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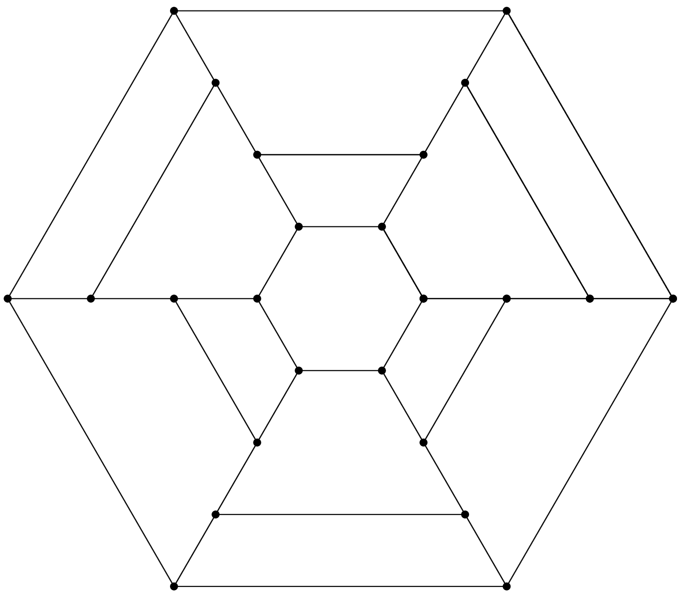

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Interesting representations of squares in base B

Bryan Orman

In this article we address the problem of representing the square of a two-digit number in base B as a four-digit number in base B. If the two-digit number is $(m\ n)$ and the four-digit number is $(p\ q\ r\ s)$, then we require $(m\ n)^2 = (p\ q\ r\ s)$ or, written out in terms of the base B,

$$pB^3 + qB^2 + rB + s \ = \ (mB + n)^2 \ = \ m^2B^2 + 2mnB + n^2.$$

We write this as

$$\begin{aligned} p(B^3 + 1) + q(B^2 - 1) + r(B + 1) - p + q - r + s & \\ = \ (m - n)^2 + m^2(B^2 - 1) + 2mn(B + 1).& \end{aligned}$$

Here all the numbers are in decimal form and, for our analysis, we will usually require $0 < p, q, r, s, m, n < B$.

Now since

$$B^3 + 1 \ = \ (B + 1)(B^2 - B + 1),$$

the above equation shows that $(B + 1)$ divides $(m - n)^2 + p - q + r - s$; so if this $(B + 1)$-free term is zero then the investigation will be simplified considerably. To this end we look at the square of the two-digit number $(m\ m)$ and its representation as one of the four-digit numbers $(a\ a\ a\ a)$, $(a\ a\ b\ b)$ or $(a\ b\ b\ a)$.

An examination of a list of the squares of two-digit numbers in base 10 reveals one really interesting square that's a four-digit number, namely $88^2 = 7744$. Here $m = n = 8$, $p = q = 7$ and $r = s = 4$; so the $(B + 1)$-free term is conveniently zero. (It is evident that $B + 1$ will play a crucial role in the investigations.) Additionally, there are three not so obvious squares, $55^2 = 3025$, $66^2 = 4356$ and $99^2 = 9801$. These have $m = n$ and $p+r = q+s$, again making the $(B + 1)$-free term zero.

Even with the above patterns the analysis will be quite involved; so we will restrict our investigations to just five problems and these will concern techniques that can be used to examine other patterns. The first one, $(m\ m)^2 = (a\ a\ a\ a)$, has just three parameters (B, a and m) and will be fairly straightforward. The second one has the additional parameter b, $(m\ m)^2 = (a\ a\ b\ b)$, as will the third one, $(m\ m)^2 = (a\ b\ b\ a)$. The fourth one will involve yet another parameter n through the two equations $(m\ m)^2 = (a\ a\ b\ b)$ and $(n\ n)^2 = (b\ b\ a\ a)$, to be satisfied simultaneously. The fifth pattern has six parameters, $(m\ m)^2 = (a\ b\ c\ d)$.

Pattern $(m\ m)^2 = (a\ a\ a\ a)$

Written out in full, this gives $aB^3 + aB^2 + aB + a = (mB + m)^2$, which reduces to

$$a(B^2 + 1) \ = \ m^2(B + 1).$$

If we assume that $a|B + 1$, that is $B = ka - 1$, then by eliminating a, a Fermat–Pell equation $B^2 - km^2 = -1$ is obtained. The cases $k = 1$, 3 and 4 give no solutions and so we examine the case $k = 2$. The equation $B^2 - 2m^2 = -1$ has solutions identified by $F_2(-1)$ in the Appendix, so that $B = u_n + 2v_n$ and $m = u_n + v_n$, where the pair (u_n, v_n) is given by the list $P_2(1)$, also in the Appendix. Taking $(u_1, v_1) = (3, 2)$ gives $B = 3 + 2 \cdot 2 = 7$ and $m = 3 + 2 = 5$, whilst $a = \frac{1}{2}(B + 1) = 4$. Further application of the list leads to the following results:

$$\begin{array}{rcll} B &=& 7: & (5\ 5)^2 \ = \ (4\ 4\ 4\ 4), \\ B &=& 41: & (29\ 29)^2 \ = \ (21\ 21\ 21\ 21), \\ B &=& 239: & (169\ 169)^2 \ = \ (120\ 120\ 120\ 120). \end{array}$$

Note that if we eliminate B from the two equations $B = 2a - 1$, $B^2 - 2m^2 = -1$, then we get $m^2 = (a - 1)^2 + a^2$, a familiar Pythagorean requirement, namely

$$5^2 = 3^2 + 4^2, \qquad 29^2 = 20^2 + 21^2, \qquad 169^2 = 119^2 + 120^2, \ldots$$

reflected in the numbers appearing in the above examples.

Pattern $(m\ m)^2 = (a\ a\ b\ b)$

Written out in full, this gives $aB^3 + aB^2 + bB + b \ = \ (mB + m)^2$, which reduces to $(a^2B + b)(B + 1) = m^2(B + 1)^2$, or

$$aB^2 + b \ = \ m^2(B + 1).$$

Using $B^2 = (B + 1)(B - 1) - 1$ this gives $(B + 1)(m^2 - a(B - 1)) = a + b$; so that $B + 1|a + b$ and we assume that $B = a + b - 1$, in keeping with $B = 2a - 1$ obtained previously where $a = b$. Finally, we have $m^2 = a(B - 1) + 1$ or, equivalently, $m^2 = (a - 1)^2 + ab$. Solving for b, we have $b = (m - 1 + a)(m + 1 - a)/a$.

There are two possibilities here; either a is prime or a is not prime.

