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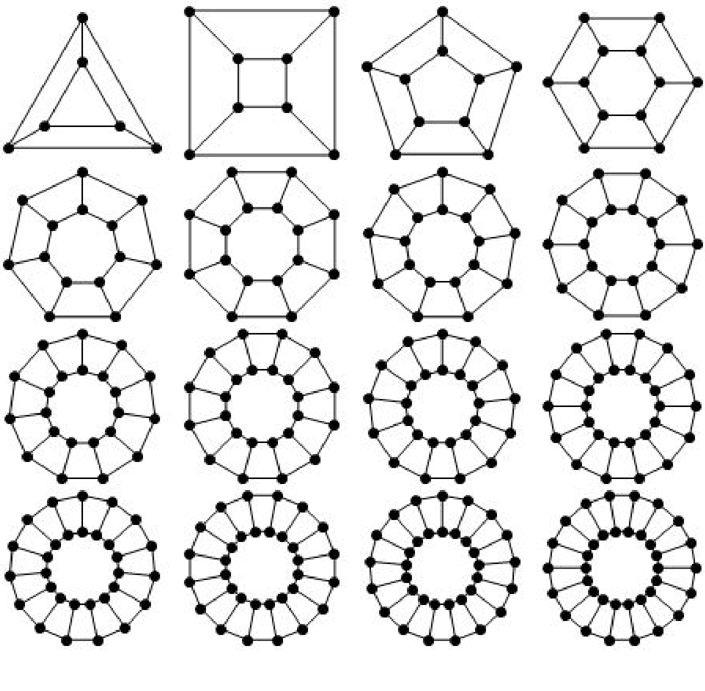
M500 243



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file to tony@m500.org.uk.

TONY HUNTINGTON

With great regret we have to report the sudden death of M500 supporter, quizmaster and friend Tony Huntington at his home on 14th December 2011.

We first heard of Tony from his critical analysis of some of our procedures. We decided that he was too formidable to be our adversary, so we asked if we might be permitted to count him as our ally, a request which he granted. He turned out to be such a felicitous colleague and friend, so full of humour, that we came to wonder if his initial interactions with us had been as much mischievous as serious. Wherever the truth lies, the day he made that first contact was a good day for everybody connected with the M500 Society.

Although he was on the M500 Committee for only a brief period in 1997 as Publicity Officer, before the demands of employment unexpectedly took him to the Middle East, all of you who knew him will regard Tony with affection as an essential part of the M500 establishment. Tony and Sonia have been volunteer supporters of M500 since 1996 and their valuable assistance in the running of the revision and winter weekends—with Tony commuting from Oman in the years before his permanent return to England—has always been much appreciated. Winter weekenders will remember Tony's many interesting and entertaining presentations, and his contributions to the M500 magazine have been a regular source of delight and amusement over the past fifteen years.

Tony's presence at future M500 events, especially his lively winter weekend sessions, will be sadly missed. Our sympathies go to his family and friends, especially Sonia.

Russian peasant multiplication

Sebastian Hayes

Ogilvy and Andersen, in their excellent book *Excursions in Number Theory*, recount the true story of an Austrian colonel who wanted to buy seven bulls in a remote part of Ethiopia some sixty or so years ago. Although the price of a single bull was set at 22 Maria Theresa dollars, no one present could work out the total cost of the seven bulls—and the peasants, being peasants, didn't trust the would-be buyer to do the calculation himself. Eventually the priest of a neighbouring village and his helper were called in.

> The priest and his boy helper began to dig a series of holes in the ground, each about the size of a teacup. These holes were ranged in two parallel columns; my interpreter said they were

> called houses. The priest's boy had a bag full of little pebbles. Into the first cup of the first column he put seven stones (one for each bull), and twenty-two pebbles into the first cup of the second column. It was explained to me that the first column was used for doubling; that is, twice the number of pebbles in the first house are placed in the second, then twice that number in the third, and so on. The second column is for halving: half the number of pebbles in the first cup are placed in the second, and so on down until there is just one pebble in the last cup. If there is a pebble remaining when doing the halving it is thrown away.
>
> The division column (the right one) is then examined for odd or even numbers of pebbles in the cups. All even houses are considered to be evil ones, all odd houses good. Whenever an evil house is discovered, the pebbles in it are thrown out and not counted, and the pebbles in the corresponding 'doubling' column are also thrown out. All pebbles left in the cups of the left, 'doubling' column are then counted, and the total is the answer.
>
> [C. S. Ogilvy & J. T. Andersen, *Excursions in Number Theory*]

The working on paper would be as follows.

Doubling column	Halving column
7	22
14	**11**
28	**5**
56	2
112	**1**
154	

The priest worked out the result using holes and pebbles in the way I have demonstrated though instead of using different coloured beans the helper simply removed the stones from right-hand holes opposite ones with an even number in them. The colonel duly paid up, astounded to note that the crazy system 'gave the right answer'.

Let us go further back in time. We suppose that a 'primitive' society had grasped the principle of numerical symbolism at the most rudimentary level, namely that a chosen *single* object such as a shell or bean could be used to represent a *single* different object, such as a tree or a man, and that clusters of men or trees could be represented by appropriate clusters of shells—the 'appropriateness' to be checked by the time-honoured method of 'pairing off'. This society has not, however, necessarily attained the stage of realizing that a single 'one-symbol' will do for every singleton, let alone reaching the stage of evolving a base such as our base ten. Now suppose the chief wants each of the villages in a certain area to provide '*nyaal*' or ●●●●●●● young men for some public works or warlike purpose. We have '*nyata*' or ○○○○○○ villages from which to draw the task force. The chief relies on two shamans to carry out numerical calculations, both of whom are adept in the practice of 'pairing off' but one has specialized in 'doubling' imaginary or actual quantities, the other in 'halving' imaginary or actual quantities. Although both shamans know that every quantity can be doubled, the 'halving' shaman knows that this procedure does not always work in reverse. He gets round this by simply throwing away the extra bean or shell—the equivalent of our 'rounding off' a quantity to a certain number of decimal places.

