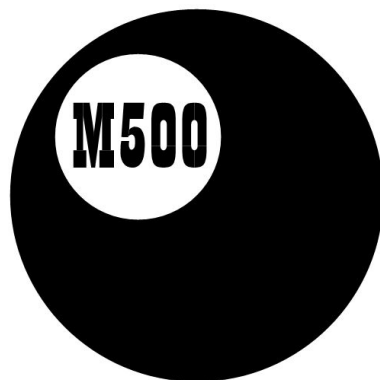
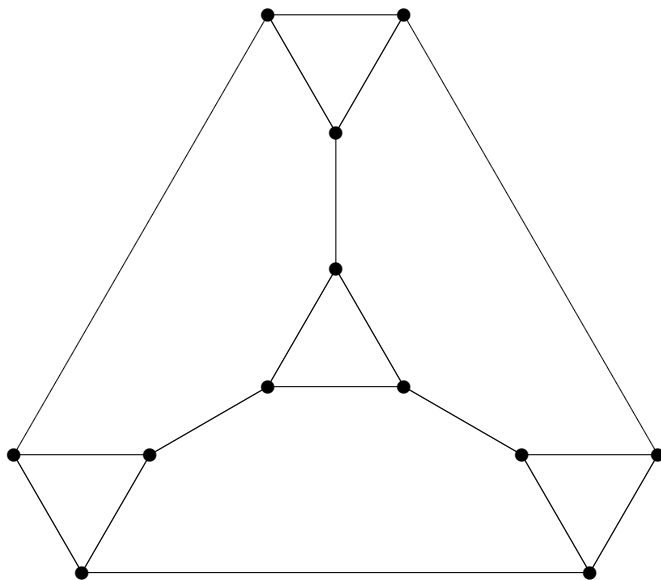


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M500 240



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards.

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Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation.

Solution 233.5 – Croquet

A croquet hoop made of wire of diameter 1 has an opening of width w and is set into the (x, y) plane with the opening occupying the interval from $(-w/2, 0)$ to $(w/2, 0)$. A croquet ball has diameter $d < w$. What is the set of points from which the ball, when struck in a non-spin-inducing manner, will eventually go through the hoop, possibly after bouncing off its uprights a number of times.

Robin Marks

Suppose that, on a curious croquet ground, one of the Queen of Hearts's soldiers doubles himself up to stand on his hands and feet and transforms himself into a perfectly resilient hoop with perfectly round surfaces. The hedgehog, who will be the ball, drinks a shrinking potion and curls up so that he is perfectly round before shrinking to a diameter d which is negligible compared with 1.

The flamingo, acting as the mallet, is sufficiently well behaved for a few seconds to allow Alice to hit this exceedingly small hedgehog towards the hoop, at an angle θ_1 to the line joining the soldier's hands and feet. Let the soldier's feet, on the left, and his hands, on the right, be numbered 1 and 2 respectively. Let the centres of circles 1 and 2 be O_1 and O_2 .

Now suppose that the hedgehog rolls in a peculiarly straight line, only just missing circle 2. He travels on, heading towards point S inside circle 1, but before he reaches S he hits circle 1 at point C_1 and bounces off, heading towards the point C_2 where he bounces off again. After many more bounces he eventually emerges from the other side of the hoop.

The question we have to answer is: What is this angle θ_1 such that the hedgehog only just makes it through the hoop?

Let the radius of the hoop wire be r . To calculate what happens, first choose the value of s , between 0 and 1. Point S is on the x -axis at a distance sr from O_1 . A line passing through S which is a tangent to circle 2 has point of tangency T .

Referring to Figure 1, in right-angled triangle TSO_2 , the angle TSO_2 is θ_1 , the hypotenuse has length $h = 2r + w - sr$, and the side TO_2 has length r . Hence distance $TS = \sqrt{h^2 - r^2}$; hence

$$\tan \theta_1 = r / \sqrt{h^2 - r^2}.$$

The first contact point C_1 is at one of the two points where the line going through T and S intersects circle 1. We find C_1 by solving, for x and y , the simultaneous equations for circle 1 and for the line ST :

$$\begin{aligned} r^2 &= y^2 + \left(x + \frac{w}{2} + r\right)^2, \\ y &= \frac{r}{\sqrt{h^2 - r^2}} \left(x + \frac{w}{2} + r(1 - s)\right), \end{aligned}$$

giving two solutions. We choose C_1 to be the solution with the greater x value.

Having found C_1 we can now calculate angle C_1O_1S , which is labelled ψ_1 . The angle of incidence for the hedgehog is angle $TC_1N = \text{angle } O_1C_1S = \theta_1 - \psi_1$. Because the hedgehog is not spinning we expect the angle of incidence to equal the angle of reflection; the angle NC_1C_2 should also be $\theta_1 - \psi_1$. Thus after the first bounce the hedgehog changes direction by an angle of $2(\theta_1 - \psi_1)$.

