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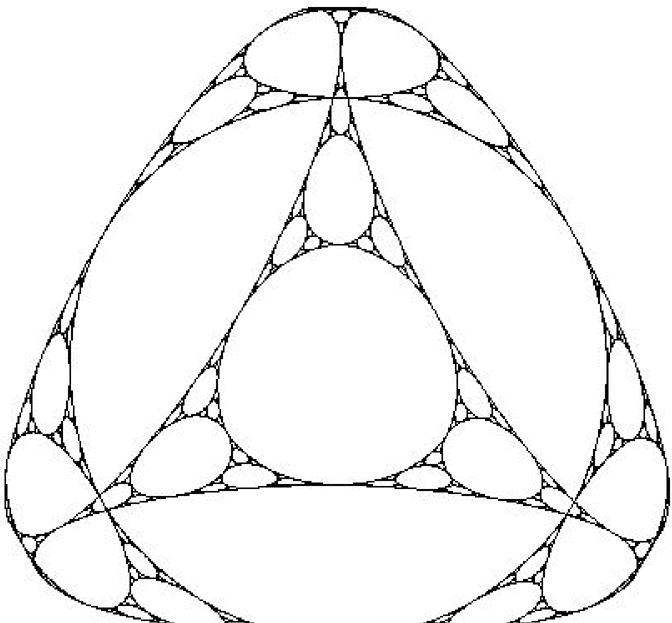
M500 246



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The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

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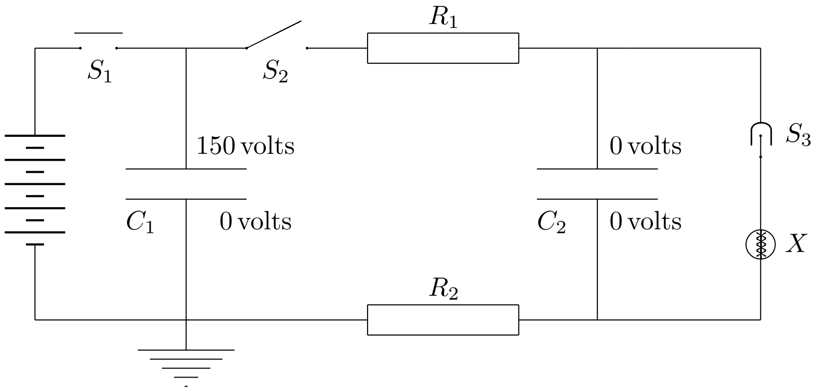
Solution 243.7 – Circuit

Behold a simple circuit containing two capacitors and two resistors. (Typical values might be something like $C_1 = 0.7\,\mu$F, $C_2 = 0.3\,\mu$F and $R_1 = R_2 = 5\,\mathrm{M}\Omega$.) The diagram represents the initial state, with 150 volts across C_1. What happens when switch S_2 is closed? In particular, what are the voltages on each side of C_2 as functions of time?



Mike Lewis

The circuit (redrawn using the ISO symbol for the resistors [and with some extra bits added on the left and right—TF]) is as above. The capacitor C_1 initially holds a charge of V volts at the moment that the switch S_2 is closed. The problem is to find the functions of time that describe the voltages on the two plates of C_2.

Solution as a differential equation

Some fundamental definitions. Capacity: $C = Q/V$, $C =$ capacity in farads, $Q =$ charge in coulombs, $V =$ voltage across the capacitor plates in volts; resistance: $R = V/I$, $R =$ resistance in ohms, $I =$ current in amperes; current: $I = dQ/dt$; that is, current is the rate of flow of charge around the circuit.

Using lower case letters to denote functions of time, at the moment of switch closure, t_0, the initial current flow, i_0, is governed by the resistors, since capacitor C_2 is initially uncharged, and is

$$i_0 = \frac{V}{R_1 + R_2}.$$

From the above relationships, the voltage across a capacitor, in terms of the current flow into or out of it is

$$v(t) \;=\; \frac{q(t)}{C} \;=\; \frac{1}{C}\int_0^t i(\tau)\,d\tau.$$

Using Kirchoff's Voltage Law that the sum of the directed EMFs around a loop is zero:

$$0 \;=\; v(t) + i(t)R_1 + \frac{1}{C_2}\int_0^t i(\tau)\,d\tau + i(t)R_2.$$

Substituting for $v(t)$ to give an equation in terms of current and charge:

$$0 \;=\; \frac{1}{C_1}\int_0^t i(\tau)\,d\tau + i(t)R_1 + \frac{1}{C_2}\int_0^t i(\tau)\,d\tau + i(t)R_2.$$

Differentiating to give a differential equation in terms of current:

$$0 \;=\; \frac{1}{C_1}i(t) + R_1\frac{d}{dt}i(t) + \frac{1}{C_2}i(t) + R_2\frac{d}{dt}i(t).$$

Rearranging and grouping terms:

$$0 \;=\; (R_1 + R_2)\,\frac{C_1C_2}{C_1 + C_2}\,\frac{d}{dt}i(t) + i(t).$$

This is a first order homogeneous differential equation the solution of which is $i(t) = Ae^{\lambda t}$, where A is a constant and, by inspection,

$$\lambda \;=\; -\,\frac{C_1 + C_2}{(R_1 + R_2)C_1C_2}.$$

We express the current as $i(t) = Ae^{-t/T}$, where $T = -1/\lambda$, which would commonly be referred to as the 'time constant' of the RC circuit.