The halving shaman works with a column of holes on the left-hand side of a 'numbering area' (a flat piece of ground with holes in it) and he has a store of short sticks, shells or some other common object, which he places in the holes, or simply in a cluster on the ground. The doubling shaman works with a similar column of holes on the right but he has a store of beans or shells which are in two colours, light and dark. (The use of colour to distinguish two different types of quantities, or to distinguish between males and females, was the invention of a revered mathematical shaman who taught the two current shamans.)

The halving shaman sets out the sticks or shells representing the villages and tries, if possible, to have two matching rows. The doubling shaman watches carefully and, if the amount on the left can be arranged in two rows exactly, as in this case, he starts off with a set of dark coloured beans to represent the young men to be co-opted for the task at hand from each village. We thus have the following.

Villages	Young males
○○○ ○○○	●●●● ●●●

Now the halving shaman selects half the quantity in the first hole, i.e. a single row, and arranges it as evenly as possible in two rows. In this case, there is a bean left over, and the doubling shaman, noticing this, doubles the original amount on the right but also changes the colour of the beans.

○○ ○	○○○○○○○ ○○○○○○○

The halving shaman discards the extra unit on the left and once again throws away a row. This leaves just a single bean ○ and, since we are not allowed to split a bean or shell, this signals the end of the procedure as far as he is concerned. The doubling shaman doubles his quantity and since the quantity on the right is 'odd' (since it cannot be arranged in a two matching rows) he once again chooses light coloured beans.

○	○○○○○○○○ ○○○○○○○○ ○○○○○○○○ ○○○○○○○○

The two shamans collaborate to combine all the light coloured beans (but not the dark coloured ones), giving a total of the following.

○	○○○○○○○○ ○○○○○○○○ ○○○○○○○○ ○○○○○○○○ ○○○○○○○○ ○○○○○○○○

The chief is given this amount of beans and thus knows how many young men he can expect to get for the task at hand. From experience, the chief will have a pretty good idea of the size of this quantity and, if it seems inadequate for the task, may decide to increase the quota of young men impressed from each village. When preparing for battle, the chief might use human beings as counters, pair them off against the beans, then have them form square formations to judge whether he has a large enough army or raiding force.

If asked by a time traveller why the dark-coloured beans—which are always opposite an *even* number—are rejected, the doubling shaman would probably say that even amounts are female (because of breasts) and the chief doesn't want effeminate men or boys who were still living with their mothers.

The multiplicative system just demonstrated is very ancient indeed; it is probably the very earliest mathematical system worthy of the name and was doubtless invented, reinvented and forgotten innumerable times throughout human history. Since it does not require any form of writing and involves only three operations, pairing off, halving and doubling, which are both easy to carry out and are not troublesome conceptually, the system remained extremely popular with peasants the world over and became known as *Russian Multiplication* because, until recently, Russia was the European country with by far the largest proportion of innumerate and illiterate peasants. It is actually such a good method that I have seriously considered using it myself, at any rate as a visual aid in doing mental arithmetic—it is one of the tools employed by traditional 'lightning calculators' and mathematical *idiot savants*.

Actually, one could say that the three mathematical procedures predate not only the earliest tribal societies but even the existence of mammals! Viruses, the lowest form of 'life'—if indeed they are to be considered alive at all, which is still a matter of debate—are incapable of doubling, i.e. cannot reproduce, let alone halving and have to get the DNA of another cell to do the work for them. They may be considered capable of 'pairing off' however, since a virus seeks out the nucleus of a cell on the basis of one virus, one nucleus. Bacteria, a much more advanced life form, reproduce by mitosis, basically duplicating everything within the cell and splitting in two, the 'daughter' cell being an exact replica (clone) of the 'mother' cell. So they are capable of doubling. Some 'advanced' eukaryotes, including mammals, are also capable of halving since this procedure is involved in sexual reproduction (but not in other forms of reproduction). Animal and plant cells are said to be diploid since they contain homologous pairs of chromosomes; in humans we have $2n = 46$. However, the sex cells during meiosis not only double in number but manage to halve the chromosome count, producing so-called haploid cells (gametes) which, in our human case, come in two kinds. Fusion of 'egg' and 'sperm' cells restores the diploid number and incidentally introduces a further mathematical operation, combination, which may be considered the distant ancestor of Set Theory. It is thus maybe not at all surprising that peasants the world over have felt at home with 'Russian' multiplication, being closer to Nature and thus to the three basic processes of Nature, pairing off, doubling and halving on which human reproduction depends.

A good written notation is not at all essential for Russian Multiplication, but it does speed things up. Using our Hindu/Arabic notation, suppose you want to multiply 147 by 19. This is a somewhat tedious enterprise if you are not allowed a calculator and these days two students out of three would probably come up with the wrong answer. So here goes.

19	×	147	
9		294	
4		**588**	147
2		**1176**	294
1		2352	2352
			2793

Now do it with a calculator. The result: 2793.

Why does the system work? You might like to think about this for a moment before reading on. (It personally took me a long time to cotton on though someone I mentioned it to saw it at once.)

Russian Multiplication works because any number can be represented as a sum of powers of two (counting the unit as the 0th power of any number).

Algebraically we have

$$N = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x^1 + A_0$$

with $x = 2$. In practice there are only two choices of coefficient for the A_n, A_{n-1}, …, A_0 namely 0 and 1 because once we get to a remainder of 2 we move to the next column. When 0 is the coefficient this term is not reckoned in the final count and is discounted just like the pebbles in the hole opposite an even cluster. Since $1 \times x^n = x^n$, we can simply dispense with coefficients altogether—which is not true for any other base.