The hedgehog's new direction is along a line passing through C_1 at angle to the x -axis of $\theta_1 - 2(\theta_1 - \psi_1) = 2\psi_1 - \theta_1$. The second contact point for the hedgehog, C_2 , is at one of the two points where this line intersects circle 2. To make calculations easier we find not C_2 but the reflection of C_2 in the y -axis, by solving simultaneously the equation for circle 1 and for the equation for the line passing through reflections of both C_1 and C_2 . The angle of this reflected line to the x -axis is $\theta_2 = \theta_1 - 2\psi_1$. Therefore we solve, for x and y ,

$$\begin{aligned} r^2 &= y^2 + \left(x + \frac{w}{2} + r\right)^2, \\ y &= (\tan \theta_2)x + (y \text{ coordinate of } C_1) - (\tan \theta_2)(x \text{ coordinate of } C_1), \end{aligned}$$

again giving two solutions. We choose C_2 to be the solution with the greater x value.

Now we can calculate the angle ψ_2 , the angle subtended at O_1 by the reflection of C_2 , and hence we find angle

$$\theta_3 = 2(\theta_2 - \psi_2) - \theta_2 = \theta_2 - 2\psi_2 = \theta_1 - 2\psi_1 - 2\psi_2.$$

Continuing in this manner we calculate successively ψ_3, ψ_4, \dots . At each step we calculate

$$\theta_n = \theta_1 - 2\psi_1 - 2\psi_2 - \dots - 2\psi_{n-2} - 2\psi_{n-1}.$$

As the hedgehog zigzags towards the x -axis values of ψ_i are positive and decreasing. If and when the hedgehog crosses the x -axis the ψ_i are negative; successive values increase in magnitude before the hedgehog finally makes it through the hoop.

The value chosen for s is 0.708164, giving initial angle $\theta_1 = 33.9236^\circ$ and the following calculated values of ψ_i .

$\psi_1 = 10.6441^\circ$	$\psi_9 = 0.00131825^\circ$
$\psi_2 = 3.9129^\circ$	$\psi_{10} = -0.00686379^\circ$
$\psi_3 = 1.4867^\circ$	$\psi_{11} = -0.0219096^\circ$
$\psi_4 = 0.56741^\circ$	$\psi_{12} = -0.0588651^\circ$
$\psi_5 = 0.216647^\circ$	$\psi_{13} = -0.154687^\circ$
$\psi_6 = 0.0825937^\circ$	$\psi_{14} = -0.405219^\circ$
$\psi_7 = 0.0311374^\circ$	$\psi_{15} = -1.06137^\circ$
$\psi_8 = 0.0108185^\circ$	$\psi_{16} = -2.78623^\circ$

In Figure 1, we show the path of the vanishingly small hedgehog up to the second contact point C_2 .

But now the shrinking potion has started to wear off; the hedgehog grows to a visible size, diameter d . “How does this affect the calculations?” Alice wonders. “Not a lot,” says the Knave of Hearts. “We only need replace the hoop wire radius $1/2$ with $1/2 + d/2$, and also substitute $w - d$ for w . Then do the calculations again as above.”

In Figure 2, we show the path of the hedgehog now diameter $d = 1/10$ going through the gap $w = 3/8$. The shaded area between the circles shows the area covered by the moving hedgehog’s body. The value chosen for s is 0.708164, giving an initial approach angle θ_1 of 33.9236° . Each of the points C_1, C_2, \dots lies on a dotted circle and indicates the position of the hedgehog’s centre when his exterior is in contact with a hoop.

Approaching the x -axis he makes very slow progress, bouncing back and forth almost horizontally (Figure 3). Note that many of the labels C_i have been omitted from the central part of the hedgehog’s path because of lack of space. Calculations show that his centre crosses the x -axis between contact points C_9 and C_{10} . If we choose a starting value for s which is only 10^{-7} lower and repeat the calculations, he does not get through the hoop but bounces back out into the $+y$ half-plane from whence he came.

