2 = 1$, then

$$(x_1 + \sqrt{N}y_1)(x_1 - \sqrt{N}y_1) \;=\; 1$$

and so

$$(x_1 + \sqrt{N}y_1)^n(x_1 - \sqrt{N}y_1)^n \;=\; 1.$$

Setting $x_n + \sqrt{N}y_n = (x_1 + \sqrt{N}y_1)^n$, we have $x_n - \sqrt{N}y_n = (x_1 - \sqrt{N}y_1)^n$ and hence $x_n^2 - Ny_n^2 = 1$. This means that $x_n + \sqrt{N}y_n = (x_1 + \sqrt{N}y_1)^n$ will generate all solutions of the Pell–Fermat equation from the known fundamental solution (x_1, y_1).

For our example we have the fundamental solution $(x_1, y_1) = (23, 4)$ and so $x_2 + \sqrt{33}y_2 = (23 + 4\sqrt{33})^2 = 1057 + 184\sqrt{33}$, giving $C_7 = 1057/184$, and $x_3 + \sqrt{33}y_3 = (23 + 4\sqrt{33})^3 = 48599 + 8460\sqrt{33}$, giving $C_{11} = 48599/8460$.

From the equation defining the solution (x_n, y_n) we can derive uncoupled second-order linear recurrence relations similar to those for the continued fraction pair (p_k, q_k), namely $x_{n+2} = 2x_1x_{n+1} - x_n$ and $y_{n+2} = 2x_1y_{n+1} - y_n$, for $n > 0$. The initial conditions are the trivial solution $(x_0, y_0) = (1, 0)$ and the fundamental solution (x_1, y_1). Unfortunately not all integers have convenient fundamental solutions for this method to be generally useful; $N = 33$ has (23,4) as we have seen, but $N = 13$ has $(649, 180)$ and $N = 29$ has $(9801, 1820)$. The culprit of course is the period of the continued fraction for these integers, both 13 and 29 having a period of length 5.

Second-order recurrence relations (Burton, pages 286 ff)

Since both the previous methods involved second order recurrence relations for (p_k, q_k) and (x_n, y_n), and, recalling that the classical Fibonacci sequence u_n, generated from $u_{n+2} = u_{n+1} + u_n$ with $u_1 = u_2 = 1$, has the property that $v_n = u_{n+1}/u_n \to (1 + \sqrt{5})/2$ as $n \to \infty$, it should be possible to construct a recurrence relation whose sequence leads to a rational approximation to $\sqrt{N}$.

Consider the problem: $u_{n+2} = 2\alpha u_{n+1} + \beta u_n$, with u_1 and u_2 prescribed. The trial solution $u_n = \lambda^n$ leads to the quadratic equation $\lambda^2 - 2\alpha\lambda - \beta$, with solutions $\lambda_\pm = \alpha \pm \sqrt{\alpha^2 + \beta}$, and the general solution, $u_n = A\lambda_+^n + B\lambda_-^n$. Since $|\lambda_-/\lambda_+| < 1$ we have

$$v_n \;=\; u_{n+1}/u_n \to \lambda_+ \;=\; \alpha + \sqrt{\alpha^2 + \beta},$$

or

$$v_n - \alpha \;=\; u_{n+1}/u_n - \alpha \to \sqrt{\alpha^2 + \beta} \;=\; \sqrt{N},$$

say. The rate of convergence depends on $|\lambda_-/\lambda_+| < 1$, and the decomposition $N = \alpha^2 + \beta$, with $|\beta| < \alpha$, will produce the fastest convergence of v_n to $\sqrt{N}$. Since $v_n \approx 2\alpha$ for large n, the most convenient starting values u_1 and u_2 should mirror this and so $u_1 = 1$, $u_2 = 2$ would be the natural choice.

In our example, $N = 33 = 6^2 - 3$, so that $\alpha = 6$ and $\beta = -3$. The recurrence relation becomes $u_{n+2} = 12u_{n+1} - 3u_n$, with $u_1 = 1$ and $u_2 = 12$. Simple evaluations lead to $u_3 = 141$, $u_4 = 1656$ and $u_5 = 19449$. Furthermore $v_1 = 6$, $v_2 = 23/4$, $v_3 = 270/47$ and $v_4 = 1057/184$, and we see that this simpler method has produced the known convergents. This method works for all N and so is certainly superior to both the previous methods.

The Newton–Raphson method

From the Pell–Fermat method we have

$$x_{2n} + \sqrt{N}y_{2n} \;=\; (x_n + \sqrt{N}y_n)^2 \;=\; x_n^2 + Ny_n^2 + 2x_n y_n\sqrt{N}.$$

Thus $x_{2n} = x_n^2 + Ny_n^2$ and $y_{2n} = 2x_n y_n$ and the convergent $z_{2n} = x_{2n}/y_{2n}$ is equal to $(x_n^2 + Ny_n^2)/(2x_n y_n)$, or

$$z_{2n} = \frac{1}{2}\left(z_n + \frac{N}{z_n}\right).$$

This is precisely the Newton–Raphson method applied to the function $f(x) = x^2 - N$.