If we look back at the pattern of black and grey in the right-hand column and write 0 for black and 1 for grey, we have the representation of the number on the left in binary notation (though it is in reverse order compared to our system). Take the multiplication of 19 and 147 on page 5. The pattern in the right-hand column is, from the bottom upwards, as follows.

grey
black
black
grey
grey $= 10011 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$
$= 19_{10}$

A hole in the ground functions as a 'House of Numbers' and can only be in two states: either it is empty or it has something in it (i.e. is non-empty). The Abyssinian priest's assistant who removed the stones from a house opposite one with an even number of stones in it was placing the House in the zero state. The right-hand column Houses were in fact functioning in two different though related roles: on the one hand they were in binary (empty or non-empty) while on the other hand they gave the quantities to be added in base one.

Did people using the system know what they were doing? In most cases probably not although, judging by their confidence in handling arithmetical operations, the Egyptian scribes, using a very similar method I shall perhaps write about in a subsequent article, almost certainly did: the peasants using the system just knew it worked. There is nothing surprising or shocking about this—how many people who use decimal fractions without a moment's thought realize that the system only works because we are dealing with an indefinitely extendable geometric series which converges to a limit because the common multiple is less than unity?

One might wonder whether it would be possible to extend the principle of Russian Multiplication to tripling, quadrupling and so on? You might like to think about this for a moment. Worked example: Take 19×23 using 3 as divisor and multiplier.

19	**23**
6	69
2	**207**
?	

We have already run into difficulties since we cannot get back to the unit. On the analogy with modulus 2 Russian Multiplication, we might decide we have to take into account the final entry on the right nonetheless, plus all entries which are not opposite an exact multiple of 3. This means the answer is $207 + 23 = 230$, which is way off since $10 \times 23 = 230$. What has gone wrong?

A little thought should reveal that, whereas in the case of modulus 2 we only had to neglect at most a unit on the left-hand side, in the case of modulus 3 there are two possible remainders, 1 and 2. If we are opposite a number on the left which is 1 (mod 3) we include the number on the right in the final addition. However, if we are opposite a number which is 2 (mod 3) we must double the entry on the right since it is this much that has been neglected. In the above $19 = (6 \times 3) + 1$ and so it is 1 (mod 3) but 2 at the bottom is $(0 \times 3) + 2$ and so is 2 (mod 3). Applying the above we obtain $23 + (2 \times 207) = 23 + 414 = 437$, which is correct.

To make the system work properly we need not *one* but *two* ways of marking entries in the right-hand column to show whether they just have to be added on or have to be doubled first. This is an annoying complication, and even apart from this it is not that easy to divide into three and to treble integers. And if we move onto higher moduli there are much greater complications still. The Russian way of doing things ceases to be simple and user-friendly.

Russian Multiplication is a good example of an invention excellent in itself but which does not lead on to further inventions and discoveries: it remains all on its own like an island in the middle of the Pacific Ocean. Once the crucial improvement of distinguishing the entries to be added from the others was made, there was nothing much that could be done in the way of improvements except possibly the introduction of colour coding, my distinction between dark and light coloured beans. To actually find a better multiplication system you have to make a giant leap in time to the ciphered Greek system of numerals or the full place value Indian system—and even so the advantages would not have been apparent to peasants. If you are only dealing with relatively small quantities, Russian Multiplication is quite adequate, is easier to comprehend, and there are fewer opportunities for making mistakes. In such a case we see that there is indeed a 'simplicity cut off point' beyond which it is not worth extending existing techniques, since the disadvantages outweigh the advantages. However, there may also be a 'second time round point' when technology has become so sophisticated that it has become 'simple' (= 'user-friendly') once more. Computers, being as yet relatively unintelligent creatures, have reverted to base 2 arithmetic though I believe 16 is also used. Wolfram's cellular automata based on simple rules which specify whether a given 'cell' repeats or doesn't can perform complicated operations like taking square roots of large numbers.

This cycle of invention, stasis, disappearance and reinvention happens all the time; it is more often than not impossible to improve on an early invention without making a giant leap, a leap requiring not only new ideas but large-scale social and economic changes which are usually felt to be undesirable because they are disruptive, or are quite simply out of the question given the available technology. Short of hiring expensive modern haulage equipment the best way to move large heavy objects across uneven ground is the time-honoured Egyptian system of wooden rollers which are repeatedly brought round to the front. (I have often had occasion to use this system myself in inaccessible places and it is surprising how well it works.) The longbow made of yew and animal gut more than held its own against the far more advanced crossbow: the English bowmen won Agincourt against axe-wielding French knights and Genoese crossbowmen largely because the crossbow is slow to reload and its effectiveness is much reduced in wet weather (the English kept their catgut dry until the battle began). In point of fact the longbow, an extremely rudimentary weapon, was only superseded in speed, range and accuracy by the repeating rifle—one of Wellington's military advisors seriously suggested re-introducing the longbow against Napoleon's *Grande Armée*. And the horse as a means of transport was only superseded by the railway; messages were not transmitted much faster across Europe (if at all) under Napoleon than under Augustus Caesar.

Tony Forbes writes — Curiously, Russian peasant multiplication does have an interesting modern application. It is the system that is used to perform computations on elliptic curves. In the Abelian group associated with an elliptic curve one often wants to 'multiply' a point X on the curve by a positive integer j to give $j\,X = X + X + \cdots + X$, where $+$ is the group operation. Indeed, I included this topic in a series of talks on elliptic curves, which I delivered at the London South Bank University in 2007. However, I did make the change to the more politically correct form: *Russian agricultural community multiplication.*

The computation of $j\,X$ is done with just the two procedures: (i) doubling and (ii) the addition of X to something. Start with $Y = O$, the identity element of the group, and assume the base 2 representation of j is available—as it would be on a typical modern computer. Then scan the binary digits of j from left to right. If you see a 1, double Y and add X. If you see a 0, just double Y. For example, 42 is 101010 in base 2; so $42X$ gets computed as $Y = 2(2(2(2(2(2\,O + X)) + X)) + X)$.