Figure 1: Path of the vanishingly small hedgehog up to C_2

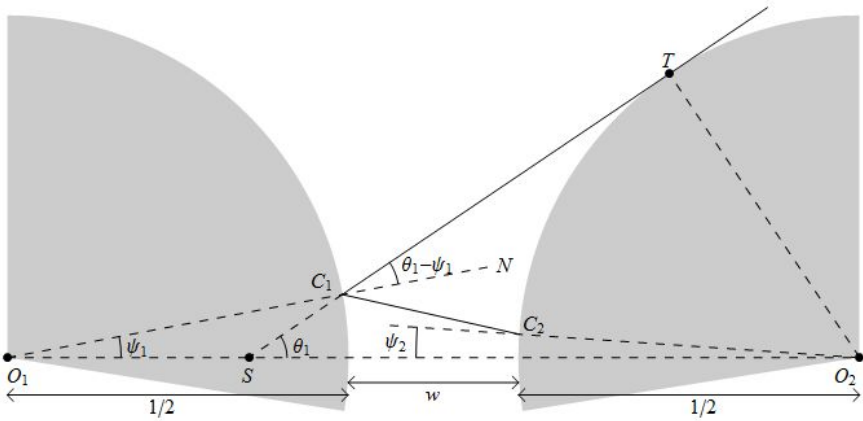


Figure 2: Path of the visible hedgehog

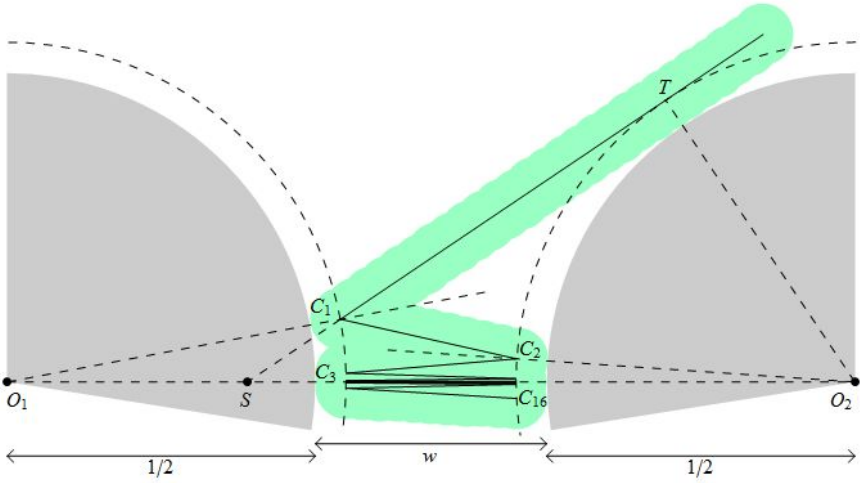
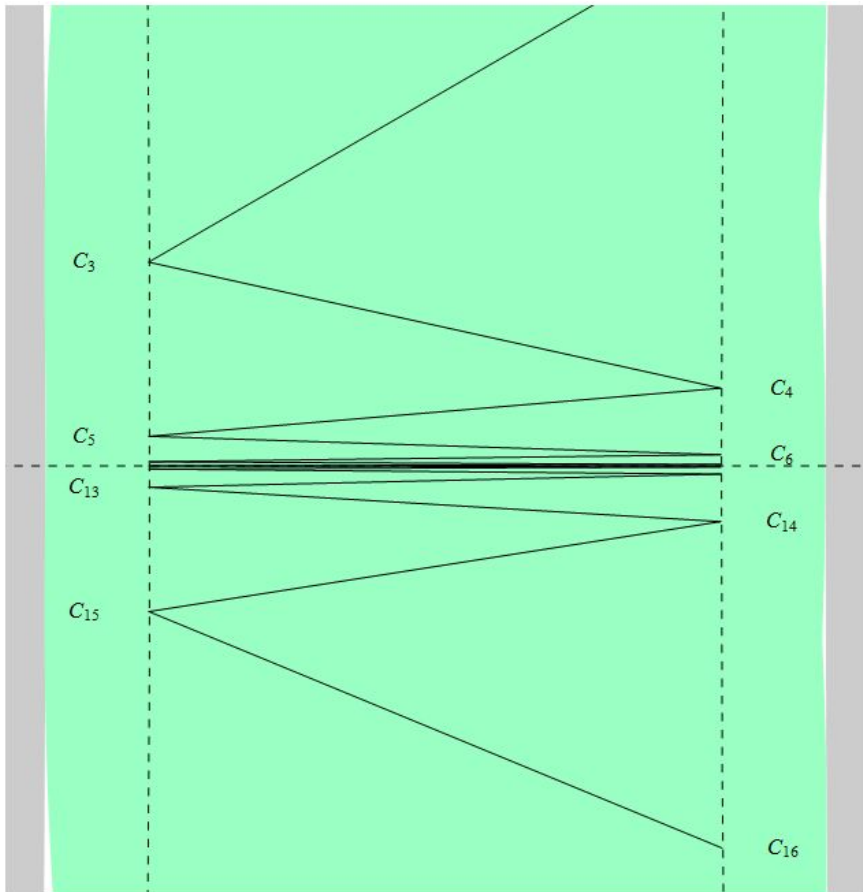


Figure 3: Magnified view of the path of the visible hedgehog



Problem 240.1 – Two tins of biscuits

Rex Watson

There are two tins, each containing $n > 0$ biscuits. Take a biscuit from a tin chosen at random. Keep doing this until one tin is empty. What is the expected number of biscuits that remain in the other tin?

[The problem also appeared on the wall of the OU Mathematics Department Common Room.]

Solution 233.3 – Six tans

Show that $\prod_{j=1}^6 \tan \frac{\pi j}{13} = \sqrt{13}$.