The solution has to satisfy the initial conditions, and from earlier work, $i_0 = i(0) = V/(R_1 + R_2)$, from which it follows that

$$i(t) \;=\; \frac{V}{R_1 + R_2}\,e^{-t/T}, \qquad T \;=\; \frac{(R_1 + R_2)C_1C_2}{C_1 + C_2}.$$

To complete the solution to the problem as posed, the voltage on the lower plate of capacitor C_2 will be

$$v_L(t) \;=\; \frac{VR_2}{R_1 + R_2}\,e^{-t/T}.$$

For the upper plate, the voltage will be

$$\begin{aligned} v_U(t) &= v_L(t) + \frac{1}{C_2}\int_0^t i(\tau)\,d\tau \;=\; \frac{V R_2\, e^{-t/T}}{R_1+R_2} + \frac{1}{C_2}\int_0^t \frac{V\, e^{-\tau/T}}{R_1+R_2}\,d\tau \\ &= \frac{V}{R_1+R_2}\left(\left(R_2 - \frac{T}{C_2}\right)e^{-t/T} + \frac{T}{C_2}\right). \end{aligned}$$

Note that at $t = 0$ the voltage on both plates will be equal since there will be no charge on C_2 and is

$$v_U(0) \;=\; v_L(0) \;=\; \frac{R_2}{R_1+R_2}\,V.$$

In a similar way, as $t \to \infty$,

$$v_L(\infty) \;=\; 0, \qquad v_U(\infty) \;=\; \frac{V}{R_1+R_2}\frac{T}{C_2} \;=\; \frac{C_1}{C_1+C_2}\,V.$$

Since the current flow approaches zero as $t \to \infty$, and thus the voltage drops across the resistors approach zero, the voltage across C_1 approaches that across C_2.

Solution by means of the Laplace Transform

In electrical, electronic and control engineering, where linear differential equations are common the process can be speeded up considerably. The Laplace Transform was applied by Hendrik Bode to problems in electronics, in particular the problems associated with failures in amplifier repeaters in long distance telephone circuits. His book *Network Analysis and Feedback Amplifier Design* (van Nostrand, Princeton, New Jersey, 1945) contains the series of 'after hours' lectures he gave whilst at Bell Telephone Labs.

The Laplace Transform is based on the fact that the exponential function is the eigenfunction of differentiation. This has been seen in the first part of this article and is illustrated by:

$$\frac{d}{dt}\,e^{\lambda t} \;=\; \lambda e^{\lambda t}.$$

The Laplace Transform of a function of time, $f(t)$, is defined as

$$\mathcal{L}\big(f(t)\big) \;=\; F(s) \;=\; \int_0^\infty f(t)e^{-st}\,dt.$$

In practice the transforms of many functions are 'well known' and can be found from tables. Thus it is not usual to perform the integration. Two important transforms are those of the derivative and integral of a function of time:

$$\mathcal{L}\left(\frac{d}{dt}\,f(t)\right) \;=\; sF(s) \quad \text{and} \quad \mathcal{L}\left(\int f(t)\,dt\right) \;=\; \frac{1}{s}\,F(s).$$

The earlier equation,

$$0 \;=\; \frac{1}{C_1}\int_0^t i(\tau)\,d\tau + i(t)R_1 + \frac{1}{C_2}\int_0^t i(\tau)\,d\tau + i(t)R_2,$$

can be rewritten in Laplace Transform form by treating the charged capacitor C_1 as an uncharged capacitor in series with a battery of voltage V, the combination of battery and capacitor is treated as one unit and not as two separable components:

$$V(s) \;=\; \frac{1}{sC_1} I(s) + R_1 I(s) + \frac{1}{sC_2}\,I(s) + R_2 I(s).$$

The Laplace Transform $V(s)$ encompasses the closing of the switch and is $V(s) = V/s$. The equation now becomes

$$\frac{V}{s} \;=\; \frac{1}{sC_1}\,I(s) + R_1 I(s) + \frac{1}{sC_2}\,I(s) + R_2 I(s).$$

Multiplying through by s, equivalent to differentiation, gives

$$V \;=\; \left(\frac{C_1 + C_2}{C_1 C_2} + s(R_1 + R_2)\right) I(s)$$

and in final form

$$I(s) \;=\; \frac{V}{\dfrac{C_1 + C_2}{C_1 C_2} + s(R_1 + R_2)}.$$

The final stage is to invert the transform to obtain $i(t)$. In Bode's book this done by means of integration around the Bromwich Contour. In engineering problems the aforementioned tables are used. The relevant standard form is

$$\frac{1}{s - a} \;\to\; e^{at}.$$

A little rearrangement gives

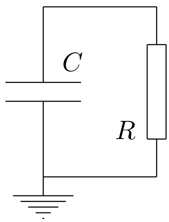
$$I(s) \;=\; \frac{V}{R_1 + R_2}\,\frac{1}{\dfrac{C_1 + C_2}{C_1 C_2 (R_1 + R_2)} + s}$$

and applying the relationship gives the previous result,

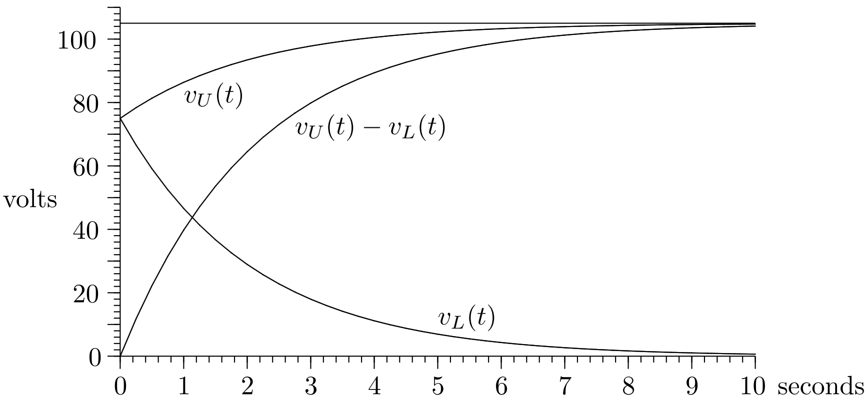
$$i(t) \;=\; \frac{V}{R_1+R_2}\,e^{-t/T}, \qquad T \;=\; \frac{(R_1+R_2)C_1C_2}{C_1+C_2},$$

where T is the previously found time constant. The various voltages can be found in a similar manner.