Another way to describe this method is 'squaring the rectangle', since a rectangle with sides z_n and N/z_n has area N. A new rectangle, with one side equal to the average of these two sides, that is, z_{2n}, and the other equal to N/z_{2n}, will have area N. These rectangles become more and more square-like, and in the limit they become a square of side $\sqrt{N}$. In our example, $N = 33$ and $z_1 = 6$, the α in the previous method. The sequence produced by the iteration is simply 23/4, 1057/184, 2234497/388976, ... coinciding with the familiar convergents C_3, C_7 and, unexpectedly, C_{15}.

Conclusion

The Newton–Raphson method is the superior method for finding rational approximations to $\sqrt{N}$ and would be used in all circumstances. But what of the other methods? They are all linked as they reflect the quadratic nature of the problem and the appearance of second-order recurrence relations is therefore no surprise. What has been presented here is a modest unification of four related methods through a quite simple problem.

Last words

Did you know that the last two words are sufficient (and sometimes necessary) to identify a Shakespeare play? See how many you can recognize before you look them up. (This is like something we did in M500 **207**.)

say Amen!	restore amends.	before another.	memory. Assist.
lead away.	us befall!	untimely bier.	'em clap.
every day.	happy day.	my diseases.	has ending.
me farewell.	Mistress Ford.	me free.	mutual happiness!
our hearts.	lasting joy.	should know.	so long.
good night.	shot off.	a peace.	up, pipers!
take pity.	heart relate.	Nerissa's ring.	and realm.
her Romeo.	at Scone.	great solemnity.	tam'd so.
drums strike.	acceptance take.	but true.	this way.
be won.			

'A new galaxy spotted by the Hubble telescope is 13.1 billion light years away and would take the space shuttle, at 17,600 mph, 1.35 million years to reach it, astronomers said.' [*Telegraph* (21 Oct 2010), spotted by Ralph Hancock. OK, we give up! Where did the 1,350,000 years come from?]

Solution 231.5 – Four cos and four tans

Prove that

$$\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} \;=\; 1 \quad\Rightarrow\quad \frac{\cos^4 B}{\cos^2 A} + \frac{\sin^4 B}{\sin^2 A} \;=\; 1.$$

Tony Forbes

The obvious solution to

$$\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} \;=\; 1 \tag{$*$}$$

is $A = B$. If, as one reader suggested, this is the only solution of $(*)$ then the problem becomes trivial. However, I'm not so sure.

Anyway, I thought I would have a go—on the assumption that there really do exist non-trivial solutions to $(*)$. As is usual with problems like this the difficult part is trying to discover the right approach.

Writing $(*)$ as

$$(\cos^4 A)(\sin^2 B) + (\sin^4 A)(\cos^2 B) \;=\; (\cos^2 B)(\sin^2 B)$$

and using $\cos^2 B + \sin^2 B \;=\; 1$ to expand the right-hand side to either $\cos^2 B - \cos^4 B$ or $\sin^2 B - \sin^4 B$, we obtain

$$\begin{aligned} \cos^4 B &= \cos^2 B - (\cos^4 A)(\sin^2 B) - (\sin^4 A)(\cos^2 B), \\ \sin^4 B &= \sin^2 B - (\cos^4 A)(\sin^2 B) - (\sin^4 A)(\cos^2 B). \end{aligned}$$

Now replace $\sin^4 A$ by $(1 - \cos^2 A)^2$ in the first equality, replace $\cos^4 A$ by $(1 - \sin^2 A)^2$ in the second and simplify:

$$\begin{aligned} \cos^4 B &= -(\cos^2 A)(\cos^2 A - 2\cos^2 B), \\ \sin^4 B &= (\sin^2 A)(1 + \cos^2 A - 2\cos^2 B). \end{aligned}$$

Hence $\cos^4 B/\cos^2 A + \sin^4 B/\sin^2 A = 1$, as required.

Solution 233.4 – Three tans

Show that the cubic $k^3 - 21k^2 + 35k - 7 = 0$ has roots $\tan^2 \frac{1}{7}\pi$, $\tan^2 \frac{2}{7}\pi$ and $\tan^2 \frac{3}{7}\pi$.

Norman Graham

See page 11 for my solution to Problem 233.2 – Three secs. The two problems are the same!