Norman Graham

Consider the equation $\cos 6\theta = \cos 7\theta$. This is satisfied by $6\theta = 2n\pi = 7\theta$, n integral, $13\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi, 12\pi$. Let $c = \cos \theta$, $s = \sin \theta$. Then

$$\begin{aligned} \cos 6\theta &= \Re e^{6i\theta} = \Re(c + is)^6 \\ &= c^6 - 15c^4s^2 + 15c^2s^4 - s^6 = 32c^6 - 48c^4 + 18c^2 - 1 \end{aligned}$$

and

$$\begin{aligned} \cos 7\theta &= \Re e^{7i\theta} = \Re(c + is)^7 \\ &= c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6 = 64c^7 - 112c^5 + 56c^3 - 7c. \end{aligned}$$

Therefore $64c^7 - 32c^6 - 112c^5 + 48c^4 + 56c^3 - 18c^2 - 7c + 1 = 0$ and hence

$$(c - 1)(64c^6 + 32c^5 - 80c^4 - 32c^3 + 24c^2 + 6c - 1) = 0,$$

the solutions of which are $c = 1$ and $c = c_j = \cos 2\pi j/13$, $j = 1, 2, \dots, 6$.

Let $t_j = \tan^2 \pi j/13$. Since

$$\begin{aligned} \tan^2 \theta &= \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}, \\ t_j &= \frac{1 - c_j}{1 + c_j} \quad \text{and} \quad c_j = \frac{1 - t_j}{1 + t_j}. \end{aligned}$$

Therefore t_j , $j = 1, 2, \dots, 6$, are the solutions to the equation

$$\begin{aligned} 64(1 - t)^6 + 32(1 - t)^5(1 + t) - 80(1 - t)^4(1 + t)^2 - 32(1 - t)^3(1 + t)^3 \\ + 24(1 - t)^2(1 + t)^4 + 6(1 - t)(1 + t)^5 - (1 + t)^6 = 0. \end{aligned}$$

The product $\prod_{j=1}^6 t_j$ is the constant in this equation, which is $64 + 32 - 80 - 32 + 24 + 6 - 1 = 13$. Therefore

$$\prod_{j=1}^6 \tan \frac{\pi j}{13} = \sqrt{\prod_{j=1}^6 \tan^2 \frac{\pi j}{13}} = \sqrt{13},$$

as required.

Note that this solution uses the same methods as my solution to ‘Problem 233.2 – Three secs’ in M500 238.

Solution 236.4 – Real function

Suppose A and θ are real. Show that $(A + iA \tan \theta)^{\log(A \sec \theta) - i\theta}$ is also real.

Steve Moon

Let $f(A, \theta) = (A + iA \tan \theta)^{\log(A \sec \theta) - i\theta}$. Take logs to enable powers to be handled more easily. Thus

$$\begin{aligned} \log f(A, \theta) &= (\log(A \sec \theta) - i\theta)(\log A + \log(1 + i \tan \theta)) \\ &= (\log(A \sec \theta) - i\theta) \left(\log A + \log \left(1 + \frac{i \sin \theta}{\cos \theta} \right) \right) \\ &= (\log(A \sec \theta) - i\theta)(\log A + \log(\cos \theta + i \sin \theta) - \log \cos \theta), \end{aligned}$$

and since $-\log \cos \theta = \log \sec \theta$ and $\log(\cos \theta + i \sin \theta) = \log e^{i\theta} = i\theta$, we have

$$\begin{aligned} \log f(A, \theta) &= (\log A + \log \sec \theta - i\theta)(\log A + \log \sec \theta + i\theta) \\ &= (\log(A + \log \sec \theta))^2 + \theta^2, \end{aligned}$$

Hence $f(A, \theta) = e^{(\log(A + \log \sec \theta))^2 + \theta^2}$, which is real for real A and θ .

Problem 240.2 – One

Bob Bertuello

(1) If you put the numbers from 1 to 10 in alphabetical order, in which position would ONE be placed.

(2) If you put the numbers from 1 to 100 in alphabetical order, in which position would ONE be placed?

(3) If you put the numbers from 1 to 1000 in alphabetical order, in which position would ONE be placed now?

...

Problem 240.3 – Double sum

Show that

$$\sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \frac{1}{r^2 s^2} = \frac{\pi^4}{120}.$$

Solution 234.4 – Tetrahedron

Three sides of a tetrahedron form an equilateral triangle of side a . The other three sides have length 1. Show that the diameter of the sphere that circumscribes the tetrahedron is

$$\frac{\sqrt{3}}{\sqrt{3-a^2}}.$$

What if three sides have length 1 and three sides have length $a \neq 1$, but no face of the tetrahedron is equilateral?

Stuart Walmsley

The centre of the circumscribed sphere will lie on the threefold symmetry axis of the tetrahedron, that is, the line joining the unique vertex to the centre of the equilateral triangle.

If ABC denotes the base equilateral triangle and P its centre, that is, the point equidistant from A , B and C , then elementary trigonometry shows that the length of AP is $a/\sqrt{3}$.

[It was at about this point when the Editor started to draw a diagram on a convenient piece of scrap paper. He recommends that the reader do likewise. ... — TF]

If D is the fourth vertex, the centre of the circumsphere, O , is on the line DP in the right angled triangle APD , such that $AO = OD = r$, where r is the radius of the circumsphere.