If a is prime then either $a|m - 1$ or $a|m + 1$; so that $m = ka \pm 1$, and this leads to $b = (k^2 - 1)a + 2 \pm 2k$ and $B = k^2a + 1 \pm 2k$. Note that for a given value of a this procedure generates an infinite number of solutions. The following table illustrates these results.

k	1	1	2	2	3	3	$\dots$
m	$a+1$	$a-1$	$2a+1$	$2a-1$	$3a+1$	$3a-1$	
b	4	0	$3a+6$	$3a-2$	$8a+8$	$8a-4$	
B	$a+3$	$a-1$	$4a+5$	$4a-3$	$9a+7$	$9a-5$	

Example: Setting $a = 5$ and taking $k = 3$ gives either $m = 14$, $b = 36$ and $B = 40$, so that $(5\ 5\ 36\ 36) = (14\ 14)^2$ in $B = 40$, or $m = 16$, $b = 48$ and $B = 52$, so that $(5\ 5\ 48\ 48) = (16\ 16)^2$ in $B = 52$.

If a is not a prime then: if we set $a = a_1a_2$ with $a_1|m-1$ and $a_2|m+1$, so that $m-1 = a_1k_1$ and $m+1 = a_2k_2$, then $a_1k_1 - a_2k_2 + 2 = 0$, it follows that $b = k_1k_2 - a + 2$ and $B = k_1k_2 + 1$.

Procedure: First factorize $a = a_1a_2$ and choose k_1 and k_2 to satisfy $a_1k_1 - a_2k_2 + 2 = 0$. If $k_1 = \alpha$ and $k_2 = \beta$ gives one solution then $k_1 = pa_2 + \alpha$, $k_2 = pa_1 + \beta$, p an integer, gives the general solution.

Example: $a = 15 = 3{\cdot}5$, so that $a_1 = 5$ and $a_2 = 3$ with $5k_1 - 3k_2 + 2 = 0$. This equation has the general solution $k_1 = 2 + 3p$, $k_2 = 4 + 5p$, and so taking $p = 1$ gives $k_1 = 5$ and $k_2 = 9$. It follows that $m = 26$, $b = 32$ and $B = 46$, so that $(15\ 15\ 32\ 32) = (26\ 26)^2$ in $B = 46$.

Optional Exercises: Put $a = 16$, and take $k = 2$ to obtain $(16\ 16\ 54\ 54) = (33\ 33)^2$ in $B = 69$ and $(16\ 16\ 46\ 46) = (31\ 31)^2$ in $B = 61$. Put $a = 16 = 2 \cdot 8$ and take either $a_1 = 8$, $a_2 = 2$, $k_1 = 3$, $k_2 = 13$ or $a_1 = 8$, $a_2 = 2$, $k_1 = 2$, $k_2 = 9$, so that $a_1k_1 - a_2k_2 + 2 = 0$ is satisfied. Thus $(16\ 16\ 25\ 25) = (25\ 25)^2$ in $B = 40$ and $(16\ 16\ 4\ 4) = (17\ 17)^2$ in $B = 19$ are obtained.

Pattern $(m\ m)^2 = (a\ b\ b\ a)$

Written out in full, this gives $aB^3 + bB^2 + bB + a = (mB + m)^2$, which reduces to

$$a(B+1)^2 - (3a-b)(B+1) + 3a - b = m^2(B+1),$$

so that $B+1|3a-b$. If we assume that $B+1 = 3a-b$ then we have to solve $m^2 - 1 = (a-1)(B+1)$ with b given by $b = 3a - B - 1$. We note that the solution will require $3a > B+1$.

Examples: First, $m = 13$ requires $(a-1)(B+1) = 168 = 8{\cdot}21$. So $a = 9$ and $B = 20$ would do and then $b = 3{\cdot}9 - 20 - 1 = 6$. Thus $(9\ 6\ 6\ 9) = (13\ 13)^2$ in $B = 20$. Also $m = 17$ requires $(a-1)(B+1) = 288 = 12 \cdot 24$. So $a = 13$ and $B = 23$ would do and then $b = 15$. Thus $(13\ 15\ 15\ 13) = (17\ 17)^2$ in $B = 23$.

Patterns $(m\ m)^2 = (a\ a\ b\ b)$ **and** $(n\ n)^2 = (b\ b\ a\ a)$

Employing the results from the $(m\ m)^2 = (a\ a\ b\ b)$ investigation gives

$$\begin{aligned} aB^2 + b &= m^2(B+1), & bB^2 + a &= n^2(B+1), & (1)\\ B &= a+b-1, & B &= b+a-1, & (2)\\ m^2 &= (a-1)^2 + ab, & n^2 &= (b-1)^2 + ba, & (3)\\ m^2 &= a(B-1)+1, & n^2 &= b(B-1)+1. & (4) \end{aligned}$$

Adding m^2 and n^2 from (4) gives $m^2+n^2 = (a+b)(B-1)+2$ and substituting $a+b = B+1$ from (2) gives $m^2+n^2 = B^2+1$.

The approach employed for the previous pattern will be used for this system and so we set $m = ka \pm 1$, but to avoid cluttered expressions the $m = ka+1$ case will be investigated first and the $m = ka-1$ case will be quoted.

Substituting $m = ka+1$ into m^2 from (4) gives $B = k^2a + 2k + 1$ and then using this B in $b = B - a + 1$ from (2) gives $b = (k^2-1)a + 2(k+1)$. Finally, substituting this in n^2 from (3) gives

$$n^2 \;=\; (k^2-1)k^2a^2 + 2k(k+1)(2k-1)a + (2k+1)^2.$$

This is a quadratic equation for a, with positive solution

$$\begin{aligned} a &= \frac{-(k+1)(2k-1) + \sqrt{(k^2-1)n^2 + 2(k+1)}}{k(k^2-1)}\\ &= \frac{x-(k+1)(2k-1)}{k(k^2-1)}, \end{aligned}$$

where $x^2 - (k^2-1)n^2 = 2(k+1)$.

Procedure: First decide on a value for k and then solve this Fermat–Pell equation for the general expressions of x and n in terms of (u, v). Then use the value of x to determine a, and finally m, b and B.