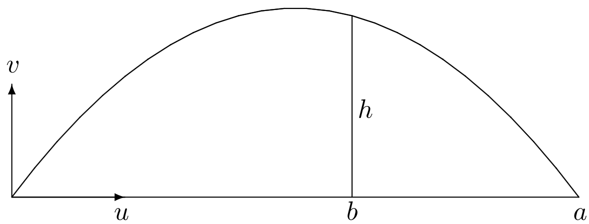
Solution 233.1 – Hill

A cannon of mass M fires a shot of mass m to hit a target at distance a. At distance b in the line of fire there is a hill of height h. Assuming that the shell just clears the hill before going on to strike the target, prove that the gun must have been aiming at an angle of

$$\arctan\left(\frac{M}{M+m}\cdot\frac{ah}{b(a-b)}\right)$$

to the horizontal.

Norman Graham



The standard formula is $s = ut + \frac{1}{2}\alpha t^2$, where t = time, s = distance, u = initial velocity and α = uniform acceleration.

Horizontally to the hill, $b = ut$. Vertically, $h = vt - \frac{1}{2}gt^2$. Hence $h = vb/u - \frac{1}{2}g(b/u)^2$. Similarly, at the target, $0 = va/u - \frac{1}{2}g(a/u)^2$. Therefore

$$\frac{1}{2}\,\frac{g}{u^2} \;=\; \frac{v}{au} \qquad \text{and} \qquad h \;=\; \frac{vb}{u} - b^2\,\frac{v}{au}.$$

Hence

$$h \;=\; \frac{v}{u}\left(b - \frac{b^2}{a}\right) \qquad \text{and} \qquad \frac{v}{u} \;=\; \frac{ah}{b(a-b)}.$$

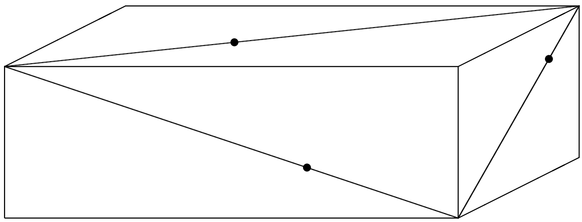
If the gun recoils at velocity u_1, conservation of momentum gives $Mu_1 = mu$. Relative to the gun, the horizontal velocity is $u + u_1 = u(1 + m/M)$. Therefore relative to the gun, the elevation is

$$\tan^{-1}\frac{v}{u+u_1} \;=\; \tan^{-1}\left(\frac{M}{m+M}\,\frac{ah}{b(a-b)}\right).$$

I do not think there is a formula which allows for R, the radius of the Earth, which would involve a variable direction of gravity. In practice, this does not matter since $a \ll R$ for guns. However, it is relevant to the launch of satellites, for which the path is traced by computer simulation.

Solution 232.5 – Three points on a cuboid

Take a cuboid and mark a point on each of three mutually orthogonal faces. Show how to construct the lines on each of the three faces at the intersections of the plane that passes through the three points.



Dick Boardman

We restate the problem as follows. Given a set of axes and given three points, one in the (x, y) plane, P, one in the (y, z) plane, Q, and one in the (x, z) plane, R, find points M on the x axis, N on the y axis and V on the z axis such that V, R and M are collinear, V, Q and N are collinear, and N, P and M are collinear.

The basis of the method is as follows. Let the coordinates of the given points be $P = (P_x, P_y, 0)$, $Q = (0, Q_y, Q_z)$ and $R = (R_x, 0, R_z)$. Choose a variable point V on the z axis, $V = (0, 0, V_z)$. Let N be the point where the line VQ meets the y axis: $N = (0, N_y, 0)$. Let M_1 be the point where the line VR meets the x axis: $M_1 = (M_{1x}, 0, 0)$ and let M_2 be the point where the line NP meets the x axis: $M_2 = (M_{2x}, 0, 0)$. We need to choose V such that $M_1 = M_2$.