Solved in a similar manner (using differential equations) by **Edward Stansfield** — To deduce the current in the circuit, an equivalent circuit (right) is obtained by noting that the resistors and capacitors operate in series. The equivalent resistance is $R = R_1 + R_2$ and the equivalent capacitance is $C = C_1C_2/(C_1+C_2)$. Hence the time constant for the circuit is $T = CR = C_1C_2(R_1+R_2)/(C_1+C_2)$.



Tony Forbes — The values stated in the problem were used in typical Luftwaffe bomb fuze circuits during the Second World War. Initially in our somewhat simplified diagram switches S_1, S_2 and S_3 are open. When the bomb leaves the aircraft S_1 momentarily closes to charge the storage capacitor, C_1, with 150 volts, and then S_2 is (permanently) closed. As the seconds tick away while the bomb is falling the charge on C_1 leaks on to the firing capacitor, C_2. The bomb becomes armed when C_2 accumulates sufficient charge to operate the igniter, X. On impact the trembler switch, S_3, closes the firing circuit to initiate detonation by delivering the charge on C_2 to X. For dive bombing a shorter arming time is desirable and this is achieved with a higher voltage, 240 instead of 150.



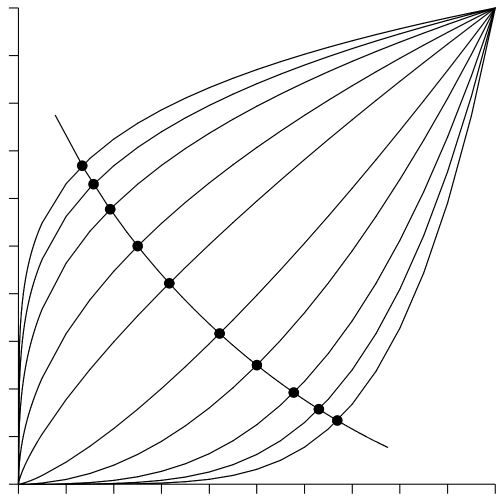
Solution 232.4 – Gradients

For each real number α in the range $(0, \infty)$, draw the graph of the function

$$x \mapsto x^{\alpha},$$

$0 \le x \le 1$, and mark the point where the gradient is 1.

What is the function traced out by these points (assuming that a suitable choice is made when $\alpha = 1$)?



Steve Moon

Let $y = x^{\alpha}$, $0 \le x \le 1$, $\alpha > 0$. Then

$$\frac{dy}{dx} = 1 \;\Rightarrow\; x = \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}, \; y = \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$$

to give the x and y coordinates of the points on the curves $y = x^{\alpha}$ where the gradient is 1. Now if $y = x^{\alpha}$, then $\alpha = \dfrac{\log y}{\log x}$ and we can eliminate α to yield

$$x \;=\; \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} \;=\; \left(\frac{\log x}{\log y}\right)^{1-\frac{\log y}{\log x}}.$$

Hence

$$x^{\frac{\log y}{\log x}-1} \;=\; \frac{\log x}{\log y}.$$

Recalling that $x^{\log y/\log x} = y$, this expression can be simplified to

$$y \log y \;=\; x \log x, \tag{1}$$

as defining the function for the points where the gradient of $y = x^{\alpha}$ is 1, $\alpha > 0$, $0 \le x \le 1$. This has two 'parts'.

One solution is $y = x$ for which $dy/dx = 1$ for all x, and $\alpha = 1$.

The other is the more interesting curve, which clearly intersects $y = x$ at some point to be determined. If we differentiate (1) implicitly, we obtain

$$\frac{dy}{dx}\log y + \frac{dy}{dx} = \log x + 1,$$

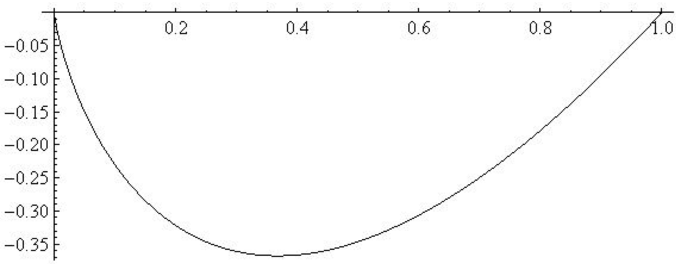
and thus

$$\frac{dy}{dx} = \frac{1+\log x}{1+\log y}. \tag{2}$$

Since $0 \le x, y \le 1$, $\log x \le 0$, $\log y \le 0$, and we could infer that as $x \to 0$, $dy/dx \to -\infty$ and as $y \to 0$, $dy/dx \to 0$. But this argument might lack rigour as it ignores the behaviour of the other variable, y or x. However, having the y and x axes as asymptotes seems consistent with the points $(0,1)$ and $(1,0)$ being the limits as $\alpha \to 0$ and $\alpha \to \infty$ respectively.

What happens when $y = x$ ($\alpha = 1$)?

If $y\log y = x\log x$, then $x\log x = c$ for $0 \le x \le 1$. If we sketch $c = x\log x$, we see that the curve has a minimum, and when $0 \ge c \ge c_{\min}$, there are two distinct solutions for $0 \le x \le 1$ except when $c = c_{\min}$.



If we call these β and γ, then $\beta\log\beta = \gamma\log\gamma$ and we see that the plot of $x\log x = y\log y$ passes through (β,γ) and (γ,β). The curve is symmetric on reflection in $y = x$.

At the minimum, $\beta = \gamma$ and a simple calculation shows that $x = y = 1/e$ and $c_{\min} = -1/e$. So the point on $y\log y = x\log x$ where $y = x$ is crossed by the curve is $(1/e, 1/e)$ and we can see why the expression (2) for dy/dx is unhelpful since it reduces to $0/0$ here.