Example: Setting $k = 2$ we have $m = 2a+1$ and $a = (x-9)/6$. The Fermat–Pell equation becomes $x^2 - 3n^2 = 6$, corresponding to $F_3(6)$ in the Appendix. So we have $x = 3u+3v$ and $n = u+3v$, where the pair (u, v) is given by $P_3(1)$. The other formulas give $a = \frac{1}{2}(u+v-3)$, $m = u+v-2$, $b = \frac{3}{2}(u+v+1)$ and finally $B = 2u+2v-1$. The pair (u, v) will generate an infinite set of representations; we will present just two of them. From $P_3(1)$ we choose $(7, 4)$ giving $n = u+3v = 7+3\cdot 4 = 19$, $a = \frac{1}{2}(7+4-3) = 4$, $m = u+v-2 = 7+4-2 = 9$, $b = \frac{3}{2}(u+v+1) = \frac{3}{2}(7+4+1) = 18$

and $B = 2u + 2v - 1 = 2 \cdot 7 + 2 \cdot 4 - 1 = 21$. So we have the coupled pair $(4\ 4\ 18\ 18) = (9\ 9)^2$ and $(18\ 18\ 4\ 4) = (19\ 19)^2$ in $B = 21$. Choosing $(26, 15)$ from $P_3(1)$ gives the base $B = 81$ with the coupled pair $(19\ 19\ 63\ 63) = (39\ 39)^2$ and $(63\ 63\ 19\ 19) = (71\ 71)^2$.

Optional Exercise: Set $k = 3$. Then $x^2 - 2(2n)^2 = 8$. Use $F_2(8)$ and take $(17, 12)$ from $P_2(1)$ to generate $(4\ 4\ 40\ 40) = (13\ 13)^2$ and $(40\ 40\ 4\ 4) = (41\ 41)^2$, both in $B = 43$.

We now give the results for $m = ka - 1$. Examples: First, $k = 2$ gives $x^2 - 3n^2 = -2$. This requires $F_3(-2)$ and, using $(7, 4)$ from $P_3(1)$, the pair $(4\ 4\ 10\ 10) = (7\ 7)^2$ and $(10\ 10\ 4\ 4) = (11\ 11)^2$ in $B = 13$ is obtained. Next, $k = 3$ gives $x^2 - 8n^2 = -4$. This requires $F_2(-4)$ and, using $(17, 12)$ from $P_2(1)$, the pair $(4\ 4\ 28\ 28) = (11\ 11)^2$ and $(28\ 28\ 4\ 4) = (29\ 29)^2$ in $B = 31$ is obtained.

Pattern $(m\ m)^2 = (a\ b\ c\ d)$

For this final investigation we examine the pattern for which $a+c = b+d$ and for convenience we write $a-b = d-c = p$, with $p < \min(a, d)$, $1 \le p \le B-1$. Our usual manipulation of the basic equation results in

$$m^2 \;=\; a(B-1) - p + \frac{a+p+d}{B+1},$$

and we assume therefore that either $a+p+d = B+1$ or $a+p+d = 2(B+1)$.

The first possibility, $a + p + d = B + 1$ gives $m^2 + (p - 1) = a(B - 1)$ and if $p = 3$ say, then we need to solve $m^2 + 2 = a(B - 1)$. Taking $m = 5$ means that $a(B - 1) = 27 = 3 \cdot 9$ and so $a = 3$ and $B = 10$ would be one choice. Also $a + p + d = B + 1$ gives $d = 5$ and $b = a - p$, $c = d - p$ give $b = 0$ and $c = 2$ respectively. Thus $(3\ 0\ 2\ 5) = (5\ 5)^2$ in $B = 10$, which is the decimal result already noted, that is, $55^2 = 3025$!

The second possibility, $a+p+d = 2(B+1)$, gives $m^2+(p-2) = a(B-1)$. For example, with $m = 20$ and $p = 16$ we have $a(B - 1) = 414 = 18 \cdot 23$ and so $a = 18$, $B = 24$ and $b = 2$, $c = 0$ and $d = 16$ follow. Thus $(18\ 2\ 0\ 16) = (20\ 20)^2$ in $B = 24$.

Other patterns

The patterns analysed involve the square of $(m\ m)$ and the next step would be to examine cases in which $(m\ n)$ is involved. Other patterns are possible. For example, digits and their reverses; $(48\ 48\ 84\ 84) = (79\ 79)^2$ in $B = 131$ and $(28\ 28\ 82\ 82) = (55\ 55)^2$ in $B = 109$ are examples.

The fact that we have a cubic in B could be exploited through the digits, and $(a\ 3a\ 3a\ a) = (6a\ 6a)^2$ in $B = 36a - 1$ is an example. An even better example is $(m^2\ m^2-1\ m^2-1\ m^2) = (mn\ mn)^2$ in $B = 2m^2 = n^2+1$. The values $m = 5$, $n = 7$ give $B = 50$, and produce $(25\ 24\ 24\ 25) = (35\ 35)^2$. Expressing the squares of three digit numbers as six digit numbers in base B will employ techniques similar to those already introduced and will lead to interesting results. As a final example, we state that $B = 68$ is the smallest base in which $(a\ b\ a\ b\ a\ b)$ is a three digit square, and $a = 17$, $b = 53$ gives $(34\ 52\ 45)^2$. The only other base with this property is $B = 313$. A most interesting result, but a much too difficult representation to establish here!

Appendix

$F_d(c)$: the positive solutions of the equation $x^2 - dy^2 = c$

If (u_1, v_1) is the fundamental solution of the associated Pell equation, $u^2 - dv^2 = 1$, then the general solution (u_{n+1}, v_{n+1}) is given inductively by the formulas $u_{n+1} = u_1u_n + dv_1v_n$ and $v_{n+1} = u_1v_n + v_1u_n$, for $n = 1, 2, 3, \ldots$, and these are given explicitly under $P_d(1)$, below.

If the Fermat–Pell equation $x^2 - dy^2 = c$ is solvable, it has infinitely many solutions (x_n, y_n) and these are given by the formulas $x_n = x_0u_n + dy_0v_n$ and $y_n = x_0v_n + y_0u_n$, for $n = 1, 2, 3, \ldots$, where (x_0, y_0) is one solution of the given equation. These are given below by the formulas under $F_d(c)$.

Pell equations

$P_2(1) : u^2 - 2v^2 = 1$;

$(u_1, v_1) = (3, 2)$, producing $u_{n+1} = 3u_n + 4v_n$ and $v_{n+1} = 2u_n + 3v_n$:

u_n	3	17	99	577	$\ldots,$
v_n	2	12	70	408	$\ldots.$

$P_3(1) : u^2 - 3v^2 = 1$;

$(u_1, v_1) = (2, 1)$, producing $u_{n+1} = 2u_n + 3v_n$ and $v_{n+1} = u_n + 2v_n$:

u_n	2	7	26	97	$\ldots,$
v_n	1	4	15	56	$\ldots.$

$P_5(1) : u^2 - 5v^2 = 1$;

$(u_1, v_1) = (9, 4)$, producing $u_{n+1} = 9u_n + 20v_n$ and $v_{n+1} = 4u_n + 9v_n$:

u_n	9	161	2889	51841	$\ldots,$
v_n	4	72	1292	23184	$\ldots.$

Fermat–Pell equations

$F_2(-1)$: $x^2 - 2y^2 = -1$; $(x_0, y_0) = (1, 1)$, producing $x_n = u_n + 2v_n$ and $y_n = u_n + v_n$, with the (u_n, v_n) given by $P_2(1)$.