Then if in the isosceles triangle AOD , the angle ADO is denoted by ω , angle AOD is $\pi - 2\omega$. Remembering that $AD = 1$, by the sine rule

$$r = \frac{\sin \omega}{\sin 2\omega} = \frac{1}{2 \cos \omega}.$$

But, from the full right angled triangle APD , $\sin \omega = a/\sqrt{3}$. So

$$\cos^2 \omega = 1 - \frac{a^2}{3} = \frac{3-a^2}{3}$$

and

$$d = 2r = \frac{1}{\cos \omega} = \frac{\sqrt{3}}{\sqrt{3-a^2}},$$

as required. Note that if $a = 1$, d becomes $\sqrt{3/2}$, the appropriate value for the regular tetrahedron.

For the second part, let $AB = AC = a$, then $BC = 1$ and the other edge meeting A must be of length 1. The figure may be completed in two ways, mirror images of each other. Let $BD = a$. Then $CA = AB = BD = a$ and $BC = CD = DA = 1$.

[... And at this point the Editor decided that a 3-dimensional cardboard model would serve better than any diagram on paper. — TF]

The line joining the midpoints of the centre members of these trios, AB and CD , is a two-fold axis of symmetry, the only symmetry element if $a \neq 1$. The circumcentre must therefore lie on this line. Choose this as the z -axis. Let the origin be at the midpoint of AB and let AB lie along the x -axis. Let the angle the line CD makes with the x -axis be ϕ with $c = \cos \phi$ and $s = \sin \phi$, and let j be the common z coordinate of C and D . Then we have the following.

$$\begin{aligned} \text{Coordinates of } A: & \quad \left(\frac{1}{2}a, 0, 0\right) \\ \text{Coordinates of } B: & \quad \left(-\frac{1}{2}a, 0, 0\right) \\ \text{Coordinates of } C: & \quad \left(\frac{1}{2}c, \frac{1}{2}s, j\right) \\ \text{Coordinates of } D: & \quad \left(-\frac{1}{2}c, -\frac{1}{2}s, j\right) \end{aligned}$$

The coordinates ensure that $CA = BD$ and $BC = DA$. Then

$$AB^2 = CA^2 \quad \text{if} \quad a^2 = \frac{1}{4}(a-c)^2 + \frac{1}{4}s^2 + j^2$$

and

$$CD^2 = CB^2 \quad \text{if} \quad 1 = \frac{1}{4}(a+c)^2 + \frac{1}{4}s^2 + j^2,$$

which simplify to

$$4a^2 = 1 + a^2 - 2ac + 4j^2 \quad \text{and} \quad 4 = 1 + a^2 + 2ac + 4j^2.$$

Adding leads to

$$j^2 = \frac{1}{4}(1 + a^2).$$

Subtracting leads to $c = (1 - a^2)/a$.

The centre of the circumsphere, G , lies on the z -axis by symmetry. Let the coordinates of G be $(0, 0, g)$. Then $GA = GB$ and $GC = GD$ by symmetry, so it is required that $GA^2 = GC^2$. Now

$$GA^2 = \frac{1}{4}a^2 + g^2 \quad \text{and} \quad GC^2 = \frac{1}{4} + (j - g)^2.$$

Simplification using $j^2 = \frac{1}{4}(1 + a^2)$ leads to

$$g = \frac{1}{4j} = \frac{1}{2\sqrt{1 + a^2}}.$$

The radius of the circumsphere, r say, is GA so that

$$r^2 = \frac{a^2}{4} + g^2 = \frac{a^2}{4} + \frac{1}{4(1+a^2)} = \frac{1+a^2+a^4}{4(1+a^2)},$$

giving the diameter

$$d = \sqrt{\frac{1+a^2+a^4}{1+a^2}},$$

which may also be written as

$$d = \sqrt{\frac{1-a^6}{1-a^4}}.$$

As before, if $a = 1$, d becomes $\sqrt{3/2}$, the appropriate value for the regular tetrahedron.

The restrictions on the value of a imposed by the geometry are not made clear through d (in contrast to the first problem). For two isolated triangles with sides $\{1, 1, a\}$ and $\{1, a, a\}$, the restriction on a is obviously $\frac{1}{2} < a < 2$. A further restriction is associated with the cosine c : $c = (1-a^2)/a = 1/a - a$. Its limits are $-1 \leq (1-a^2)/a \leq 1$. The second of these leads to

$$1 - a^2 - a \leq 0.$$

The quadratic is recognized as that of the golden mean (which is therefore associated with one more improbable problem). In this way the restriction on a can be written

$$0.618\dots = \frac{\sqrt{5}-1}{2} \leq a \leq \frac{\sqrt{5}+1}{2} = 1.618\dots$$

Final comment. The differences in the detailed forms of the two tetrahedra considered here can be traced to the following feature. In the first tetrahedron, the two sets of three edges are geometrically distinct: one set forms a triangle and the other a tripod. There is one type of isosceles triangle, implying that $a < 2$. The limit in which the apex of the tripod is in the same plane as the base triangle imposes the more restrictive condition $a < \sqrt{3}$. In contrast, in the second tetrahedron, the two sets of three edges are geometrically equivalent, forming as they do a set of three edges with two common vertices. The value of a is then restricted by the relative orientation of the two sets—that is, through the cosine parameter.