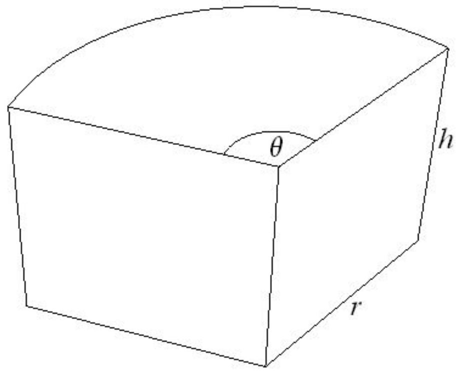
Problem 246.1 – Euler's function

Recall that Euler's function $\phi(n)$ is the number of integers k such that $1 \le k < n$ and $\gcd(k,n) = 1$. Show that $\phi(n) > 0.37\,n^{0.9}$.

Problem 246.2 – Cake

Tony Forbes

A slice of cake, a sector of a cylinder of radius r, height h and subtending an angle of θ, has total surface area 1. Determine r, h and θ to maximize its volume.



The diagram actually represents what I believe to be the optimum volume. However, because of the three-dimensional nature of the drawing and in contrast to the coffee cup problem (M500 **242**), I'm pretty sure that making accurate measurements with a ruler and a protractor would not provide an easier alternative to solving the problem by a proper mathematical analysis.

Problem 246.3 – Calculus assortment

Some traditional A-level calculus problems to try.

(i) Let S be the shape bounded by $\pm\cos x$, $x \in [-\pi/2, \pi/2]$. Let T be S rotated by 90 degrees about the origin. What is the area of $S \cup T$?

(ii) Let S be as in (i). Does the ellipse with axes $-\pi/2 \le x \le \pi/2$, $y = 0$ and $x = 0$, $-1 \le y \le 1$ enclose S?

(iii) Let $U = S \cup T$ with S and T as in (i). Let V be U rotated by 45 degrees about the origin. What is the area of $U \cup V$?

(iv) Find the distance of closest approach between the curves $y = e^x$ and $y = \log x$.

(v) Compute $\displaystyle\lim_{n\to\infty}\left(\sum_{k=2}^{n}\frac{1}{\log k} - \int_2^n \frac{dx}{\log x}\right)$.

Problem 246.4 – Stack sort

Prove that a sequence of distinct positive integers can be stack sorted if and only if it does not contain a subsequence $b * c * a$ with $a < b < c$ and where the asterisks indicate the possible presence of intervening numbers.

This came up in a lecture by Einar Steingrímsson at the Open University Winter Combinatorics Colloquium on 25 January this year. The problem was solved by Donald Knuth in 1968. In particular, $(2, 3, 1)$ is the only exception out of the six permutations of $(1, 2, 3)$.

To understand stack sorting one can imagine the sequence represented by a pile of numbered dinner plates, I, with the first number at the top. There are two further piles, initially empty: the output sequence, O, and a temporary storage facility called the stack. You transfer a plate from the top of the stack to O if its number is less than the number at the top of I, or if I is empty. If this is not possible, you transfer the plate at the top of I to the stack. You stop when both I and the stack are empty.

Here is an example with the sequence $(3, 2, 1, 4)$.

3								4
2	2		1			3	3	3
1	1	1 2	2	2	2	2	2	2
4 - -	4 3 -	4 3 -	4 3 -	4 3 1	4 3 1	4 - 1	- 4 1	- - 1

And you can try $(1, 8, 3, 2, 4, 5, 6, 7, 15, 9, 14, 12, 10, 11, 13)$ for a more substantial sequence where the $b * c * a$ pattern is absent. On the other hand, sorting $(2, 3, 1)$ in this manner doesn't work.

2						3
3	3	3		1	1	1
1 - -	1 2 -	1 - 2	1 3 2	- 3 2	- 3 2	- - 2

Student: "What's this backslash sign between A and B mean?"

Tutor: "It's the set difference; $A \setminus B$ is all elements of A not in B."

Student: "I understand. Thanks."

Tutor: "I expect you are familiar with the use of the minus sign for this operation, $A - B$, as used by some authors."

Student: "Yes ... of course."

Tutor: "Anything else?"

Student: "What's this cup sign between C and D?"

Two times five equals ten

Bryan Orman

It is clear that all three integers in the title are of the form n^2+1, where n is a non-zero integer, and that the product can be written as $(1^2+1)(2^2+1) = 3^2+1$. This multiplicative structure can be examined in general terms; given a positive integer x, is it possible to determine positive integers y and z such that $(x^2+k)(y^2+k) = z^2+k$, where k is a fixed non-zero integer?

The left-hand side can be rewritten to give

$$(xy+k)^2 + k(y-x)^2 \;=\; z^2+k$$

so that $xy+k = z$ and $y-x = 1$. Thus, given x, it follows that $y = x+1$ and $z = x^2+x+k$, and the multiplicative identity becomes

$$(x^2+k)\big((x+1)^2+k\big) \;=\; (x^2+x+k)^2+k.$$

This will generate iterations in a straightforward manner, as illustrated by the further development of $2\times 5 = 10$. Since $2\times 5 = 10$ is $(1^2+1)(2^2+1) = 3^2+1$ and

$$(3^2+1)(4^2+1) \;=\; (3^2+3+1)^2+1 \;=\; 13^2+1,$$

$10\times 17 = 170$. It follows that $2\times 5\times 17 = 170$, although this disguises the underlying structure given by $(1^2+1)(2^2+1)(4^2+1) = 13^2+1$. This process can be continued indefinitely, as will be shown later.

There is an alternative method of generating the product. Suppose that the representation $P_n = N_1N_2N_3\ldots N_n$ is known, where $P_n = z_n^2+k$ and $N_n = x_n^2+k$, with $x_{n+1} = z_n+1$ and $z_{n+1} = z_n^2+z_n+k$. The next product P_{n+1} is given by $P_{n+1} = P_nN_{n+1}$ and if P_n is known in its product form then only N_{n+1} is required to determine P_{n+1}. The following calculations should be self evident:

$$\begin{aligned} N_{n+1} &= x_{n+1}^2+k \;=\; (z_n+1)^2+k \;=\; z_n^2+2z_n+k+1 \\ &= (z_n^2+k)+2z_n+1 \;=\; P_n+2z_n+1. \end{aligned}$$

Now

$$\begin{aligned} P_{n-1}+N_n &= (z_{n-1}^2+k)+(x_n^2+k) \;=\; z_{n-1}^2+k+(z_{n-1}+1)^2+k \\ &= 2(z_{n-1}^2+z_{n-1}+k)+1 \;=\; 2z_n+1 \end{aligned}$$

and the working formula for N_{n+1} is then $N_{n+1} = P_n+P_{n-1}+N_n$.