Problem 243.1 – Cheese

Tony Forbes

You have a $1\,\mathrm{m}^3$ cube of cheese to divide equitably among k people. (Perhaps you are hosting a cheese and wine party for a large gathering.) So you slice off $1/k$ m from one end by a plane cut parallel to one of the faces of the cube, leaving a cuboid of dimensions $1\,\mathrm{m} \times 1\,\mathrm{m} \times (1 - 1/k)\,\mathrm{m}$. However, for the second and subsequent pieces you must make the cut through a cross-section of minimum area. (Obviously you want to save wear on the cheese cutter.)

Denote by d_i the distance (in metres) of the ith cut from the nearer of the two faces parallel to it. For example, if $k = 7$, then $d_1 = 1/7$, $d_2 = 1/6$ since the second slice is made $1/6$ m from one of the $1\,\mathrm{m} \times 6/7\,\mathrm{m}$ faces, and $d_3 = 1/5$ since the third slice is made $1/5$ m from one of the $5/6\,\mathrm{m} \times 6/7\,\mathrm{m}$ faces of the $1\,\mathrm{m} \times 6/7\,\mathrm{m} \times 5/6\,\mathrm{m}$ cuboid left by the second cut.

Find a formula for the sequence of numbers $d_1, d_2, d_3, \ldots, d_{k-1}$.

Problem 243.2 – Cosh integral

Tony Forbes

Let n be a positive integer. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{\cosh^n x \, \cosh^{n+1} y} = \frac{2\pi}{n}. \qquad (1)$$

Of course, one can split it up, evaluate each integral separately and then multiply. However I feel that because of the truly elegant nature of (1) there might be an alternative and more enlightening proof of appropriate simplicity.

If it makes life easier, you may assume that

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh x} = \pi \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{dx}{\cosh^2 x} = 2.$$

Can you get from **SHIP** to **DOCK** without using an intermediate word containing two consecutive vowels? Usual rules apply. (i) Substitute one letter at a time. (ii) Do not deviate from English as she is spoke.

Solution 234.4 – Tetrahedron

> Three sides of a tetrahedron form an equilateral triangle of side a. The other three sides have length 1. Show that the diameter of the sphere that circumscribes the tetrahedron is
>
> $$\frac{\sqrt{3}}{\sqrt{3-a^2}}.$$
>
> What if three sides have length 1 and three sides have length $a \neq 1$, but no face of the tetrahedron is equilateral?

Dick Boardman

I offer here an alternative way of solving the second part of the problem, which is in some ways easier to comprehend (at least for me) than Stuart Walmsley's solution in M500 **240**.

Consider an equilateral triangle ABC with $|AB| = 1$ and $|AC| = |BC| = a$ in the (x, y)-plane and with coordinates

$$A = \left(-\frac{1}{2},\ 0,\ 0\right), \quad B = \left(\frac{1}{2},\ 0,\ 0\right), \quad C = \left(0,\ \frac{\sqrt{4a^2-1}}{2},\ 0\right).$$

Let $D = (u, v, w)$ be the fourth point of the tetrahedron and suppose $|AD| = a$ and $|BD| = |CD| = 1$. The coordinates of D are determined by solving

$$\begin{aligned} u^2 + u + \tfrac{1}{4} + v^2 + w^2 &= a^2, \\ u^2 - u + \tfrac{1}{4} + v^2 + w^2 &= 1, \\ u^2 + \left(v - \tfrac{1}{2}\sqrt{4a^2-1}\right)^2 + w^2 &= 1, \end{aligned}$$

and choosing the solution with $w > 0$ to obtain:

$$D = \left(\frac{a^2-1}{2},\ \frac{3a^2-2}{2\sqrt{4a^2-1}},\ \sqrt{\frac{a^6-2a^4-2a^2+1}{1-4a^2}}\right).$$

If $S = (x, y, z)$ is the centre of the circumcircle and d its diameter, then $|AS| = |BS| = |CS| = |DS| = d/2$. Solving for x, y, z and d then yields

$$S = \left(0,\ \frac{1-2a^2}{2\sqrt{4a^2-1}},\ \frac{1}{2}\sqrt{\frac{1-3a^2+a^4}{1-3a^2-4a^4}}\right), \qquad d = \sqrt{\frac{a^4+a^2+1}{a^2+1}}.$$

As before, this works only when $\sqrt{5} - 1 \leq 2a \leq \sqrt{5} + 1$.

Problem 243.3 – Odd sequence

Robin Whitty

Alexander Sharkovsky defined an ordering on the positive integers by virtue of the fact that each may be uniquely specified in the form $2^r p$ where r is a non-negative integer and p is a positive odd number. Sharkovsky's famous theorem on limit cycles in iterated functions is based on this ordering, which is usually specified informally thus:

$$3, 5, 7, \ldots, 2\cdot 3, 2\cdot 5, 2\cdot 7, \ldots, 2^2\cdot 3, 2^2\cdot 5, 2^2\cdot 7, \ldots, \quad \ldots, 2^3, 2^2, 2^1, 2^0.$$

Give a precise definition of this ordering.

Solution 240.3 – Double sum

Show that $\displaystyle\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{r^2s^2} = \frac{\pi^4}{120}$.