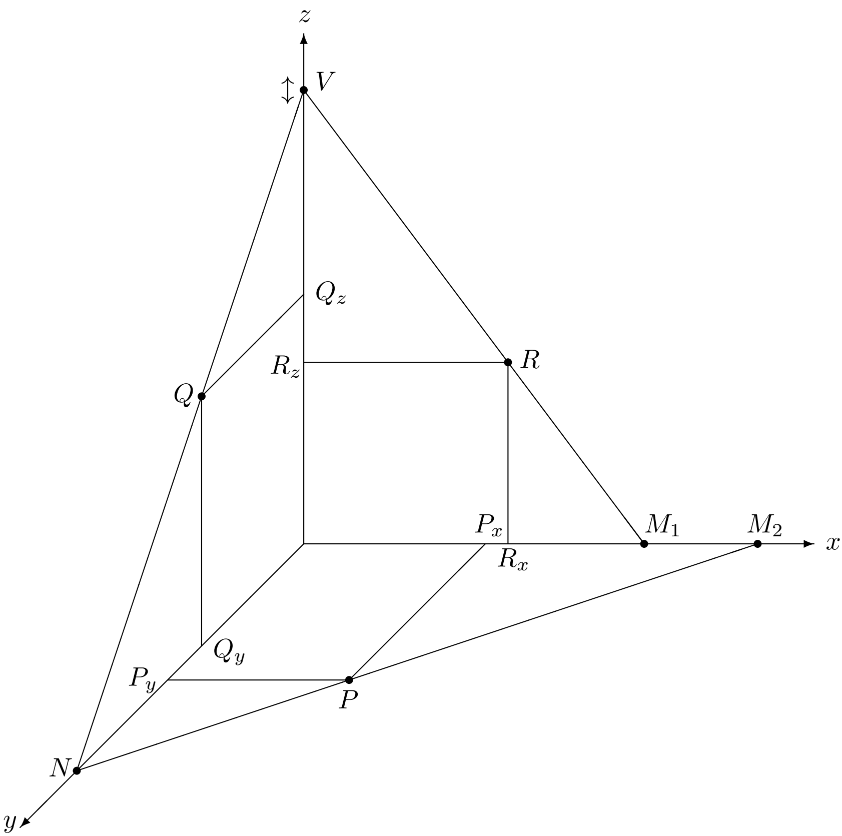
By similar triangles,

$$\frac{Q_y}{V_z - Q_z} = \frac{N_y}{V_z}, \qquad \frac{R_x}{V_z - R_z} = \frac{M_{1x}}{V_z}, \qquad \text{and} \qquad \frac{P_x}{N_y - P_y} = \frac{M_{2x}}{N_y}.$$

Solving $M_1 = M_2$ and eliminating N_y then gives

$$V_z = \frac{P_y Q_z R_x + P_x Q_y R_z}{P_x Q_y + P_y R_x - Q_y R_x}, \qquad N_y = \frac{P_y Q_z R_x + P_x Q_y R_z}{Q_z R_x - P_x Q_z + P_x R_z},$$

$$M_{1x} = M_{2x} = \frac{P_y Q_z R_x + P_x Q_y R_z}{P_y Q_z - P_y R_z + Q_y R_z}.$$



Robin Marks

Starting with the equation of a plane P of the form $ax + by + cz + d = 0$, divide through by a to get an alternative form: $x + a_2 y + a_3 z = a_1$, where a_1, a_2 and a_3 are constants to be determined. To find the point x_P where P intersects the x axis, substitute $y = 0$ and $z = 0$, giving $x_P = a_1$. Similarly $y_P = a_1/a_2$ and $z_P = a_1/a_3$.

Let the three points on the faces of the cuboid be $p_1 = (0, p_{12}, p_{13})$, $p_2 = (p_{21}, 0, p_{23})$, $p_3 = (p_{31}, p_{32}, 0)$. Substitute these into the equation for P to get three simultaneous equations:

$$a_2 p_{12} + a_3 p_{13} = a_1, \quad p_{21} + a_3 p_{23} = a_1 \quad \text{and} \quad p_{31} + a_2 p_{32} = a_1.$$

Solve these three equations for a_1, a_2 and a_3. Intersections of the axes with the plane P are

$$\begin{aligned} x_P &= a_1 = \frac{p_{12}p_{23}p_{31} + p_{13}p_{21}p_{32}}{p_{12}p_{23} + p_{13}p_{32} - p_{23}p_{32}}, \\ y_P &= \frac{a_1}{a_2} = \frac{p_{12}p_{23}p_{31} + p_{13}p_{21}p_{32}}{p_{13}(p_{21} - p_{31}) + p_{23}p_{31}}, \\ z_P &= \frac{a_1}{a_3} = \frac{p_{12}p_{23}p_{31} + p_{13}p_{21}p_{32}}{p_{12}(p_{31} - p_{21}) + p_{21}p_{32}}. \end{aligned}$$

We construct the desired lines by joining x_P, y_P and z_P in pairs.

Problem 238.1 – Disc

Choose a point at random in the unit disc. Choose a direction at random. What is the expected distance from the point to the unit circle in the chosen direction? (Thanks to **Emil Vaughan** for communicating this to me.)

Problem 238.2 – Zeros

Let $f(z)$ be a quadratic in the complex plane. Suppose the zeros of $f(z)$, z_1 and z_2 (which not necessarily distinct), lie in the closed unit disc. Prove that the zero of $f'(z)$ lies in the intersection of the closed unit discs centred on z_1 and z_2.

This is a special case of *Sendov's conjecture*, which you can find in Robin Whitty's web site, http://www.theoremoftheday.org.