In the example, $N_1 = 2$, $P_1 = 2$, $N_2 = 5$ and $P_2 = 10$. Applying the formula with $n = 2$ gives $N_3 = P_2 + P_1 + N_2$ so that $N_3 = 10 + 2 + 5 = 17$, and therefore $P_3 = P_2N_3 = 10 \times 17 = 170$, as before. Continuing,

$$N_4 \;=\; P_3 + P_2 + N_3 \;=\; 170 + 10 + 17 \;=\; 197$$

and so

$$P_4 \;=\; P_3N_4 \;=\; 170 \times 197 \;=\; 33490;$$

$$N_5 \;=\; P_4 + P_3 + N_4 \;=\; 33490 + 170 + 197 \;=\; 33857$$

and so

$$P_5 \;=\; P_4N_5 \;=\; 33490 \times 33857 \;=\; 1133870930.$$

Putting these together gives the factorization

$$2 \times 5 \times 17 \times 197 \times 33857 \;=\; 1133870930.$$

All these numbers are of the form $n^2 + 1$; explicitly,

$$(1^2 + 1)(2^2 + 1)(4^2 + 1)(14^2 + 1)(184^2 + 1) \;=\; 33673^2 + 1,$$

as are the intermediate Ps, which can be checked.

This iterative procedure has bypassed the original multiplicative identity and it is useful to recalculate the products from the repeated use of this identity. To this end it is convenient to introduce the notation,

$$\{x\} \;=\; x^2 + k,$$

so that the identity becomes

$$\{x\}\{x + 1\} \;=\; \{x^2 + x + k\}.$$

With $k = 1$, so that $\{x\} = x^2 + 1$, and taking $x = 1$, the identity gives $\{1\}\{2\} = \{1^2+1+1\} = \{3\}$, which is just $2 \times 5 = 10$. The next choice for x is 3, and it gives $\{3\}\{4\} = \{3^2 + 3 + 1\} = \{13\}$. Multiplying $\{1\}\{2\} = \{3\}$ on the right by $\{4\}$ gives $\{1\}\{2\}\{4\} = \{3\}\{4\} = \{13\}$.

The next choice for x is 13 and $\{13\}\{14\} = \{13^2 + 13 + 1\} = \{183\}$. Continuing the right multiplication, $\{1\}\{2\}\{4\}\{14\} = \{13\}\{14\} = \{183\}$. Now $\{183\}\{184\} = \{183^2 + 183 + 1\} = \{33673\}$; so, finally,

$$\{1\}\{2\}\{4\}\{14\}\{184\} \;=\; \{33673\},$$

which converts to the $2 \times 5 \times 17 \times 197 \times 33857 = 1133870930$, obtained previously.

Two observations can be made concerning this result. The first is to note that all the factors on the left hand side are prime numbers. Will the next iteration produce a further prime factor? That is, is $N_6 = \{33674\}$ a prime number? Now $N_6 = 33674^2 + 1 = 1133938277 = 373 \times 3040049$, and is therefore not a prime number.

The next thing to note is that the four numbers 17, 197, 33857 and 1133938277 all end with a 7, as will all the other factors if this product is continued. The reason for this is clear from the examination of the multiplicative identity, $\{x\}\{x+1\} = \{x^2+x+1\}$. If x ends in a 3 then $x+1$ ends in a 4 and x^2+x+1 ends in a 3. This is cyclic and all the above factors correspond to the $\{\ldots 4\}$ term, which is just $(\ldots 4)^2 + 1 = \ldots 7$.

Since it has been conjectured that there are an infinite number of primes of the form $n^2 - 2$, it might be fruitful to derive a product formula for these numbers (with $k = -2$), as it might generate more than just five primes in the product. The iteration formula in this case is $\{x\}\{x+1\} = \{x^2+x-2\}$ with $\{x\} = x^2 - 2$, and the first product is $2 \times 7 = 14$, that is, $\{2\}\{3\} = \{4\}$ and it generates the following sets:

$$\begin{aligned}
\{4\}\{5\} &= \{18\},\\
\{18\}\{19\} &= \{340\},\\
\{340\}\{341\} &= \{115938\},\\
\{115938\}\{115939\} &= \{13441735780\}.
\end{aligned}$$

Combining the above produces

$$\{2\}\{3\}\{5\}\{19\}\{341\}\{115939\} = \{13441735780\}$$

and the first five factors on the left hand side are

$$\{2\} = 2,\ \{3\} = 7,\ \{5\} = 23,\ \{19\} = 359,\ \{341\} = 116279.$$

All these are primes but the remaining factor is not, since

$$\{115939\} = 115939^2 - 2 = 13441851719 = 23831 \times 564049.$$

The two examples examined here produced a product of five primes, namely, $2 \times 5 \times 17 \times 197 \times 33857$ for $k = 1$ and $2 \times 7 \times 23 \times 359 \times 116279$ for $k = -2$. The number of primes depends on both the value of k and the starting prime, which was taken to be the smallest prime allowed, that is, 2. All other starting primes are odd but, if $\{x\}$ is odd then $\{x+1\}$ is even, so no further prime factors would then occur. Furthermore, all the many other values of k examined, but not recorded here, did not produce as many as five prime factors, so a fortuitous choice of the two k values for the illustrative examples!

This is where this particular investigation ends since it is known that there is no non-constant polynomial $f(n)$ with integral coefficients which takes on just prime values for integral n. The iterative process considered would produce polynomials of increasing degrees, with the ones of lower degree generating products of primes, but all too soon the iteration would fail.