Also solved by Steve Moon.

Solution 231.6 – Three arctans

Suppose $a, b, c > 0$ and let $p = a + b + c$. Prove that

$$\arctan \sqrt{\frac{ap}{bc}} + \arctan \sqrt{\frac{bp}{ca}} + \arctan \sqrt{\frac{cp}{ab}} = \pi. \quad (1)$$

What if there is no restriction on a, b and c ?

Steve Moon

For any A, B, C , we can form the identity

$$\begin{aligned} \tan(A + B + C) &= \frac{\tan A + \tan(B + C)}{1 - \tan(A)\tan(B + C)} \\ &= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}. \end{aligned}$$

Now if we set

$$A = \arctan \sqrt{\frac{ap}{bc}}, \quad B = \arctan \sqrt{\frac{bp}{ca}}, \quad C = \arctan \sqrt{\frac{cp}{ab}},$$

then

$$\begin{aligned} \tan(A + B + C) &= \frac{\sqrt{\frac{ap}{bc}} + \sqrt{\frac{bp}{ca}} + \sqrt{\frac{cp}{ab}} - \sqrt{\frac{p^3}{abc}}}{1 - \frac{p}{c} - \frac{p}{b} - \frac{p}{a}} \\ &= \frac{\sqrt{p} \left(\sqrt{\frac{a}{bc}} + \sqrt{\frac{b}{ca}} + \sqrt{\frac{c}{ab}} - \frac{p}{\sqrt{abc}} \right)}{1 - \frac{p(ab + bc + ca)}{abc}} \\ &= \frac{\sqrt{p} \left(a\sqrt{abc} + b\sqrt{abc} + c\sqrt{abc} - p\sqrt{abc} \right)}{abc - p(ab + bc + ca)} = 0. \end{aligned}$$

So

$$\tan \left(\tan^{-1} \sqrt{\frac{ap}{bc}} + \tan^{-1} \sqrt{\frac{bp}{ca}} + \tan^{-1} \sqrt{\frac{cp}{ab}} \right) = 0.$$

If $x > 0$, then $\arctan x \in (0, \pi/2)$. Moreover, $\tan x = 0 \Rightarrow x = n\pi$ for some integer n . So if $a, b, c > 0$, then $A, B, C \in (0, \pi/2)$ and therefore $A + B + C = \pi$.

If one of a, b, c is zero, the relationship still holds (because $0 + \pi/2 + \pi/2 = \pi$) as long as we take $\tan^{-1} k/0 = \pi/2$.

If more than one of a, b, c are zero, the expression is undefined since $\tan 0/0$ is undefined.

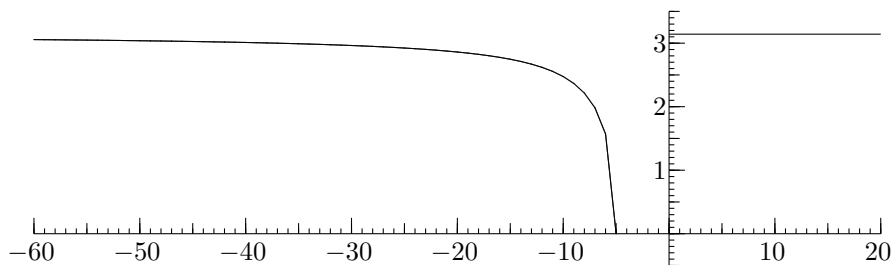
Tony Forbes

If not all of a, b and c are non-negative, the above analysis doesn't seem to work. To try to see what is going on let us put $b = 2, c = 3$. If we call the resulting expression on the left of (1) $T(a)$, then

$$T(a) = \arctan \sqrt{\frac{a(a+5)}{6}} + \arctan \sqrt{\frac{2(a+5)}{3a}} + \arctan \sqrt{\frac{3(a+5)}{2a}}.$$

We must make the assumption that the things being square-rooted are non-negative and that for finite real x the function $\arctan(x)$ takes its principal value, the one in the range $(-\pi/2, \pi/2)$. As we have seen, $T(a) = \pi$ when $a \geq 0$, and we are interested in what happens when $a < 0$. Clearly $T(-5) = 0$. In between, when $-5 < a < 0$, we have to leave $T(a)$ undefined since it involves imaginary quantities.

However, $T(a)$ is well defined for $a \leq -5$ but it is not equal to π , as you can see from the following plot.



Interestingly, if we negate the first term in $T(a)$ to get

$$U(a) = -\arctan \sqrt{\frac{a(a+5)}{6}} + \arctan \sqrt{\frac{2(a+5)}{3a}} + \arctan \sqrt{\frac{3(a+5)}{2a}},$$

then it turns out that $U(a)$ does have a constant value, namely zero, for $a < -2$. Unfortunately I can't see a good reason for making this transformation.