$F_2(-2)$: $x^2 - 2y^2 = -2$; $(x_0, y_0) = (4, 3)$, producing $x_n = 4u_n + 6v_n$ and $y_n = 3u_n + 4v_n$, with the (u_n, v_n) given by $P_2(1)$.

$F_2(-4)$: $x^2 - 2y^2 = -4$; $(x_0, y_0) = (2, 2)$, producing $x_n = 2u_n + 4v_n$ and $y_n = 2u_n + 2v_n$, with the (u_n, v_n) given by $P_2(1)$.

$F_2(8)$: $x^2 - 2y^2 = 8$; $(x_0, y_0) = (4, 2)$, producing $x_n = 4u_n + 4v_n$ and $y_n = 2u_n + 4v_n$, with the (u_n, v_n) given by $P_2(1)$.

$F_3(-2)$: $x^2 - 3y^2 = -2$; $(x_0, y_0) = (1, 1)$, producing $x_n = u_n + 3v_n$ and $y_n = u_n + v_n$, with the (u_n, v_n) given by $P_3(1)$.

$F_3(6)$: $x^2 - 3y^2 = -6$; $(x_0, y_0) = (3, 1)$, producing $x_n = 3u_n + 3v_n$ and $y_n = u_n + 3v_n$, with the (u_n, v_n) given by $P_3(1)$.

$F_5(4)$: $x^2 - 5y^2 = 4$; $(x_0, y_0) = (3, 1)$, producing $x_n = 3u_n + 5v_n$ and $y_n = u_n + 3v_n$, with the (u_n, v_n) given by $P_5(1)$.

Solution 237.5 – Another sum

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} = \frac{\pi^2 - 9}{3}$.

Bryan Orman

Clearly,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} &= \sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{2}{n+1} - \frac{2}{n} \right) \\
&= 2\sum_{n=1}^{\infty} \frac{1}{n^2} - 1 - 2 = \frac{2\pi^2}{6} - 3 = \frac{\pi^2 - 9}{3}.
\end{aligned}$$

Something to ponder. Take a cube and slice off each vertex to a distance half-way along the edges that meet there. The result is a *cuboctahedron*. If you do the same with a regular octahedron, the dual of the cube, you also get a cuboctahedron. And an *icosidodecahedron* is the solid that you obtain by removing the vertices in a similar manner from either the regular icosahedron or its dual, the regular dodecahedron. The regular tetrahedron is its own dual. So what's a *tetratetrahedron*?

Solution 236.5 Harmonic quotients

Let
$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \;=\; \frac{a(n)}{b(n)},$$
the nth harmonic number expressed as a fraction in its lowest terms (so that $\gcd(a(n), b(n)) = 1$). Let $s(n) = b(n)/b(n-1)$. For which n do we have $s(n) < 1$. In particular, show that $s(2 \cdot 3^n) = 1/3$.

Tommy Moorhouse

In investigating this problem I found some interesting related results, probably well known but intriguing nonetheless. I have drawn up a quite loose proof of a condition on n that leads to $s(n) < 1$, but I am sure more advanced arguments will supply a full proof.

One starting point for the investigation is the expression, following directly from the definition of the harmonic sum,
$$\frac{a(n)}{b(n)} \;=\; \frac{a(n-1)}{b(n-1)}+\frac{1}{n}=\frac{na(n-1)+b(n-1)}{nb(n-1)}.$$
As our first result we have $s(p) = p$ whenever p is prime. To prove this write
$$\frac{a(p)}{b(p)} \;=\; \frac{a(p-1)}{b(p-1)}+\frac{1}{p}=\frac{pa(p-1)+b(p-1)}{pb(p-1)}.$$
Since the numerator and denominator of the right hand side have no common factors (p does not divide $b(p-1)$ and $\gcd(a(p-1), b(p-1)) = 1$) we see that $b(p) = pb(p-1)$ so that $s(p) = p$.

This is a useful 'warm-up' for further investigation, and we can show, for example, that if $s(n)$ is an integer then n divides $b(n)$. This follows since
$$na(n) \;=\; na(n-1)s(n)+b(n);$$
for if $s(n)$ is an integer then n must divide the right-hand side and so must divide $b(n)$. Thus if $b(n)$ is not divisible by n then $s(n)$ is not an integer. This is a useful condition on n.

Taking this further, examining a list of $s(n)$ and $b(n)$ (mod n) confirms that $b(n) \not\equiv 0 \pmod{n}$ whenever $s(n) < 1$. This means not only that n has cancelled in the sum
$$\frac{na(n-1)+b(n-1)}{nb(n-1)},$$

leading to the important relation $b(n-1) \equiv 0 \pmod{n}$ but that some additional cancellation has occurred. We will consider $n = mp^k$ for some integer k with $p \not| m$ and $1 < m < p$.

To make progress we take what looks like a step towards over complicating the issue. We write the nth harmonic sum as

$$\frac{1}{n!}\left(n! + \frac{n!}{2} + \cdots + \frac{n!}{n}\right). \tag{1}$$

This is so that we can decompose $n!$ into a product of primes as

$$n! \;=\; 2^{\alpha_2(n)} 3^{\alpha_3(n)} \ldots p^{\alpha_p(n)}$$

where p is the largest prime factor of $n!$. The numbers $\alpha_q(n)$ are not hard to work out and are given explicitly by

$$\alpha_q(n) \;=\; \sum_{r=1}^{\infty} \left[\frac{n}{q^r}\right],$$

where the sum is in fact finite because the integer-part function $[x]$ vanishes for $x < 1$. The largest power of a prime q dividing integers less than or equal to n is $[\log_q(n)]$, which we denote by $\beta_q(n)$. Thus the sum (1) has a common factor

$$2^{\alpha_2(n)-\beta_2(n)} 3^{\alpha_3(n)-\beta_3(n)} \ldots p^{\alpha_p(n)-\beta_p(n)},$$

which we cancel to get

$$\frac{a(n)}{b(n)} \;=\; \frac{T(n)}{2^{\beta_2(n)} 3^{\beta_3(n)} \ldots p^{\beta_p(n)}}.$$

Note that

$$\frac{a(n-1)}{b(n-1)} \;=\; \frac{T(n-1)}{2^{\beta_2(n-1)} 3^{\beta_3(n-1)} \ldots p^{\beta_p(n-1)}}.$$

We can see that $\beta_p(n-1) = \beta_p(n)$ unless n is a power of p, which is not the case here, so that the denominators of the expressions involving $T(n)$ and $T(n-1)$ are equal before any cancellations with the numerators. This means that if p divides $T(n)$ and does not divide $T(n-1)$ then $s(n)$ has a factor of the form $1/p^r$ for some r. But p cannot divide $T(n-1)$ since p^k divides $b(n-1)$. We have $\beta_p(n-1) = k$ and this factor survives when we cancel down from $T(n-1)$ to $a(n-1)$.