Problem 238.3 – Sums

Let

$$S_n(k) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^k,$$

where $k \geq 2$ is an integer. Prove that

$$\frac{S_1(k)}{1} - \frac{S_2(k)}{2} + \frac{S_3(k)}{3} - \cdots \pm \frac{S_k(k)}{k} = 0.$$

Answers to quiz on page 5: R3, MND, CoE, Cor; WT, H62, R2, H8; 12N, JC, T&C, PPoT; AYLI, MWoW, T, 2GoV; AWTEW, H63, MfM, KL; H42, H, Cym, MAaN; TA, O, MoV, H61; R&J, tSp, A&C, TotS; ToA, H5, KJ, LLL; H41.

Solution 233.2 – Three secs

Show that

$$\sec^4\frac{\pi}{7}+\sec^4\frac{2\pi}{7}+\sec^4\frac{3\pi}{7} \;=\; 416.$$

Norman Graham

To show that $\sec^4\pi/7+\sec^4 2\pi/7+\sec^4 3\pi/7=416$ or, equivalently,

$$\bigl(1+\tan^2\tfrac{1}{7}\pi\bigr)^2+\bigl(1+\tan^2\tfrac{2}{7}\pi\bigr)^2+\bigl(1+\tan^2\tfrac{3}{7}\pi\bigr)^2 \;=\; 416,$$

it suffices to prove that

$$3+\sum_{j=1}^{3}t_j+2\sum_{j=1}^{3}t_j^2 \;=\; 416,$$

where $t_j=\tan^2 j\pi/7$. Moreover, since

$$\tan^2\theta \;=\; \frac{\sin^2\theta}{\cos^2\theta} \;=\; \frac{1-\cos 2\theta}{1+\cos 2\theta},$$

we have $t_j=(1-c_j)/(1+c_j)$, where $c_j=\cos 2\pi j/7$.

Consider the equation $\cos 4\theta=\cos 3\theta$. This is satisfied by $4\theta=2n\pi\pm 3\theta$ (n integral), that is, $7\theta=0,2\pi,4\pi,6\pi,\ldots$. But, writing c for $\cos\theta$,

$$\cos 4\theta \;=\; 2\cos^2 2\theta-1 \;=\; 2(2c^2-1)^2-1 \;=\; 8c^4-8c^2+1,$$

$$\cos 3\theta \;=\; \cos 2\theta\cos\theta-\sin 2\theta\sin\theta \;=\; (2c^2-1)c-2c(1-c^2) \;=\; 4c^3-3c.$$

Hence $8c^4-4c^3-8c^2+3c+1=0$; that is,

$$(c-1)(8c^3+4c^2-4c-1) \;=\; 0.$$

Now $c=1$ has solution $\theta=0$ and $8c^3+4c^2-4c-1=0$ has solutions c_j. But $c_j=(1-t_j)/(1+t_j)$; so the t_j are the solutions to

$$8(1-t)^3+4(1-t)^2(1+t)-4(1-t)(1+t)^2-(1+t)^3 \;=\; 0.$$

This gives $t^3-21t^2+35t-7=0$ on collecting like terms. Thus $\sum t_j=21$ and $\sum t_jt_k=35$. Also $\bigl(\sum t_j\bigr)^2=(t_1+t_2+t_3)^2=\sum t_j^2+2\sum t_jt_k$. Therefore

$$3+2\sum t_j+\sum t_j^2 \;=\; 3+2\cdot 21+21^2-2\cdot 35=416,$$

as required.

Solution 232.3 – Three degrees

Devise a ruler-and-compasses geometric construction for

$$\frac{1}{16}\left(\left(\sqrt{6}+\sqrt{2}\right)\left(\sqrt{5}-1\right)-2\left(\sqrt{3}-1\right)\left(\sqrt{5+\sqrt{5}}\right)\right).$$

If you have a calculator handy, you can verify that this expression and $\sin\pi/60$ are the same—hence the title of the problem.

Dick Boardman

We construct angles of 36° and 30°, substract them to get 6° and bisect 6° to get 3°. To illustrate the method, sketch an isosceles triangle ABC where $AB = AC$ and $\angle BAC = 36°$. Then $\angle ACB = \angle ABC = 72°$. Draw the bisector of $\angle ABC$ and let it meet AC at D. Then $\triangle BDA$ is isosceles with base angles 36° and $\triangle BCD$ is isosceles with base angles 72°. Hence $AD = DB = BC$. Triangles ABC and BCD are similar and hence $AB/BC = BC/(AC - AD) = BC/(AB - BC)$. Therefore $AB/BC = (1 + \sqrt{5})/2$.

Now for the actual construction. Construct a right-angled triangle PQR (with right angle at Q), where $PQ = 1$ and $QR = 4$. Extend PR to S where $RS = PQ = 1$. This constructs the length $1 + \sqrt{5}$. Note that $PQ = 1$ is the unit length for the rest of the construction.