Solution 237.5 – Another sum

Show that
$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} = \frac{\pi^2 - 9}{3}.$$

Steve Moon

Expanding $1/(n^2(n+1)^2)$ using partial fractions,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} &= \sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{2}{n+1} - \frac{2}{n} \right) \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 - 2 = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 3. \end{aligned}$$

We now consider $\sum_{n=1}^{\infty} 1/n^2$.

Using the Taylor series for $\sin x$, we have

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \dots$$

Now $(\sin x)/x = 0$ for $x = n\pi$, $n = \pm 1, \pm 2, \pm 3, \dots$, and we can write $(\sin x)/x$ as an infinite product of linear terms given by these roots:

$$\begin{aligned} \frac{\sin x}{x} &= \left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{\pi}\right) \left(1 - \frac{x}{2\pi}\right) \left(1 + \frac{x}{2\pi}\right) \left(1 - \frac{x}{3\pi}\right) \left(1 + \frac{x}{3\pi}\right) \dots \\ &= \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \dots \end{aligned}$$

Multiply out and equate the coefficient of x^2 in the expansion of $(\sin x)/x$ to obtain

$$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \dots\right) = -\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{3!} = -\frac{1}{6}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 3 = \frac{\pi^2 - 9}{3}.$$

Also solved by Bryan Orman (in M500 239) and Tommy Moorhouse.

Shannon entropy

Ralph Hancock

The piece on Shannon entropy led me to look at the research on those two famous and maddening unsolved things, the Voynich manuscript and the Phaistos Disc, both of which have been suspected of being fakes. Both have had their entropy analysed.

The Voynich MS, of the 15th–16th century, is a single book, 170 characters, 35 words, 20–30 letters in alphabet (uncertainty as to which are duplicates) plus a few dozen rarer symbols. Its entropy comes out about the same as Latin, midway between the literary style of Julius Caesar and the commoner style of the Vulgate; a striking result. The illustrations, of equally maddening incomprehensibility, show that (if it is meaningful) it is about a restricted range of subjects, such as plants. Code or anagrams are suspected by some researchers. All attempts at a solution so far have been feeble and ridiculous.

The Phaistos Disc, very roughly *c.* 1700 BC, is a unique clay disc, bearing 241 characters stamped into the clay in a spiral on each side by using punches bearing a set of 45 symbols having the character of ideograms. The entropy of this very small sample suggests that it is meaningful. Again, no attempted solution has made a shred of sense. Suggestions have been made that the disc is actually not a text but a ludo-like board game, but the entropy test suggests otherwise.

The disc is of the same date and geographical area as another undeciphered script, Linear A, which was written in Crete between 1800 and 1450 BC, but has no apparent relation to it. There are several hundred clay tablets in Linear A, mostly small and, as you would expect, the script evolves over time. Apparently with 60–100 symbols for syllables plus a range of illustrative ideograms, it is fairly similar to the slightly later Linear B, which has been deciphered because it was a way of writing Greek, but the language used here is still unknown, despite attempts at matching all the local languages to it.

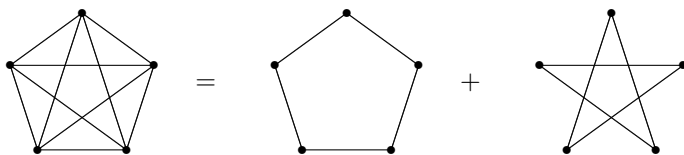
It is noticeable what broad hints you need to solve an undeciphered script. In the case of Linear B it was a line of text with a picture of a tripod at the end and the decipherer Michael Ventris's guess that the language was Greek, and that the first four syllables of the text were *ti-ri-po-de*. Egyptian hieroglyphics needed more: a bilingual inscription, the Rosetta Stone (it has three inscriptions, but two or them are Egyptian in different scripts, so the third one in Greek was the key); plus the correct guess that the hieroglyphs

enclosed in oblong boxes were royal names, which of course also occurred in the Greek version; plus the fact that a language derived from ancient Egyptian, Coptic, had only just died out and was still understood.

Shannon entropy also reminds me of the WinZip test for authorship. The idea is that different authors' texts will compress by different amounts. To prove that Shakespeare was not written by Christopher Marlowe, take ten random excerpts from each author, all of roughly the same length, reduce them to plain text files having exactly the same characteristics, and crunch them with WinZip. Then examine the percentage by which each is compressed (which WinZip tells you in the zip file contents list). The Shakespeare texts will fall within a range, the Marlowe ones within a different range. I am told that it is quite accurate, though of course there is no reason why two different authors shouldn't crunch by the same amount.

Problem 240.4 – Cycles

Show that the complete graph on n vertices, K_n , can be partitioned into cycles of length n if and only if n is odd. Here is what the case $n = 5$ looks like.