We know that $T(n)$ may still be divisible by a factor occurring in the denominator and if this is the case then $s(n)$ will turn out to be less than 1.

Let us consider the prime p. Now, p cancels from every pth term in the sum (1) for $T(n)$, but divides all the other terms. We therefore have an expression made up from all the terms not divisible by p and having the form (recalling that $n = mp^k$)

$$S(n)\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right) \;=\; S(n)\frac{a(m)}{b(m)},$$

where p does not divide $S(n)$ and $T(n) = pB(n) + S(n)a(m)/b(m)$ for some integer $B(n)$. Thus if $a(m) \equiv 0$ modulo p then p divides $T(n)$, and cancels between the numerator and the denominator. The power of p dividing $T(n)$ may be greater than 1. If p^t divides $b(n-1)$ we must have $b(n)/b(n-1) = 1/p^{(t-r)}$ for some $t > r > 0$. Thus if $n = mp^k$ and $a(m) \equiv 0 \pmod{p}$ then $s(n)$ is less than 1.

This suggests a strategy for finding n such that $s(n) = 1/p^r$ for some r: take $a(m)$ for some m. This is a product of primes larger than m since all smaller primes divide $b(m)$. If $a(m)$ itself is prime or a prime power, say p_j^t, form the numbers mp_j^r for $r > 1$. Each of these numbers has $s(mp_j^r) = 1/p^t$ with $t > 0$. In particular we find that since $a(2) = 3$, $s(2 \cdot 3^m) = 1/3$. Similarly since $a(4) = 25$, $s(4 \cdot 5^m) = 1/25$ $(m > 1)$.

If $a(m)$ is not a prime (which is almost always the case) we take any of the prime factors p of $a(m)$ and form mp^k for integer $k > 0$ and note that $s(mp^k) < 1$.

The cautious reader will have spotted some holes in the above arguments, but hopefully they are not hard to patch up.

Mathematics in the kitchen – VIII

Ken Greatrix

We have recently increased the amount of recycling, but our local council only wants clean tins and bottles. This means rinsing them out.

The question: How much water do you put in a bottle to achieve the most efficient rinsing?

When shaken, a quantity of water in a bottle will impact with the other end of the bottle. If there's too little water, the effect is reduced, too much and there's not enough room to gain sufficient momentum and kinetic energy. A bottle can be idealized as a uniform cylinder and the shaking can be idealized as a uniform force or uniform acceleration/deceleration.

Readers are welcome do a detailed analysis of the problem.

Letters

Prime pathway

Dear Tony,

I was surprised and delighted to see the 'prime pathway' featured again on the front cover of M500 **236**; surprised that you should give this further consideration and delighted to see the two prime-free century blocks.

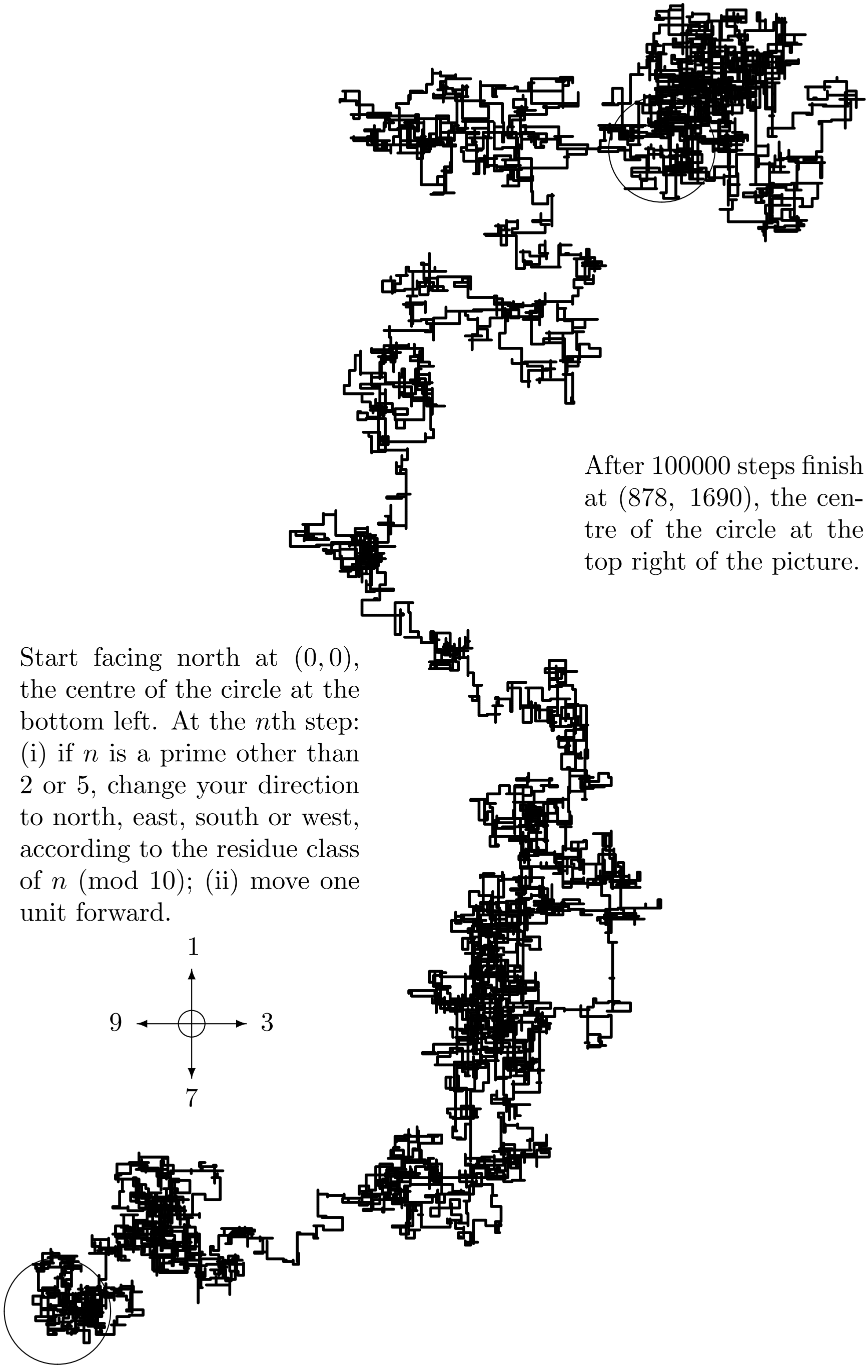
I expected to see a prime-free century much earlier and my own attempts were not very methodological and limited to primes below about $16 \cdot 10^6$ with my modest BASIC program. On receiving M500 **236** I amended the precision and confirmed the double century. You are no doubt used to finding huge gaps between enormous primes but I enjoy working with numbers that I find easier to comprehend. You do not say this is the first occurrence of a prime-free block. (Like the proverbial No. 11 bus, after a lengthy absence two come along together.)