Next, construct triangle ABC with $AB = AC = 1 + \sqrt{5}$ and $BC = 2$. This creates an angle of 36° at A. On base AC construct an equilateral triangle ACF and bisect angle FAC to meet CF at G. This produces an angle of 30° at A. Then $\angle BAC - \angle GAC = 36 - 30 = 6 = \angle BAG$. Bisect $\angle BAG$ to give an angle of 3°. Construct a triangle with angles 3°, 90°, 87° and hypotenuse 1. The other sides will be $\sin 3°$ and $\cos 3°$

This needs to be related to the expression given for $\sin 3°$.

Tony Forbes

From the above (or otherwise), we see that $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, $\sin 36° = \sqrt{\frac{1}{8}(5-\sqrt{5})}$ and $\cos 36° = \frac{1}{4}(1+\sqrt{5})$. Therefore

$$\sin 6° \;=\; \sin 36°\cos 30° - \cos 36°\sin 30° \;=\; \frac{1}{8}\left(\sqrt{30-6\sqrt{5}}-\sqrt{5}-1\right).$$

Plug this value into $\sin^2 6° = 4\sin^2 3°(1-\sin^2 3°)$ and solve. The main difficulty will be in convincing yourself that your solution is the same as the one given in the statement of the problem.

Solution 186.3 – Two hands

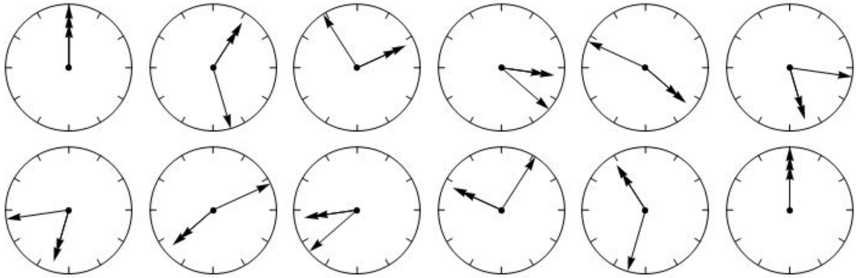
(i) At what times do the hour and minute hands overlap on a normal analogue clock? (ii.a) When do the hour, minute and second hands overlap exactly? (ii.b) Apart from that special case, at what times do the three hands of a clock overlap as closely as possible?

Tony Forbes

This came up in a conversation recently. So, although it has been a long time since it first appeared, I thought it might be of sufficient interest to resurrect it for M500 in case you too get involved in a similar discussion. The interesting part is (ii.b). However, we require the solution to (i), which is not too difficult to obtain with the help of the back of a small envelope: $60h/11$ minutes past h for $h = 0, 1, 2, \ldots, 10$, as shown below together with the case $h = 11$ added at the end. I have also drawn the second hand, and you can read off the answer to part (ii.a) from the first (or last) clock.

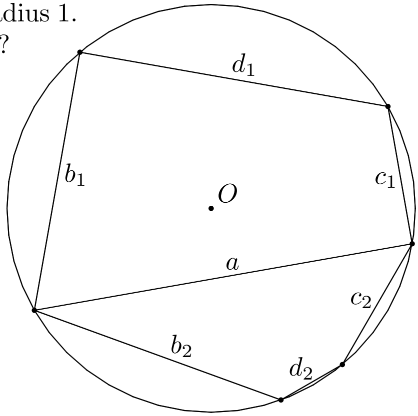
For part (ii.b) we look at the clocks where the second hand is near the other two. There seem to be two candidates: just after 3:16, where the second hand has gone a little too far and must be backed up by about 5 seconds to meet the hour hand, and just before 8:44, where we need to wind the second hand forwards. Surprisingly (at least to me) both cases qualify.

I won't bother with all the tedious analysis. Nevertheless you can easily verify that the results are correct. When the time is 3:16 and 11760/719 seconds exactly, the hour and second hands line up at $(70560/719)^\circ \approx 98.1363^\circ$ (measured clockwise from midday) but the minute hand is slightly behind at $(70200/719)^\circ \approx 97.6356^\circ$. And at 8:43:(31380/719) the directions are $(188280/719)^\circ \approx 261.864^\circ$ and $(188640/719)^\circ \approx 262.364^\circ$ respectively. In each case the angle between the hour and minute hands is $(360/719)^\circ \approx 0.500695^\circ$.



Solution 233.7 – Cyclic quadrilateral

A convex quadrilateral of sides a, b, c and d is inscribed in a circle of radius 1. What is d in terms of a, b and c?



Robin Marks

For ease of description, let a, b and c be arranged in descending size order so that $a \geq b \geq c$. In a unit circle, centre O, draw a chord of length a. Label it a. The chord a divides the circle into a major and a minor segment except for the case $a = 2$ when there are two equal segments (semicircles).