And now a novelty. Suppose that the three non negative integers in the multiplicative identity are Pythagorean triples, that is,

$$(a^2 - k)(b^2 - k) \;=\; c^2 - k$$

with $a^2 + b^2 = c^2$. This leads to $b = a + 1$ and $c = a^2 + a - k$, and so $k = a^2 + a - c$. The first triple is $(3, 4, 5)$ and this requires k to be 7, giving $(3^2 - 7)(4^2 - 7) = 5^2 - 7$. Other triples can be generated from the result that if $(x, x+1, z)$ is a Pythagorean triple then so is $(3x+2z+1, 3x+2z+2, 4x+3z+2)$. The next three results concerning these Pythagorean triples are

$$\begin{aligned}
(20^2 - 391)(21^2 - 391) &= 29^2 - 391,\\
(119^2 - 14111)(120^2 - 14111) &= 169^2 - 14111,\\
(696^2 - 484127)(697^2 - 484127) &= 985^2 - 484127.
\end{aligned}$$

Further results would involve quite large numbers!

M500 Society Committee – call for applications

The M500 Society invites applications from M500 members for posts as Officers of the Society, that is, Secretary, Membership Secretary, Treasurer, Publisher, Week-end Organizer and two others who form the Editorial Board. The Officers who currently hold these posts are willing to stand again for another year. Anyone interested in an Officer's post should apply to the Secretary by 1st October. It should be noted that we are particularly seeking someone to manage Publicity. Applications should include name of applicant, address, telephone no., e-mail address, applicant's statement of aims and qualifications and signature.

Applications should be sent to the Secretary.

Letters

Coins, ships and docks

Hello Tony,

My copy of the mag came this morning and I was pleased to see the three coins article.

Problem 243.6 [There are 10 piles of 10 coins each. Nine piles are good and one pile is counterfeit. Good coins weigh 10 and dud coins weigh 9. You have a kitchen scale which tells you the weight in the pan, and you need to identify the dud pile in as few weighings as possible. How many weighings?] is very easy. You put the piles in a line. Take one coin from the first pile, 2 from the second, k from the kth up to $k = 10$. You put them all on the scale. Then you only have to subtract the weight from 550 to find the number of the pile.

So only one weighing is needed.

My solution for SHIP to DOCK: SHIP, SLIP, SLAP, SLAY, SPAY (neuter), SPRY, SARY (a fruit native to Cambodia), SARK (a shirt or chemise), SACK, SOCK, DOCK.

The only one not in *Chambers Concise* is SARY, but *Wikipedia* gives this definition.

Regards,

Vincent Lynch

Dear Tony,

This problem [Problem 243.6 – Piles of coins] can be solved with one weighing. Take one coin from pile 1, [. . . see above]. I suspect that this was adapted from an old Arabic problem known as the riddle of the gold bars.

Looking through old copies of M500 I got the impression we had more pages and variety in the past. When contributions are low could you consider printing some of the old material again for the benefit of new members. After you have thought about that for a few minutes try the following tongue-twisters.

A bloke's back brake-block broke.

The sixth sick sheik's sixth sheep's sick.

Barbara Lee

Dear Tony,

SHIP, SHIN, THIN, THEN, THEY, TREY, TROY, TROD, PROD, PROS, PRYS, PAYS, RAYS, RATS, RATE, RACE, RACK, ROCK, DOCK. These are all words in my edition of *Chambers Dictionary*.

Best wishes,

Francis McDonnell

Dear Eddie,

Just thought of a solution to the thing at the foot of page 9. Y is not a vowel. 'Skys' is a verb, as in 'to sky a ball'. SHIP, SKIP, SKIS, SKYS, SAYS, SACS, SACK, SOCK, DOCK.

Ralph Hancock

Now try again assuming Y is a vowel, or prove that it can't be done. Don't forget that words with two consecutive vowels are forbidden. — TF

Minimal Sudoku

Hi Tony,

A couple of years ago, I seem to remember you published a question about what is the minimum population of a Sudoku puzzle that it's solvable without repeats. I saw this recently:

http://www.nature.com/news/mathematician-claims-breakthrough-in-sudoku-puzzle-1.9751

[in which Gary McGuire of University College, Dublin offers a proof that 17 starter digits are necessary to create a valid Sudoku puzzle] and was wondering whether you received an answer back then and if so what was it?

Best regards,

Martin Orman

The issue in question was M500 **206** in which we also asked readers for a puzzle with 9 empty regions (rows, columns, 3×3 boxes). That, too, has been solved (right, or look up 'Mathematics of Sudoku' in *Wiki*). With McGuire's result it is easy to see that 11 empty regions cannot be achieved. So we now ask: *Is there a Sudoku puzzle with 10 empty regions?* — TF

			1		2	3	4	
			3		4	8	5	
	5	8				1	3	
	6	1				4	2	
	7	5	2		6			
	8	6	9		1			

Solution 184.4 – Three real numbers

Find three real numbers, a, b, c such that

$$a+b+c \;=\; ab \;=\; \frac{70+26\sqrt{13}}{27} \quad \text{and} \quad \frac{a}{b} \;=\; \frac{b}{c}.$$

A similar problem appeared in *IEE News*; the only difference was that they had $a+b+c = ab = 25$. However, I (TF) found it too difficult; so I 'simplified' the problem by changing 25 to $(70+26\sqrt{13})/27$. But that was a long time ago and we did actually publish a solution in issue **186**. Nevertheless, we now think the original *IEE News* problem is not without interest and therefore we are pleased to offer an exact solution here.