Bryan Orman

We have

$$\sum_{r=1,\ s=1}^{\infty}\frac{1}{r^2s^2} = \sum_{r=1}^{\infty}\frac{1}{r^2}\sum_{r=1}^{\infty}\frac{1}{s^2} = \frac{\pi^2}{6}\frac{\pi^2}{6} = \frac{\pi^4}{36}$$

and

$$\sum_{r=1,\ s=1}^{\infty}\frac{1}{r^2s^2} = \sum_{r=1,\ s=r+1}^{\infty}\frac{1}{r^2s^2} + \sum_{s=1,\ r=s+1}^{\infty}\frac{1}{r^2s^2} + \sum_{r=s=1}^{\infty}\frac{1}{r^2s^2}$$

$$= 2\sum_{r=1,\ s=r+1}^{\infty}\frac{1}{r^2s^2} + \sum_{r=1}^{\infty}\frac{1}{r^4} = \frac{\pi^4}{36}.$$

Therefore

$$\sum_{r=1,\ s=r+1}^{\infty}\frac{1}{r^2s^2} = \frac{1}{2}\left(\frac{\pi^4}{36} - \frac{\pi^4}{90}\right) = \frac{\pi^4}{120}.$$

Note that we have used $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r^2} = \frac{\pi^2}{6}$ from the Fourier series of x^2, $|x| \le \pi$ with $x = \pi$ and $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r^4} = \frac{\pi^4}{90}$ from the Fourier series of $(x^2-\pi^2)^2$, $|x| \le \pi$ with $x = \pi$.

Solution 236.2 – Series

Find a closed expression for $S(M) = \sum_{n=1}^{M} \frac{1}{n(n+1)}$ and show that $\sum_{n=1}^{\infty} \frac{1}{n(n+N)} = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{k}$.

Reinhardt Messerschmidt

Let

$$S_N(M) \;=\; \sum_{n=1}^{M} \frac{1}{n(n+N)}.$$

We have:

$$\begin{aligned} S_N(M) \;&=\; \sum_{n=1}^{M} \frac{1}{N}\left[\frac{1}{n} - \frac{1}{n+N}\right] \;=\; \frac{1}{N}\left[\sum_{n=1}^{M} \frac{1}{n} - \sum_{n=1}^{M} \frac{1}{n+N}\right] \\ &=\; \frac{1}{N}\left[\sum_{n=1}^{N} \frac{1}{n} + \sum_{n=N+1}^{M} \frac{1}{n} - \sum_{n=1}^{M} \frac{1}{n+N}\right] \qquad \text{(if } M > N\text{)} \\ &=\; \frac{1}{N}\left[\sum_{n=1}^{N} \frac{1}{n} + \sum_{n=1}^{M-N} \frac{1}{n+N} - \sum_{n=1}^{M} \frac{1}{n+N}\right] \\ &=\; \frac{1}{N}\left[\sum_{n=1}^{N} \frac{1}{n} - \sum_{n=M-N+1}^{M} \frac{1}{n+N}\right]. \qquad (1) \end{aligned}$$

Suppose k is an integer greater than 0. From (1):

$$S_1(10^k - 1) \;=\; 1 - \frac{1}{(10^k - 1) + 1} \;=\; 1 - 10^{-k};$$

in other words $S_1(9999\ldots9) = 0.9999\ldots9$. The second summation in (1) has N non-negative terms that decrease as n increases; therefore

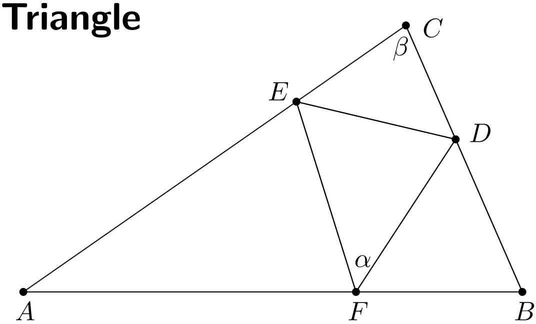
$$0 \;\le\; \sum_{n=M-N+1}^{M} \frac{1}{n+N} \;\le\; N\left[\frac{1}{(M-N+1)+N}\right] \;=\; \frac{N}{M+1}.$$

Now $\lim_{M\to\infty} N/(M+1) = 0$; therefore

$$\sum_{n=1}^{\infty} \frac{1}{n(n+N)} \;=\; \lim_{M\to\infty} S_N(M) \;=\; \frac{1}{N}\sum_{n=1}^{N} \frac{1}{n}.$$

Problem 243.4 – Triangle

Here's something that you can easily do on the train to work. Given that $AE = AF$ and $BD = BF$, show that $\beta + 2\alpha = \pi$.



Tea

An engineer, a scientist and a mathematician make tea.

The engineer goes first, watched by the others. He fills the kettle with water from the sink tap. He puts the kettle on the stove and lights the gas. Whilst waiting for the water to boil he busies himself getting the teapot, teaspoons, cups and saucers ready, retrieving the milk from the fridge and the sugar from the sugar cupboard. When the water is ready he uses some to wash out the teapot and returns the kettle to the stove. Then he adds tea-leaves to the teapot followed by boiling water. The teapot is closed and wrapped in a tea-cosy. The gas is turned off. After four minutes the tea is poured into the cups with milk and sugar added as desired. Tea is served.

The scientist goes next. The starting point is as before except that the kettle is full of water. No problem—he copies the engineer's procedure except for the first step.

Now it's the mathematician's turn. Again, the kettle is full of water. The mathematician ponders for several seconds before arriving at his solution to the problem. He empties the kettle.

By the way, I (TF) had some feedback about the filler at the bottom of M500 **241** page 10 from one or two people.

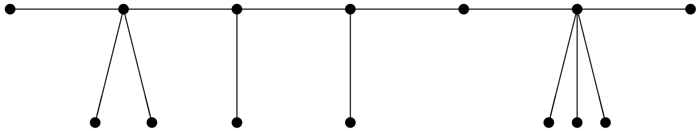
> Do the rationals form a group under addition? No. For example, $3 \in \mathbb{Q}$ and $\frac{1}{7} \in \mathbb{Q}$ but $3 + \frac{1}{7} = \frac{22}{7} = \pi$, and π is irrational.