In the major segment (or, if $a = 2$, in a semicircle), draw a chord b_1 of length b starting at one end of a, and another chord c_1 of length c starting at the other end of a. Finally join the free ends of b_1 and c_1 to form the chord d_1 of length d_1, forming a cyclic quadrilateral $ab_1d_1c_1$.

In the minor segment, similarly draw chords b_2 and c_2 from the ends of a. If b_2 and c_2 intersect or their ends touch there is no second solution. Otherwise join the free ends of b_2 and c_2 to form chord d_2 of length d_2, forming a second cyclic quadrilateral $ab_2d_2c_2$.

We need to find d_1 and, if it exists, d_2. Let us introduce the notation that $\hat{x}$ denotes the angle subtended at the centre of the circle by a chord of length x. Then $\sin \hat{x}/2 = x/2$ and hence $\hat{x} = 2 \arcsin x/2$.

The condition for the smaller solution d_2 is that the angles subtended at O by b_2 and c_2 add up to less than the angle subtended by a, that is, $\hat{a} > \hat{b} + \hat{c}$, or, in other words, $\arcsin a/2 > \arcsin b/2 + \arcsin c/2$. If this condition is true then d_2 exists and $\hat{d_2} = \hat{a} - \hat{b} - \hat{c}$, or

$$d_2 \;=\; 2 \sin \left(\arcsin \frac{a}{2} - \arcsin \frac{b}{2} - \arcsin \frac{c}{2} \right).$$

This is a satisfactory solution to the problem as set. However, my pocket calculator cannot calculate arcsin. Could the arcsin terms be eliminated? On the internet I found the addition formula

$$\arcsin x \pm \arcsin y \;=\; \arcsin \left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2} \right).$$

So

$$\begin{aligned}
&\arcsin x - \arcsin y - \arcsin z \\
&\quad = \arcsin\Big(\big(x\sqrt{1-y^2} - y\sqrt{1-x^2}\big) - z\sqrt{1-\big(x\sqrt{1-y^2}-y\sqrt{1-x^2}\big)^2}\Big) \\
&\quad = \arcsin\Big(\big(x\sqrt{1-y^2} - y\sqrt{1-x^2}\big) - z\big(\sqrt{1-x^2}\sqrt{1-y^2}+xy\big)\Big) \qquad (*) \\
&\quad = \arcsin\big(-xyz \\
&\qquad\qquad +x\sqrt{1-y^2}\sqrt{1-z^2} - y\sqrt{1-x^2}\sqrt{1-z^2} - z\sqrt{1-x^2}\sqrt{1-y^2}\big).
\end{aligned}$$

Substituting $x = a/2$, $y = b/2$ and $z = c/2$ gives an 'arcsin-free' solution

$$d_2 = \frac{1}{4}\Big(-abc+a\sqrt{4-b^2}\sqrt{4-c^2}-b\sqrt{4-a^2}\sqrt{4-c^2}-c\sqrt{4-a^2}\sqrt{4-b^2}\Big).$$

Given $2 \geq a \geq b \geq c$, the value the above expression for d_2 lies in the interval $-a$ (when $a = b = c$) to $+a$ (when $b = c = 0$); however if the second cyclic quadrilateral is to be convex, as required, d_2 has to be greater than zero.

Now we consider the solution d_1. The angle $\hat{d_1}$ subtended at the centre O by d_1 is 2π minus the sum of the angles subtended by a, b_1 and c_1; that is, $\hat{d_1} = 2\pi - \hat{a} - \hat{b} - \hat{c}$, or $\arcsin d_1/2 = \pi - \arcsin a/2 - \arcsin b/2 - \arcsin c/2$. Hence

$$d_1 = 2\sin\left(\arcsin\frac{a}{2} + \arcsin\frac{b}{2} + \arcsin\frac{c}{2}\right).$$

Using the arcsin addition formula again on this expression leads to

$$d_1 = \frac{1}{4}\Big(-abc+a\sqrt{4-b^2}\sqrt{4-c^2}+b\sqrt{4-a^2}\sqrt{4-c^2}+c\sqrt{4-a^2}\sqrt{4-b^2}\Big).$$

For convex cyclic quadrilaterals the value the above expression for d_1 lies in the interval 0 (for example when $a = b = c = \sqrt{3}$) to 2 (for example when $a = b = c = 1$). It is possible for d_1 to be negative, for example when $a = b = c = 1.9$, but then the cyclic quadrilaterals are not convex.