Steve Moon

We are given these criteria for a, b and c:

$$a+b+c \;=\; ab \;=\; 25, \qquad \frac{a}{b} \;=\; \frac{b}{c}.$$

But a, b and c are in geometric progression; so let $b = ar$ and $c = ar^2$. But then $a^2 r = 25$, suggesting that we put $r = k^2$. Hence the conditions reduce to

$$(k^2+1)^2 \;=\; 5k + k^2.$$

Now introduce a new variable z and add $2(k^2+1)z + z^2$ to each side:

$$(k^2+1+z)^2 \;=\; k^2(1+2z) + 5k + z^2 + 2z. \tag{1}$$

We need to determine z so that the right side of (1) is a perfect square. The right of (1) is a quadratic in k^2; so it must have a repeated root for it to be a square. So form the discriminant and set it equal to zero. Hence

$$25 - 4(1+2z)(z^2+2z) \;=\; 0 \qquad \Rightarrow \qquad 8z^3 + 20z^2 + 8z - 25 \;=\; 0.$$

We put this in 'depressed cubic' form with no quadratic term by the substitution $z = y - 5/6$:

$$y^3 - \frac{13}{12}y - \frac{605}{216} \;=\; 0. \tag{2}$$

Now for a depressed cubic in the form $y^3 + 3Hy + G = 0$, one root is real and two non-real if $G^2 + 4H^3 > 0$. But $(-605/216)^2 + 4(-13/12)^3 > 0$; so (2) has one real solution and we can find it as follows. We have

$$y^3 - \frac{13}{12}y \;=\; \frac{605}{216}.$$

Now find s and t such that

$$3st \;=\; -\frac{13}{12} \quad \text{and} \quad s^3 - t^3 \;=\; \frac{605}{216}. \tag{3}$$

The required solution is then given by $y = s - t$. From (3) we obtain $s = -13/(36t)$ and hence

$$46656t^6 + 130680t^3 + 2197 \;=\; 0.$$

Solving the quadratic in t^3 and simplifying gives

$$t^3 \;=\; -\frac{605}{432} \pm \frac{1}{144}\sqrt{39693}.$$

taking the positive sign (actually it doesn't matter which),

$$t \;=\; \sqrt[3]{-\frac{605}{432} + \frac{1}{144}\sqrt{39693}}, \qquad s \;=\; \sqrt[3]{\frac{605}{432} + \frac{1}{144}\sqrt{39693}}.$$

Let

$$u \;=\; \sqrt[3]{\frac{605 + 3\sqrt{39693}}{2}}, \qquad v \;=\; \sqrt[3]{\frac{605 - 3\sqrt{39693}}{2}}.$$

Then

$$z \;=\; s - t - \frac{5}{6} \;=\; \frac{u + v - 5}{6}$$

and we substitute this into (1):

$$\left(k^2 + 1 + \frac{u+v-5}{6}\right)^2 \;=\; k^2\left(\frac{u+v-2}{3}\right) + 5k + \left(\frac{u+v-5}{6}\right)\left(\frac{u+v+7}{6}\right)$$

$$= \left(\frac{k}{\sqrt{3}}\sqrt{u+v-2} + \frac{1}{6}\sqrt{(u+v+1)^2 - 36}\right)^2.$$

We can now take the square root of both sides, which are perfect squares:

$$k^2 + \frac{u+v+1}{6} \;=\; \frac{k}{\sqrt{3}}\sqrt{u+v-2} + \frac{1}{6}\sqrt{(u+v+1)^2 - 36}$$

and hence

$$k \;=\; \frac{1}{2\sqrt{3}}\left(\sqrt{u+v-2} \pm \sqrt{2\sqrt{(u+v+1)^2 - 36} - u - v - 4}\right),$$

giving $(a, b, c) \approx (3.51641, 7.10951, 14.3741)$ or $(23.9086, 1.04565, 0.0457316)$.

A card trick
Tommy Moorhouse

Sebastian Hayes's article about Russian peasant multiplication (M500 **243**) reminded me of an old 'magic' trick that I once found in a Christmas cracker. As with the Russian peasant multiplication, the binary expansion of an integer (that is, its expansion as a sum of powers of 2) is the key to the trick, as the reader may wish to investigate.

The person performing the trick (the 'magician') has a set of cards on which integers are written. The number in the top left corner is a power of 2 (that is, 1 on the first card, 2 on the second, 4 on the third, 2^{k-1} on the kth). We will call this power of 2 the 'value' of the card. The rest of the numbers appear as set out below for the example of a highest value card of 16. There is no limit to the number of cards or the number of integers on each card, except that all the cards must go up to the highest number appearing on the card of highest value (i.e. showing the highest power of 2 in the top left), and a card of each value (power of 2) from 1 to the highest must be present.

$\boxed{\begin{array}{l} 1\ 3\ 5\ 7\ 9\ 11\ 13\ 15 \\ 17\ 19\ 21\ 23\ 25\ 27\ 29\ 31 \end{array}}$

$\boxed{\begin{array}{l} 2\ 3\ 6\ 7\ 10\ 11\ 14\ 15 \\ 18\ 19\ 22\ 23\ 26\ 27\ 30\ 31 \end{array}}$

$\boxed{\begin{array}{l} 4\ 5\ 6\ 7\ 12\ 13\ 14\ 15 \\ 20\ 21\ 22\ 23\ 28\ 29\ 30\ 31 \end{array}}$

$\boxed{\begin{array}{l} 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15 \\ 24\ 25\ 26\ 27\ 28\ 29\ 30\ 31 \end{array}}$

$\boxed{\begin{array}{l} 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23 \\ 24\ 25\ 26\ 27\ 28\ 29\ 30\ 31 \end{array}}$

The magician invites the subject to silently choose a number between 1 and the largest number on the cards, and not to reveal what the chosen number is. The magician then shows the subject the cards and asks him to set aside all those cards showing the chosen number. Taking up the cards that have been set aside the magician tells the subject at once what his chosen number was. The astonished subject wonders how the magician could have checked through all the cards so quickly.

Of course, all the magician did was add up the powers of 2 in the top left of each card. To see how it works, first deduce the pattern of integers on each card and then relate this to the binary expansion of an integer appearing on the card. What is the largest number that should appear on the highest value card?

This trick can be generalized to any prime modulus, but the numbers on the card have to be marked ('coloured') in some way. For example, using the modulus 3 we have to mark some integers on each card, say with bold faced type as below. The interested reader will be able to build up the other cards without too much trouble.