Yes, this is a joke. Well, at least I think so. However, Marianne Fairthorne tells me that this (or something like it) was offered by a London university undergraduate as the solution to an assignment question!

Problem 243.5 – Counting caterpillars

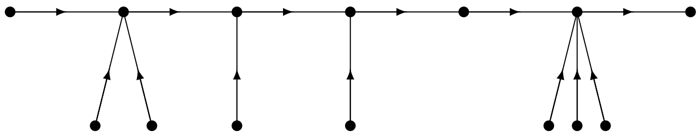
Tony Forbes

A caterpillar is a tree where there is a central path of maximal length and every vertex not on the path is at distance 1 from it. Like this, for example.

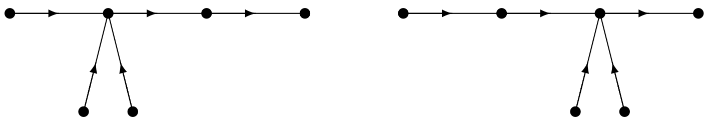


(i) How many caterpillars of n vertices are there?

(ii) If that's too difficult, consider the same problem for a specific type of directed caterpillar, where the edges of the body point towards one end and the legs point towards the body.



The effect of this complication is that, for example, these two caterpillars



are now regarded as distinct. You should get the answer 2^{n-3}.

Problem 243.6 – Piles of coins

Jeremy Humphries

There are 10 piles of 10 coins each. Nine piles are good and one pile is counterfeit. Good coins weigh 10 and dud coins weigh 9. You have a kitchen scale which tells you the weight in the pan, and you need to identify the dud pile in as few weighings as possible. How many weighings?

What does the dummy say to the ventriloquist, to make sure that they have the required equipment and paraphernalia when they are setting out for an engagement?

"Got all the gear?" [JRH]

Solution 241.3 – Multiplicative function

> Let f be a function that maps positive integers to positive integers. Suppose also that f is multiplicative; in other words, if $\gcd(x, y) = 1$ then $f(xy) = f(x)f(y)$. Suppose moreover that f is increasing; in other words, if $y > x$ then $f(y) > f(x)$. Suppose furthermore that $f(2) = 2$. Show that f must be the identity function.

Tony Forbes

Nobody has sent anything; so I might as well offer my no-frills, somewhat messy attempt to solve this problem. If there is a clever proof that can be delivered in a sentence or two, then I am unaware of it.

The first thing to recall is that, as with any multiplicative function, $f(1) = 1$. Also there is the trivial observation that the increasing property implies $f(n + 1) \geq f(n) + 1$. In particular, $f(3) \geq f(2) + 1 = 3$.

Let us put $f(3) = 3+t$ for some integer $t \geq 0$. Then by the multiplicative property we have $f(6) = 6 + 2t$. Hence

$$f(4) \in [4 + t, 4 + 2t] \quad \text{and} \quad f(5) \in [5 + t, 5 + 2t].$$

Therefore

$$\begin{aligned} f(10) = f(2)f(5) &\in [10 + 2t, 10 + 4t], \\ f(12) = f(3)f(4) &\in [12 + 7t + t^2, 12 + 10t + t^2], \\ f(15) = f(3)f(5) &\in [15 + 8t + t^2, 15 + 11t + t^2]. \end{aligned}$$

Now $f(7) \in [7+2t, 7+4t]$ since it must lie in the interval $[f(6)+1, f(10)-3]$. Also $f(7) \in [\frac{1}{2}(14 + 7t + t^2), \frac{1}{2}(14 + 11t + t^2)]$ since $f(14) = 2f(7)$ lies in $[f(12) + 2, f(15) - 1]$. However, the two intervals for $f(7)$ do not overlap unless $t = 0$ or 1.

Assume $t = 1$. Then the intervals for $f(7)$ overlap in just one value, namely $f(7) = 11$, which leads to the following deductions: $f(3) = 4$, $f(6) = 8$, $f(8) = 12$, $f(9) = 13$, $f(10) = 14$, $f(5) = \frac{1}{2}f(10) = 7$, $f(15) = f(3)f(5) = 28$ and finally $f(18) = 2f(9) = 26$, contradicting the increasing nature of f.

Hence $t = 0$ and $f(3) = 3$. Therefore $f(6) = 2f(3) = 6$, $f(4) = 4$, $f(5) = 5$, $f(10) = 2f(5) = 10$, $f(7) = 7$, $f(8) = 8$, $f(9) = 9$, $f(18) = 2f(9) = 18$ and so on.

How to solve inequalities

Tony Forbes

This came up in a previous issue, where we were puzzled by the seemingly ad hoc method of solving an equality like $x^2 < 4$. (See M500 **241**, p. 15.)

The general problem is to find a method for solving $P(x) > 0$ for x, where $P(x)$ is a polynomial, that avoids arm waving and magic tricks (such as drawing graphs). Here's one way.

We can assume without loss of generality that the leading coefficient of $P(x)$ is ± 1. Factorizing $P(x)$ over the reals, we have

$$P(x) \;=\; p_0(x)p_1(x)p_2(x)\ldots p_r(x)q_1(x)q_2(x)\ldots q_s(x) \;>\; 0,$$

where r, s are non-negative integers, $p_0(x) = \pm 1$, the other $p_i(x)$ are linear factors, the $q_j(x)$ are irreducible quadratic factors, and all factors with the possible exception of $p_0(x)$ have leading coefficient 1. The $q_j(x)$ are positive-definite quadratic forms and can therefore be cancelled to obtain

$$p_0(x)p_1(x)p_2(x)\ldots p_r(x) \;>\; 0.$$

Now for every possible way of selecting an even number of factors, solve $p_i(x) < 0$ for each of the selected factors (if any) simultaneously with $p_j(x) > 0$ for the non-selected factors (if any).