Note. To obtain $(*)$ the following has been used:

$$\begin{aligned}
&\sqrt{1-\big(q\sqrt{1-p^2} \pm p\sqrt{1-q^2}\big)^2} \\
&\qquad = \sqrt{1+2p^2q^2-p^2-q^2 \mp 2pq\sqrt{1-p^2}\sqrt{1-q^2}\big)} \\
&\qquad = \sqrt{(1-p^2)(1-q^2)+p^2q^2 \mp 2pq\sqrt{1-p^2}\sqrt{1-q^2}\big)} \\
&\qquad = \sqrt{\big(\sqrt{1-p^2}\sqrt{1-q^2} \mp pq\big)^2} = \sqrt{1-p^2}\sqrt{1-q^2} \mp pq.
\end{aligned}$$

Cellular automata

Chris Pile

I was interested to see the article by Sebastian Hayes on cellular automata (M500 **235**). I first saw this in a full-page article in *The Daily Telegraph* in 1999 (January 13th) prior to the publication of the book by Stephen Wolfram. At the time I was disappointed, as the first 'rule' that I tried was no. 140 (my house number!) and this resulted in just a vertical line.

I have not read Wolfram's *A New Kind of Science*, but I have reviewed this topic after seeing the article in M500. The eight 'generators' that give rise to the 256 rules are best numbered from the right-hand side so that the rule number can be read as the binary equivalent in powers of 2. It is a simple matter to generate the patterns on a computer but many are predictable. To give more insight into pattern generation, the presence of each generator can be shown as a different colour or symbol; as, for example, in the following table.

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
XXX	XXO	XOX	XOO	OXX	OXO	OOX	OOO
H	G	F	E	D	C	B	A

The patterns can be separated into 'odd' and 'even' rule numbers (the presence or absence of generator A). Only generators A, B, C and E can start a pattern. By inspection, it can be seen that generator C produces a vertical line, which occurs 16 times (generated by rule 4 plus multiples of 32, or rule 12 plus multiples of 32). The Sierpinski triangle occurs eight times, being generated by rule 18 plus multiples of 64, or rule 26 plus multiples of 64. Inspection of the eight generators shows that some patterns are symmetrical or exist in mirror-image pairs; e.g. the interesting rule 30 pattern is a mirror image of rule 86 since generator G is a mirror image of generator D.

The even-numbered rules give rise to 32 full triangular patterns and 16 half triangles. The odd-numbered rules produce a background of horizontal stripes, and from rule 129 onwards the pattern is 'negative'. (Rule 129 is a Sierpinski pattern.) The rules produce 36 triangles (13 mirror image pairs and 10 symmetrical). Rule 45 (or 101) gives a complex triangular pattern and rule 169 (or 225) gives a widening 'river' pattern [see back cover].

Cellular automata can produce dynamic patterns as in the game of 'Life', invented by John Conway, in which new cells are 'born' (appear on the grid) and old cells 'die' (disappear from the grid) according to the status of neighbouring cells. The grid could also be triangular or hexagonal. Maybe if the cells are part of a three-dimensional lattice, we could produce patterns to simulate the origin of the universe as the 'big bang', starting from a single cell with the appropriate rules!

Problem 238.4 – Wednesday's child

I have two children, one of whom is a boy born on a Wednesday. Assuming boys and girls as well as days of the week are equally likely, what's the probability that my other child is also a boy?

This is similar to something that appeared in *New Scientist* last year. But not identical, for it differs in one small detail. Anyway, it is possible readers of that excellent magazine will by now have forgotten the issues raised by this innocent-looking probability question.

Skidoo

Eddie Kent

An acquaintance suggested an interesting type of puzzle we could use in M500. A number is asserted as equal to an abbreviation, which has to be reconstructed to a well-known expression. An example is 26 = 'L in the A' with a solution 'letters in the alphabet.' My subconscious tells me I've seen this before, perhaps Jeremy has used something like it, but I had a look on the internet and found some examples. They are often clever, frequently baffling (even when the answer is known[1]), but ultimately trivial. One thing became clear though: 23 is always equal to S, and the S is invariably 'skidoo'. Why?

One is led to New York; to 175 Fifth Avenue at its junction with Broadway at 23rd Street: the location of the first skyscraper. Originally the Fuller but now known as the Flatiron Building, it rises 22 stories and creates a kind of wind tunnel at its base. New Yorkers used to place bets on how far the debris would extend when the wind blew it down. But young men congregated opposite at 23rd to watch ladies' skirts blowing up, sometimes as far as their knees. (See, for instance, http://www.archive.org/details/What_Happened_1901 for an engagingly cheerful piece of film.)

Since this was clearly a nuisance, policemen would regularly turn up to move them on. They were told to 'skedaddle', or more often 'skid(d)oo'. And the operation became known as 'the 23 skidoo'. This has become a common phrase in America, as in 'giving someone the 23 skidoo,' but it does not appear to have crossed the herring-pond. (Although a British punk band did adopt the name, briefly.)

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[1] If you want to tax the little grey cells, find the answer to 8675309 = J. Think US pseudo-punk, Tommy Tutone.

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Front cover: The icosahedron graph