Re: Problem 233.6 – The quartic and the golden mean

In the statement of ‘Problem 233.6 – The quartic and the golden mean’ — show that the straight line passing through the two points of inflection of a quartic meets the quartic again at the two points with x coordinates $\tau p - q/\tau$ and $\tau q - p/\tau$, where p and q are the x coordinates of the points of inflection and $\tau = \frac{1}{2}(\sqrt{5} + 1)$ is the golden ratio — in M500 **233**, or at least when Stuart Walmsley’s solution was published in M500 **237**, the Editor of this magazine should have mentioned that this intriguing connection between the quartic and τ was first discovered by **Lin McMullin**; see, for example, <http://www.theoremoftheday.org/Theorems.html>, number 165. Although Robin Whitty did in fact provide the Editor with this reference when he initially communicated the problem to him, and again when the problem appeared in M500 **233** apparently incorrectly attributed, regrettably, and for reasons that do not stand up to scrutiny, the Editor chose to ignore Robin on both occasions. Apologies — TF.

Narayana's Cows

Eddie Kent

Guten Tag, Herr Archimedes is the title of the German translation of a book on the history of mathematics by a Ukrainian scholar named Andrej Grigorewitsch Konforowitsch. This book contains many curiosities, including the following, which Konforowitsch attributed to Narayana, an Indian mathematician in the 14th century:

A cow produces one calf every year. Beginning in its fourth year each calf produces one calf at the beginning of each year. How many cows are there altogether after, for example, 17 years?

One wonders if Fibonacci ever visited India; this problem is so similar to one that he set. In the Fibonacci case each member of the sequence is arrived at by adding together the two previous numbers. In Narayana's case one adds the previous number in the sequence to the number two places before that: $N_n = N_{n-1} + N_{n-3}$.

There are many ways of looking at this pattern, like at what point do calves outnumber cows, and what is the rate of growth of the herd. Or one could investigate different periods for calves to mature. But principally one has to find a number.

To help in solving this, note that in the first, second and third years there is just the original cow and her calf, then two and finally three calves. In the fourth year the oldest calf becomes a mother and we begin a third generation of Narayana's cows.

By the eighth year the herd, that went from one to two to three to four to six to nine to thirteen to nineteen now jumps to 28 and in the ninth year 13 new calves are born. One is a daughter of the original cow; six are granddaughters, and six are great-granddaughters.

By the 15th year the herd numbers 406; this includes the original cow, 15 daughters, 78 granddaughters, 165 great-granddaughters, 126 great-great-granddaughters and 21 great-great-great-granddaughters.

In the 16th year we have one new daughter, 13 new granddaughters, 55 new great-granddaughters, 84 new great-great-granddaughters, 35 new great-great-great-granddaughters and the very first great-great-great-great-granddaughter.

Now we arrive at the 17th and final year of the problem. You have probably calculated what the population is by now, but if you haven't, or if you want to check your work, you can just count the notes at <http://music.ensembleklang.com/track/narayanans-cows>, where Ensemble Klang play the herd in a composition by Tom Johnson.

(There is a version of Narayana's Cows on YouTube, but that is in Greek, which some people might find a little too stimulating.)

Problem 240.5 – Cows

Tony Forbes

Show that the number of cows in Eddie's article on page 16 is given by

$$\frac{1}{\sqrt{93}} \left(\left(\alpha - \frac{1}{\alpha} \right) R^{n+4} + \left(\rho\alpha - \frac{1}{\rho\alpha} \right) S^{n+4} + \left(\frac{\alpha}{\rho} - \frac{\rho}{\alpha} \right) T^{n+4} \right),$$

where $\alpha = \sqrt[3]{\frac{1}{2}(29 + 3\sqrt{93})}$, $\rho = \frac{1}{2}(-1 + i\sqrt{3})$, a cube root of 1, and

$$R = \frac{1}{3} \left(1 + \alpha + \frac{1}{\alpha} \right), \quad S = \frac{1}{3} \left(1 + \rho\alpha + \frac{1}{\rho\alpha} \right), \quad T = \frac{1}{3} \left(1 + \frac{\alpha}{\rho} + \frac{\rho}{\alpha} \right)$$

are the three roots of $x^3 - x^2 - 1$. As is usual in such situations, one must also assume that the cows are immortal.

I am intrigued by the appearance of the square root of 93 in a simple problem involving cattle breeding. I wonder if herdsmen can offer an explanation. I also wonder if the farming industry is aware of the solution to this problem and in particular the value of the important constant $\alpha \approx 3.0711$.

M500 Mathematics Revision Weekend 2011

The thirty-seventh **M500 Society Mathematics Revision Weekend** will be held at

Aston University, Birmingham

over

Friday 9th – Sunday 11th September 2011.

The cost, including accommodation (with en suite facilities) and all meals from bed and breakfast Friday night to lunch Sunday is £257 (in Aston's Lakeside flats) or £307 (Aston Business School). The cost for non-residents is £123 (includes Saturday and Sunday lunch). M500 members get a discount of £10. For full details and an application form, see the Society's web site at www.m500.org.uk, or send a stamped, addressed envelope to

Jeremy Humphries, M500 Weekend 2011.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most mathematics-based OU courses, subject to sufficient numbers.

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