For example, solve $x^5 - 5x^4 + 6x^3 - x^2 + 5x < 6$. Thus

$$P(x) \;=\; -x^5+5x^4-6x^3+x^2-5x+6 \;=\; -(x-1)(x-2)(x-3)(x^2+x+1) \;>\; 0.$$

Hence we solve $(-1)(x-1)(x-2)(x-3) > 0$. Eight cases; column 1 in the table indicates the selected negative factors. Thus $2 < x < 3$ or $x < 1$.

none		$-1 > 0$, $x-1 > 0$, $x-2 > 0$, $x-3 > 0$	no solution
1st & 2nd	$-1 < 0$, $x-1 < 0$,	$x-2 > 0$, $x-3 > 0$	no solution
1st & 3rd	$-1 < 0$, $x-2 < 0$,	$x-1 > 0$, $x-3 > 0$	no solution
1st & 4th	$-1 < 0$, $x-3 < 0$,	$x-1 > 0$, $x-2 > 0$	$2 < x < 3$
2nd & 3rd	$x-1 < 0$, $x-2 < 0$,	$-1 > 0$, $x-3 > 0$	no solution
2nd & 4th	$x-1 < 0$, $x-3 < 0$,	$-1 > 0$, $x-2 > 0$	no solution
3rd & 4th	$x-2 < 0$, $x-3 < 0$,	$-1 > 0$, $x-1 > 0$	no solution
all four	$-1 < 0$, $x-1 < 0$, $x-2 < 0$, $x-3 < 0$		$x < 1$

I hope this helps. I leave it for the reader to deal with the inequality that started this discussion: $x^2 < 4$.

Solution 239.1 – Three coins

> Arthur, Ford and Marvin play a game. They try to predict the outcome of three coin tosses. As usual with coins and the tossing thereof, the probability of guessing correctly is always 1/2. After the tossings are decided, the person (or persons, if there is a tie) with the most results correct wins (or share) a Valuable Prize of £300, say. For example, if the forecasts are Arthur HHH, Ford THT, Marvin TTT and the results are HHT, then Arthur and Ford get £150 each. If the players play independently, clearly they have equal chances of winning. However, after Arthur and Ford have made their predictions and before Marvin has made his, Ford offers to show his forecast to Marvin in return for a fee of £1. What should Marvin do?

Vincent Lynch

First of all, when the forecasts are independent, the probability of 0, 1, 2, 3 correct is 1/8, 3/8, 3/8, 1/8 as it is a binomial distribution.

And when two forecasts are independent, we may multiply the probabilities to find the probability of both occurring.

First suppose that Marvin is so devoid of probability knowledge that he pays £1 and chooses the same forecasts as Ford. Then, suppose the probability of them having an equally good forecast as Arthur is p. Then they win £100 each with probability p, and if that is not the case, they have probability 0.5 of winning £150 each. So, by the law of total probability, their expectation is $\pounds(100p + 75(1-p)) = \pounds(100 - 25(1-p)) < \pounds 100$.

So, there is a good case for forecasting exactly the opposite of Ford.

But we must calculate the probabilities.

I'm studying M343 this year, but don't propose to use any fancy notation. When Marvin forecasts the opposite to Ford, he wins £300 when his forecast is better than the others. Either all three are correct, probability 1/8 and A not all correct, probability 7/8: $1/8 \cdot 7/8 = 7/64$. Or, when he has two correct and A has zero or one correct: $3/8 \cdot 1/2 = 3/16$.

He wins £150 when both his and A's are fully correct: $1/8 \cdot 1/8 = 1/64$; Or when both he and A have two correct: $3/8 \cdot 3/8 = 9/64$. Marvin cannot win with one or less correct, because then Ford has two or more correct.

With this strategy, Marvin's expectation is therefore

$$\pounds\left(300\left(\frac{7}{64}+\frac{3}{16}\right)+150\left(\frac{1}{64}+\frac{9}{64}\right)\right) \;=\; \pounds 112.50.$$

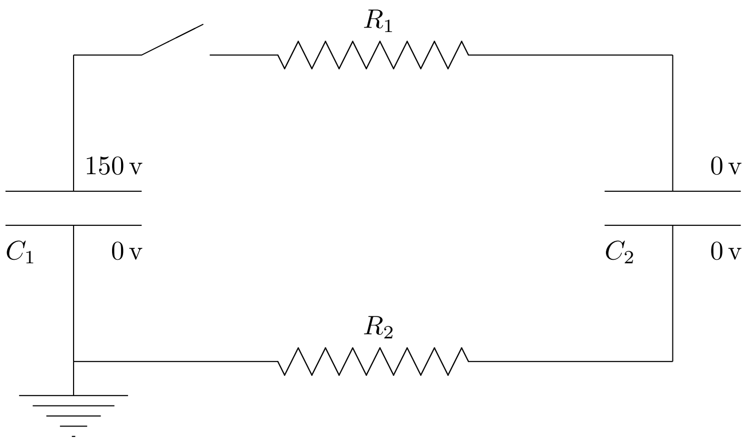
So it is certainly worth paying £1 for the increased net expectation of £111.50. But wait; crafty Ford now has, by symmetry, the net expectation of £113.50. So Marvin is in the driving seat. Instead of paying, he should be receiving. So he can make a counter-offer. But how much should Ford be prepared to pay to avoid being double-crossed? I haven't worked that one out yet. If Ford has learned his probability in the 'Restaurant at the end of the Universe', he will strike a hard bargain.

May I say I have thoroughly enjoyed the M500 magazine, and wished I had subscribed to it when I started with OU.