One aspect of my small-scale excursions into the sequence of primes (see also M500 **230**) is to discover if there is any benefit in looking at the four separate sequences of primes ending in 1, 3, 7 and 9. I have observed that there can be a long gap between primes ending in 1, for example, while the others occur with the more expected frequency. Is there any significant difference in the distribution of primes in each sequence? The object of the excursion in M500 **230** was to see if there is any bias towards primes with a particular final digit. In the 10,000 steps illustrated you have omitted the start and finish points! Even 10,000 is a small number to show any evidence of bias. The decreasing density of primes (and therefore the increasing gaps between them) would mean that the excursions would increase. But would they still be centred about the start point? I get out of my depth very quickly on this topic but I would be interested if you could comment from your knowledge and experience of prime hunting.

Yours sincerely,

Chris Pile

Tony Forbes — Yes, it was a little sloppy of me to omit the end-points on the cover of M500 **230**. So here is the pathway again with the start and finish points clearly indicated by the centres of the large circles at bottom left and top right respectively. And for variety I have extended the length by a factor of 10. Of course the detail is lost in amongst all those small black dots. However, as you can see, there is at least for the first 100000 steps a definite trend towards a north-north-easterly direction.



After 100000 steps finish at (878, 1690), the centre of the circle at the top right of the picture.

Start facing north at $(0,0)$, the centre of the circle at the bottom left. At the nth step: (i) if n is a prime other than 2 or 5, change your direction to north, east, south or west, according to the residue class of n (mod 10); (ii) move one unit forward.

According to my records, the prime pathway on the front of M500 236 shows the first occurrence of two consecutive prime-free blocks at century 4732627. The first prime-free century is at 16718. At least I think that is the case; without careful checking I have to assume my programming was perfect. The calculations were done in MATHEMATICA. I was actually trying to find *three* consecutive blocks to make a nice 4×5 rectangle to fit snugly into the space on the front cover of **236** but my computer ran out of steam. Maybe this can be achieved with a purpose-built program written in C, for example. I leave it for someone else to try.

Hyperbolic planes

Dear Eddie,

I'm a bit disappointed in the choice of *Crocheting Adventures with Hyperbolic Planes* by Daina Taimina (a Latvian mathematician at Cornell University) for the prize given for oddly-titled books. It was adjudged the winner of the annual Diagram Prize after a public vote run by *The Bookseller* magazine. Certainly it is odd, but it's also a splendid idea to crochet hyperbolic planes. There was an admiring article in *New Scientist* a couple of years ago about the mathematical confections that Taimina somehow manages to make, which in many cases are the only extant solid representations of these surfaces. So more power to her little hook.

Ralph Hancock

Problem 239.1 – Three coins

Arthur, Ford and Marvin play a game. They try to predict the outcome of three coin tosses. As usual with coins and the tossing thereof, the probability of guessing correctly is always 1/2. After the tossings are decided, the person (or persons, if there is a tie) with the most results correct wins (or share) a Valuable Prize of £300, say. For example, if the forecasts are Arthur HHH, Ford THT, Marvin TTT and the results are HHT, then Arthur and Ford get £150 each.

If the players play independently, clearly they have equal chances of winning. However, after Arthur and Ford have made their predictions and before Marvin has made his, Ford offers to show his forecast to Marvin in return for a fee of £1. What should Marvin do?

MINIMUM 50% OF RETAIL PRICE — Furniture company advert.

Solution 234.7 – Directed triangles

> Draw a directed graph as follows. Take n points, numbered 0, 1, …, $n-1$, and place them in order around the circumference of a circle. For $i = 0, 1, \ldots, n-1$ and $j = 1, 2, \ldots, \left\lfloor \dfrac{n}{2} \right\rfloor$, join point i to point $(i+j) \bmod n$ by an arrow unless an arrow going in the opposite direction is there already. A *directed triangle* is where you have three points a, b, c and arrows joining them thus: $a \longrightarrow b \longrightarrow c \longrightarrow a$.

Stuart Walmsley

Introduction

The symbol $\lfloor n/2 \rfloor$ takes the value $n/2$ when n is even and $(n-1)/2$ when n is odd. **In what follows, $\lfloor n/2 \rfloor$ is denoted by m.**

The same value of m goes with two values of n as follows.

n	2	3	4	5	6	7
m	1	1	2	2	3	3

The required number of directed triangles will be denoted by

$$d_{2m} \quad (n \text{ even}) \qquad \text{and} \qquad d_{2m+1} \quad (n \text{ odd}).$$

When n is odd, the system is cyclically symmetric so that each vertex contributes equally. If the number of directed triangles meeting at one vertex is denoted by v, the total number of directed triangles is given by

$$d_{2m+1} = \frac{(2m+1)v}{3}. \tag{1}$$

The factor of 1/3 arises because each triangle has been counted three times.

Determination of d_{2m+1} is thus reduced to finding the corresponding value of v.

Determination of v and d_{2m+1}

For $n = 2m+1$, the vertex 0 has arrows directed to vertices 1 to m and receives arrows from the remaining vertices $m+1$ to $2m$. The directed triangles meeting at 0 have to cross from the set 1 to m to the set $m+1$ to $2m$.

For the arrow 0 to m, m can reach all m members of the second set and thus contributes m directed triangles. For the arrow 0 to $m-1$, $m-1$

reaches only $m-1$ members of the second since its arrow to m does not yield a directed triangle.