1	**2**	4	**5**	7	**8**	10	**11**	13	**14**
16	**17**	19	**20**	22	**23**	25	**26**		

As before, the subject chooses a number in a given range and sets aside those cards showing this number into two piles. On one pile go the cards showing the number in pale type, on the other go those on which the number appears in bold. Taking up the selected cards the magician adds up the values of the 'pale' pile and adds this to twice the sum of the values of the 'bold' pile, announcing the mystery number. Clearly the poor magician has a tougher task, while the subject may have a longer wait for the result than is conducive to astonishment, and it seems little wonder that the former version is the only one I've ever seen in use!

Problem 246.5 – Binary sequences

Let S be a finite sequence of the symbols 0 and 1. Imagine generating a random sequence of 0s and 1s, chosen with equal probability (by tossing a coin, say), stopping as soon as S appears. Let $L(S)$ denote the expected length of such a sequence. It is actually possible to compute $L(S)$, and when one does so one obtains these results for various short sequences.

S	0	1	00	01	10	11	000	001	010	011	100	101	110	111
$L(S)$	2	2	6	4	4	6	14	8	10	8	8	10	8	14

S	0000	0001	0010	0011	0100	0101	0110	0111
$L(S)$	30	16	18	16	18	20	18	16

Like me (TF), you have probably noticed that the $L(S)$ presented above have a common property. You might even venture a guess as to how the values are calculated, and it would be very nice if you could prove it. But all we want you to do for the problem is this. Explain in a very simple manner why $L(S)$ must be an even integer.

Solution 224.4 – Integers

Given that $u = (p+1)(p+2n)/2$, where p and n are positive integers, show that values of p and n can be chosen to produce every positive integer except for those that take the form 2^{m-1} where m is a positive integer.

Steve Moon

(i) Consider first the generation of odd integers. If we set $p = 1$ then $u = 2n+1$ and we can put $n = (u-1)/2$, which is a positive integer provided that $u \geq 3$.

(ii) Now consider even number generation. Let $u = 2^a b$ with $a \geq 1$ and odd $b \geq 3$. If we let $p = 2^{a+1} - 1$, then p is a positive integer, $b = p + 2n$ and we can set $n = (b - 2^{a+1} + 1)/2$, which works provided that $b > 2^{a+1}$.

On the other hand, if $b < 2^{a+1}$, we set $p = b - 1$, which is positive since $b \geq 3$. Then $p + 2n = 2^{a+1}$ and hence $n = (2^{a+1} - b + 1)/2 > 0$.

(iii) Consider numbers of the form $u = 2^a$, $a \geq 0$, precisely those positive integers not covered by (i) and (ii). We require $(p+1)(p+2n) = 2^{a+1}$, which is easily seen to be impossible with positive integers p and n.

Solution 241.7 – Multiplicative function

Let f be an increasing, multiplicative function that maps positive integers to positive integers. Suppose also $f(2) = 2$. Show that f must be the identity function.

Dave Wild

This an alternative method to that published in M500 **243**. We will write $f(p)$ as p^*. We are told that $2^* = 2$ and, when $\gcd(m,n) = 1$, $(mn)^* = m^* n^*$. Also we can deduce that $(m+n)^* \geq m^* + n$, and $(4n+2)^* = 2^*(2n+1)^* = 2(2n+1)^*$. So

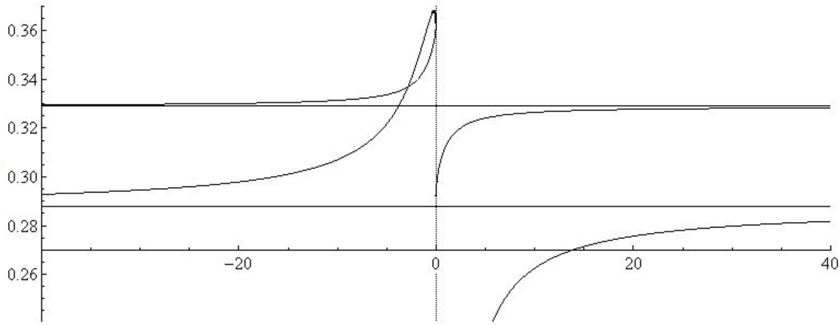
$$\begin{aligned} 5^* &= 10^*/2^* = 10^*/2 \geq (9^*+1)/2 = (18^*/2+1)/2 \\ &\geq ((15^*+3)/2+1)/2 = ((3^*5^*+3)/2+1)/2 = (3^*5^*+5)/4. \end{aligned}$$

If $3^* > 3$ then $3^* \geq 4$ and the above inequality becomes $5^* \geq 5^* + 5/4$. Therefore $3^* = 3$. If $n > 1$ and $n^* = n$ then $m^* = m$ for all integers $m \leq n$. So as $6^* = 2^* 3^* = 6$ then $5^* = 5$. Proceeding in a similar manner we see $9^* = 9$, $17^* = 17$, ... In general we have $(2n+1)^* = 2n+1$, for $n = 1, 2, 3, \ldots$. Therefore $f(n) = n$ for all n and so f is the identity function.

Problem 246.6 – Loop

Tony Forbes

The picture shows the curve $\big((\sin t)(\tan t), (\log t)/t\big)$ as t goes from 1.4 to 2π. What is the area enclosed by the little loop?



Thanks to Robin Whitty for the idea behind this problem.

M500 Mathematics Revision Weekend 2012

The thirty-eighth **M500 Society Mathematics Revision Weekend** will be held at

Aston University, **Birmingham**

over

Friday 14^{th} – **Sunday** 16^{th} **September 2012**.

The cost, including accommodation (with en suite facilities) and all meals from bed and breakfast Friday night to lunch Sunday is £265 (Aston Student Village) or £316 (Aston Business School), The cost for non-residents is £123 (includes Saturday and Sunday lunch). M500 members get a discount of £10. For full details and an application form, see the Society's web site at www.m500.org.uk, or send a stamped, addressed envelope to

Jeremy Humphries

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most mathematics-based OU courses, subject to sufficient numbers.

Contents

Front cover: A (truncated)^7 tetrahedron.