Problem 243.7 – Circuit

Tony Forbes

Behold a simple circuit containing two capacitors and two resistors. (Typical values might be something like $C_1 = 0.7\,\mu$F, $C_2 = 0.3\,\mu$F and $R_1 = R_2 = 5\,\text{M}\Omega$.) The diagram represents the initial state, with 150 volts across C_1. What happens when the switch is closed? In particular, what are the voltages on each side of C_2 as functions of time?



Problem 243.8 – Pentadecagon

Tony Forbes

Devise a nice ruler-and-compasses construction for the regular 15-gon.

Letter

Tetrahedron

Tony,

I returned from holiday to find M500 **240** and the solution to Problem 234.4 – Tetrahedron. [For the statement of the problem, see Dick Boardman's solution on page 10 of this issue.]

I echo your editorial remark about the difficulty of picturing the second case.

I did build a model, but it occurred to me after sending in the solution that the easiest way to see what is going on is to focus on the limiting 2-dimensional polygon. This is readily drawn. It is a trapezium with three equal sides $AB = BC = CD$, say, and the fourth side is equal in length to each of the two diagonals. This figure has a 2-fold rotation axis joining the mid points of BC and AD and reflection planes containing this axis in the plane and normal to the plane of the trapezium.

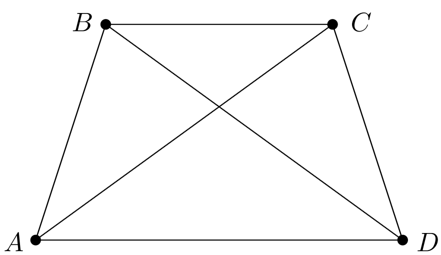
The tetrahedron is formed by twisting AD out of the plane of the paper (one up and one down!) so that the length pattern is conserved. The symmetry planes are lost but the two-fold axis remains and the centre of the circumsphere is to be found on it.

The golden mean result becomes clearer when it is recognized that the trapezium (with its diagonals) is part of a regular pentagon with its inscribed star. In fact the illustration on page 15 could have been adapted for the tetrahedron.

I must also add that I intended to delete the second expression for d on page 10, as it introduces a bogus singularity at $a = 1$, but it somehow got left in.

Best wishes,

Stuart Walmsley



Voting in M500

Eddie Kent

Judith is shortly to request nominations for new Committee members of The M500 Society. It might be of interest for members to have the method we use in elections spelled out.

Single transferable vote

There are many different ways of conducting an election. We follow the Royal Statistical Society in the version we use. It is known as Meek's method and has the approval of the Electoral Reform Society. It is used by the London Mathematical Society and other organizations. From an elector's point of view the system is simple: numbers have to be placed against candidates' names on the ballot paper to indicate the voter's order of preference (equal rankings are allowed, and not all candidates have to be given a rank).

Two basic principles govern the counting. First, if a candidate needs v votes to be elected but actually has $n > v$, then a fraction $(n-v)/n$ of each of these votes is passed on to candidates ranked lower on the ballot paper. The fact that some votes could thus become fractional causes no problem. Secondly if, after the above procedure has been iterated as far as possible, there are still vacant seats, all the votes of the candidate with the lowest total vote are redistributed in the same way. (If two candidates are tied at the lowest vote, then one is chosen at random.) A precise description of the procedure follows.

1. Each candidate, at any stage of the election, is either *elected*, *hopeful* or *excluded*. Initially everyone is hopeful.

2. At each stage of the count, each candidate x has an associated weight w_x. At this stage the candidate keeps a proportion w_x of any vote or fraction of a vote received, and the remaining proportion $(1-w_x)$ is passed on to another candidate (or in equal shares to a group of candidates if these have equal rankings). Excluded candidates have weight 0, so keep nothing. Hopeful candidates have weight 1 and keep everything which is passed to them. Elected candidates have weights between 0 and 1 determined as in §4.

3. If on a ballot paper candidates a, b, c, etc. are ranked with a first, b second, c third and so on, then, at any stage, a receives from that elector w_a of the vote, b receives $(1-w_a)w_b$ of the vote, c receives $(1-w_a)(1-w_b)w_c$ of the vote, and so on. Notice that if any candidate listed is hopeful, all fractions transferred to later candidates are 0. If any part of the vote remains

to be passed on after the whole list has been dealt with (which could happen easily if the ballot paper ranks only one candidate), that part is counted as *excess*. Initially there is no excess.

4. The *quota*—the vote a candidate must exceed at any stage in order to be elected—is defined to be $(total\ votes - total\ excess)/(number\ of\ seats + 1)$. The *weights* for elected candidates at each stage are determined (uniquely) by the requirement that the vote which remains with each of them is equal to the current quota; these weights are calculated by an iterative procedure.

5. At each stage the quota and weights are calculated according to §4, and then the procedures of §§2,3 are applied. Any candidate with more than the current quota of votes is declared elected and retains this status thereafter. If this means that at least one hopeful candidate changes to an elected candidate, the procedure is repeated.

6. If no hopeful candidate was elected in §5, the hopeful candidate with the lowest total vote at this stage (or one such chosen at random if there are many) is declared excluded, and the procedure is repeated with that candidate's weight changed to 0.

7. When the total number of elected candidates is equal to the number of seats the process stops. (Adapted from the paper 'Single transferable vote by Meek's method' by I. D. Hill, B. A. Wichmann and D. R. Woodall (*Computer J.* **30** (1987), 277–281), where more details can be found.)

How many mathematicians does it take to change a light-bulb?

1. Zero. It's a problem for engineers.
2. $O(1)$.
3. 1.000000000000000000005 approximately.
4. Three. One to hold the ladder and one to climb up the ladder to change the bulb.
5. Impossible. There is no ruler-and-compasses construction.
6. Exact value not known, but estimated to be less than $10^{10^{10^{963}}}$.

Any more? *Warning. Do not try this at home.* Changing light-bulbs is dangerous work that is best left to qualified electricians.

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