Each arrow 0 to $m-2$, $m-3$ yields one less directed triangle so that the total

$$v \;=\; m+m-1+m-2+\cdots+2+1$$

giving

$$v \;=\; \frac{m(m+1)}{2}; \tag{2}$$

whence

$$d_{2m+1} \;=\; \frac{(2m+1)m(m+1)}{6}. \tag{3}$$

Determination of d_{2m}

In going from $2m+1$ to $2m$, the value of m remains the same but the number of vertices in the directed graph is reduced by one. Comparison of the diagrams for the corresponding odd and even cases shows that the even case is identical except for the removal of one vertex when n is even. In this way $d_{2m} = d_{2m+1} - v$ which from (2) and (3) becomes

$$d_{2m} \;=\; \frac{m(m+1)(2m+1)}{6} - \frac{m(m+1)}{2},$$

which simplifies to

$$d_{2m} \;=\; \frac{(m-1)m(m+1)}{3}.$$

Conclusion

The expressions for the numbers of directed triangles are conveniently rewritten.

$$\begin{aligned} d_{2m} &= \frac{(m-1)m(m+1)}{3}, \\ d_{2m+1} &= \frac{m(m+1)}{2} + \frac{(m-1)m(m+1)}{3}. \end{aligned}$$

They may be compactly expressed as binomial coefficients

$$\begin{aligned} d_{2m} &= 2\binom{m+1}{3}, \\ d_{2m+1} &= \binom{m+1}{2} + 2\binom{m+1}{3}. \end{aligned}$$

Solution 236.2 – Series

Find a closed expression for $S(M) = \displaystyle\sum_{n=1}^{M} \frac{1}{n(n+1)}$ and show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+N)} = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{k}$.

Steve Moon

By partial fractions, $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. So

$$S(M) \;=\; \sum_{n=1}^{M}\left(\frac{1}{n} - \frac{1}{n+1}\right) \;=\; 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{M} - \frac{1}{M+1},$$

and by telescoping cancellation of terms we get

$$S(M) \;=\; \frac{M}{M+1}.$$

In particular, letting $M = 10^k - 1$ we obtain $S(99\ldots9) = 0.99\ldots9$, where each expression has k nines in it.

For $\sum_{n=1}^{\infty} 1/(n(n+N))$, we have by partial fractions

$$\sum_{n=1}^{\infty}\frac{1}{n(n+N)} \;=\; \frac{1}{N}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+N}\right).$$

And by telescoping cancellation, for arbitrarily large but finite n, before taking the limit as $n \to \infty$,

$$\sum_{n=1}^{\infty}\frac{1}{n(n+N)} \;=\; \frac{1}{N}\left(1 - \frac{1}{1+N} + \frac{1}{2} - \frac{1}{2+N} + \cdots + \frac{1}{n} - \frac{1}{n+N} + \ldots\right).$$

So we can see that for any $N > 0$ there will be N initial terms of the form $1/N(1/n)$, $n \le N$, which survive the cancellation. Since this is a sum to ∞, the limit as $n \to \infty$ of N negative terms that are relevant in a finite sum is zero. Hence

$$\sum_{n=1}^{\infty}\frac{1}{n(n+N)} \;=\; \frac{1}{N}\sum_{k=1}^{N}\frac{1}{k}.$$

Solution 232.7 – Zero

Show that $\cos\frac{1}{3}\pi + \dfrac{\cos\frac{2}{3}\pi}{2} + \dfrac{\cos\frac{3}{3}\pi}{3} + \dfrac{\cos\frac{4}{3}\pi}{4} + \cdots = 0.$

Steve Moon

The Taylor series for the principal logarithm of a complex number z is

$$\mathrm{Log}(1+z) \;=\; z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots, \qquad |z| \le 1,\; z \ne -1.$$

From which, substituting $-z$ for z,

$$\mathrm{Log}(1-z) \;=\; -\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots\right), \qquad |z| \le 1,\; z \ne 1.$$

Let $z = \cos\pi/3 + i\sin\pi/3$. Then by de Moivre's theorem,

$$\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{2}\right)^n \;=\; \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}$$

and on substituting for z, all the powers become multiple angles, thus:

$$\begin{aligned}\mathrm{Log}\left(1 - \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right) \;&=\; -\left(\cos\frac{\pi}{3} + \frac{\cos 2\pi/3}{2} + \frac{\cos 3\pi/3}{3} + \ldots\right)\\ &\quad - i\left(\sin\frac{\pi}{3} + \frac{\sin 2\pi/3}{2} + \frac{\sin 3\pi/3}{3} + \ldots\right),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Log}\left(1 - \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right) \;&=\; \mathrm{Log}\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\\ &=\; \log\left|\frac{1}{2} - \frac{i\sqrt{3}}{2}\right| + i\arg\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \;=\; \log 1 - \frac{i\pi}{3} \;=\; -\frac{i\pi}{3}.\end{aligned}$$

So, on equating real parts,

$$\cos\frac{\pi}{3} + \frac{\cos 2\pi/3}{2} + \frac{\cos 3\pi/3}{3} + \ldots \;=\; 0,$$

as required.

Incidentally, by equating imaginary parts we see that the corresponding sine series sums to $\pi/3$.

Solution 213.10 – Minor axis

> A spherical globe with a marked equator is thrown at random into the air and lands on the ground. The globe has diameter 1; hence the projection of the equator on to the ground is an ellipse with major axis 1. What is the expected length of the minor axis?

Tony Forbes

When the globe comes to rest the normal to the equatorial plane will meet the surface of the globe at a point P, say. We are interested in the angle, α, that the equatorial plane makes with the horizontal. For then $|\cos\alpha|$ will be the length of the minor axis of the elliptical projection of the equatorial disc on to the ground. So all we need do is take a sphere of radius 1/2, *choose at random* a point P on its surface and compute the mean of $|\cos\alpha|$.

Using polar coordinates, let $P = \left(\frac{1}{2}\cos\theta\sin\phi,\ \frac{1}{2}\sin\theta\sin\phi,\ \frac{1}{2}\cos\phi\right)$ where θ is longitude (measured from some fixed point on the globe) and ϕ is co-latitude (this is like latitude but measured from the north pole rather than the equator). Thus $\alpha = |\phi|$. So to solve the problem, choose θ at random from $[0, 2\pi]$ and independently ϕ from $[0, \pi]$, and compute

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\pi}\int_0^{\pi}|\cos\phi|d\phi d\theta \ = \ \frac{2}{\pi}.$$

Indeed, someone sent an answer along these lines. Unfortunately I have to say that it doesn't work. The problem is in that italicized phrase. Choosing the parameters of the polar coordinates independently at random does not produce a uniform distribution over the surface of a sphere. Points will tend to cluster around the poles. You can see this by choosing 100 points, say, at random at each of co-latitudes 10°, 20°, ..., 90°. It is clear that they will be less dense at the equator because there is more room for them to spread out.

Well, I did a little research and discovered 'Random Points on a Sphere' from *The Wolfram Demonstrations Project*, http://demonstrations.wolfram.com/RandomPointsOnASphere/. Here it is suggested (without explanation) that a uniform distribution of points over the surface of the globe can be achieved by the coordinates

$$P \ = \ \left(\tfrac{1}{2}(\cos\theta)\sqrt{1-u^2},\ \tfrac{1}{2}(\sin\theta)\sqrt{1-u^2},\ \tfrac{1}{2}u\right)$$

with $\theta \in [0, 2\pi]$ and $u \in [-1, 1]$ chosen independently at random. Now we have $|\cos\alpha| = |u|$. Therefore the answer to the problem would seem to be

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\int_{-1}^{1}|u|du d\theta \;=\; \frac{1}{2}.$$

And after a little more research I found Stan Wagon's *Problem of the Week* site at http://mathforum.org/wagon/, which considers a similar (but mathematically identical) problem. The solution given is interesting because it appears to use polar coordinates but with a fudge factor to eliminate the bias towards the poles. When integrating over polar coordinates the element of area is $r^2(\sin\phi)d\phi d\theta$. So to calculate the mean of something over a sphere, you multiply by the element of area, integrate and then divide by the surface area of the whole sphere, $4\pi r^2$. Hence for $|\cos\alpha| = |\cos\phi|$ and $r = \frac{1}{2}$, we get

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}|\cos\phi|(\sin\phi)d\phi d\theta \;=\; \frac{1}{4}\int_0^{\pi}|\sin 2\phi|d\phi \;=\; \frac{1}{2},$$

in agreement with the previous 1/2.

Finally, I think one can now see where Wolfram's coordinates come from. Put $u = \cos\phi$. Then $\sin\phi = \sqrt{1-u^2}$. So instead of choosing ϕ uniformly at random, Wolfram is really choosing ϕ such that $\cos\phi$ is uniformly distributed.

Winter Weekend experience 2011

Judith Furner

I have recently returned from the M500 Winter Weekend and what a delightful experience it was to be sure. Grateful thanks are due to Diana, who womanfully organized the whole event, and Rob and Judith, who managed to keep us happy and occupied for 48 hours. Not to mention Tony Huntington who pottered around with his camera and notebook, checking out what was going on.

When I arrived on Friday afternoon I found a bunch of friends whom I see once a year, sitting in comfortable armchairs, chatting and gossiping, and occasionally mentioning mathematics. As my formal mathematical training ceased some thirty years ago, and I have practised very little since then, I was delighted to avoid plunging into differential equations and imaginary numbers. That evening Tony persuaded us to form teams and he conducted a quiz. My normal bedtime is 10:00, but I had to stay up until after midnight just to see how it all ended. The extensive geographical, historical and cultural knowledge, especially of films and pop groups, of our competitors meant that our team was not among the first three.

We slept in comfortable study bedrooms and I was impressed with the quality of the food at lunch and dinner, especially the salads. Breakfast left something to be desired, but with a refrigerator in my room I decided to take my own orange juice and coffee next year.

We started on Saturday with some jolly good 'O' level stuff—considering Euclid's axioms, definitions, postulates and proofs. We drew triangles, looked at pictures of triangles and played with plastic triangles that we could click together (at this point I thoroughly enjoyed regressing to four years old). We constructed proofs and I won a prize for standing up and declaiming my proof which turned out to be spectacularly fallible. We considered how the angles of a triangle might not add up to 180 degrees. We looked at triangle numbers $(1, 3, 6, 10, \dots)$ and played with plastic cubes to form triangles. We proved Pythagoras's theorem and failed to prove Fermat's last theorem. We considered Pascal's triangle and how it relates to the sum of triangle numbers. We found a relationship between triangle numbers and square numbers, and fitted together our plastic-cube triangles to make squares. I remembered that I knew not only the symbols for sum and square root but (with a bit of help from my friends) the meaning of 'mathematical progression', 'recursive relationship' and 'f'.

After a good lunch we settled down to primes and squares. We saw that 13 is the sum of two squares, 4 and 9. We listed the primes up to 100 and separated them into those that were the sum of two squares and those that were not. We looked for patterns, discussed with our friends, experimented and found some interesting results. We used prime factorization to find the HCF and LCM of two numbers. We moved on to prove that the root of a prime number is irrational and considered Goldbach's conjecture, that 'every even number is the sum of two primes'. It is unproven, and remains so despite our best efforts. Finally we were encouraged to write poems about our experience. My favourite was:

Roses are red
Violets are blue
There's one even prime
And it is two.

Later on we considered polygons and found that only three regular polygons tessellate the plane. We developed our thoughts into three dimensions, with the help of yet more plastic triangles and cubes. We experimented, conjectured, convinced and proved to our hearts' delight. We considered another approach, using faces, edges and vertices, and found that $f+v-2=e$.

Rob and Judith realized that by Saturday evening we could do with a break, and sat us down to watch Andrew Wiles discussing his proving of Fermat's last theorem. We moved into supper reassured by the thought that it was indeed more difficult than Pythagoras's theorem, and it wasn't surprising that we had failed to prove it before lunch.

We spent Sunday morning investigating rotten tomatoes, and their rates of decay, depending upon the container in which they were stored. Finally, Rob discussed the most beautiful equation. He took us through some work with functions, imaginary numbers, sines and cosines, and I remembered that I had indeed come across the Taylor and Maclaurin series in a former life. He elegantly proved, to our satisfaction, that $e^{i\pi}=-1$.

During the weekend we had some singing, and we played some jolly games. We found that the total number of gifts over the twelve days of Christmas was 364, and we were impressed by the true love's generosity, and the coincidence of the number being one fewer than the days of the (non-leap) year. We shook hands with a number of friends a number of times and analysed our results. We sang an interesting version of *Supercalifragilisticexpialidocius*, where the third and fourth lines of the chorus were

We in mathematics have a far superior chorus
NewtonRaphsonEuclidPascalGaussandPythagorus.

Before we bade farewell Tony Huntington made a short presentation of his experience of the weekend, the results of his quizzing others, and some suggestions for the future.

I should like to encourage anyone with mathematical knowledge from 'O' Level to postgraduate to try the Winter Weekend. There's something there for everyone: good food, good singing, good friendship and stimulating brainwork. And many thanks again to Diana, Rob, Judith and Tony.

Problem 239.2 – Two squares

Let $r(n)$ denote the number of ways of representing n as a sum of two integer squares where, as is customary, order is relevant, so that $r(13)=8$, for example, since $13=(\pm 2)^2+(\pm 3)^2=(\pm 3)^2+(\pm 2)^2$. Compute

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} r(n) \qquad \text{and} \qquad \lim_{N\to\infty}\frac{\log N}{N}\sum_{p\le N,\ p \text{ prime}} r(p).$$

Contents M500 239 – April 2011

Front cover: Truncated octahedron graph.