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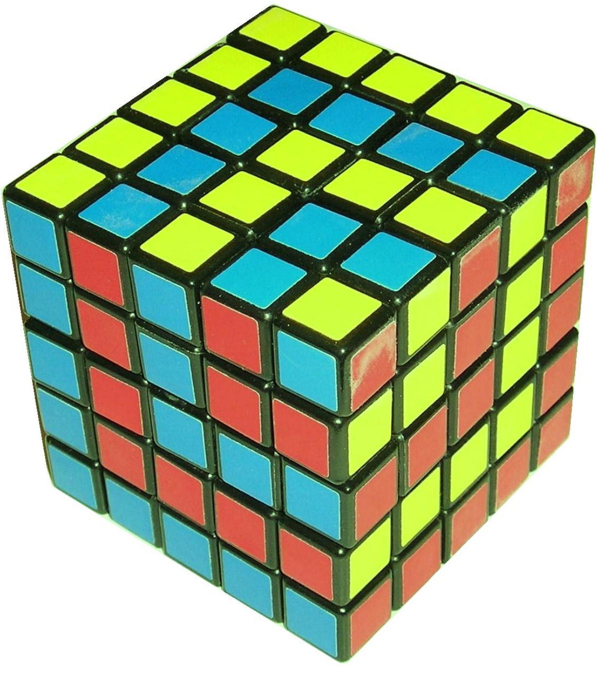
M500 248



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Integer partitions via differential equations

Tommy Moorhouse

Introduction As discussed in previous articles, the product of two integer-valued functions f and g, on the set $\{1, 2, 3, \dots\}$, given by

$$f * g(n) \;=\; \sum_{jk=n} f(j)g(k)$$

is commutative, associative and distributive over addition, very much like ordinary multiplication. We would like to deal with derivatives acting on such functions satisfying Leibniz's rule:

$$\nabla(f * g) \;=\; (\nabla f) * g + f * (\nabla g)\,.$$

The key motivation for this is that we can then deduce the solutions to differential equations by analogy. We find that the derivative given by

$$\nabla f(n) \;=\; L(n)f(n), \qquad \text{i.e. } \nabla f \;=\; L \cdot f,$$

where L (intended to bring to mind the word 'log') has the properties $L(n) > 0$ for $n > 1$ and $L(mn) = L(m) + L(n)$, is linear and satisfies Leibniz's rule. There are many integer-valued functions L satisfying this condition (see for example Section 5 of my article in M500 **218**), and here $f \cdot g$ is the function given by $f \cdot g(n) = f(n)g(n)$.

'Differential equations' can be defined in this context. The simplest such equation, introduced in M500 **236** and included here for convenience, is $\nabla f = z$ where $z(n) = 0$ for all n (in other words $\nabla f(n) = 0$ for all n). Since $L(1) = 0$ and $L(n) > 0$ for $n > 1$, we must have $f(n) = 0$ for $n > 0$ with $f(1)$ undetermined. Then f is just a multiple of I, the identity function $I(1) = 1$, $I(n) = 0$, $n > 1$. This is analogous to the constant functions in ordinary calculus having vanishing derivatives. Readers who investigated the functions $E[n]$ defined by $\nabla E[n] = nE[n]$ and intended to be analogous to e^{nx}, will have found that the desirable requirement $E[n]*E[m] = E[m+n]$ is difficult to enforce consistently (contrary to my assertion in M500 **236**). The reasons for this, and the ways around it, would take us too far afield, but we will continue to look to analogies for useful results.

The main equation In standard calculus the equation

$$\frac{df}{dx} \;=\; a(x)f(x)$$

may be solved to give

$$f(x) = A\exp\left(\int a(x)dx\right).$$

This can be deduced using only linearity and Leibniz's rule. Thus in complete analogy we can write down a solution to the equation

$$\nabla f \;=\; a * f,$$

namely

$$f \;=\; A\exp * \tilde{a}$$

if we can find a function $\tilde{a}$ such that $\nabla \tilde{a} = a$. The function $\tilde{a}$ is in some sense the analogue of the integral of $a(x)$. Here $\exp *$ is the formal sum

$$\exp * \tilde{a} \;=\; I + \tilde{a} + \frac{1}{2!}\tilde{a} * \tilde{a} + \cdots + \frac{1}{k!}\tilde{a} * \tilde{a} * \cdots * \tilde{a} + \cdots.$$

It is a simple check (using Leibniz's rule) to find that this is indeed a solution.

One key equation we want to examine is actually an identity (as we will see shortly): $L = u * K$ where L is our log function, K is a related function and $u(n) = 1$ for all $n > 0$. To make progress we need to be more specific about L. Given any integer expressed as a product of primes, say $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, and an integer-valued function $\mathbb{L}$, let $K(p^n) = \mathbb{L}(p)$ if p is prime and $n \geq 1$, $K(n) = 0$ otherwise. Let $L(n) = u * K(n) = \sum_{d|n} K(d) = \sum_{i=1}^{r} k_i \mathbb{L}(p_i)$. This defines the integer logarithm L which in turn defines our derivative operator ∇.

Now we observe that, since $u(n) = 1$ for all n, $L = L \cdot u = \nabla u$. The identity $L = L$ therefore becomes the differential equation $\nabla u = K * u$, with the solution $u = \exp * \tilde{K}$, but what is $\tilde{K}$? Considering the definition of the derivative it is easy to see that

$$\tilde{K} \;=\; \frac{K}{L} \quad \text{where} \quad \frac{K}{L}(n) \;=\; \frac{K(n)}{L(n)}$$

for $n > 1$ and $\tilde{K}(1) = 0$. Thus

$$u \;=\; \exp * \frac{K}{L} \;=\; I + \frac{K}{L} + \frac{1}{2!}\frac{K}{L} * \frac{K}{L} + \cdots.$$

There are a few things to note about this result, which the reader might like to explore further. First, it is independent of the choice of $\mathbb{L}$ used to define

K and L. It is not hard to arrive at the interesting result (see T. Apostol, *Introduction to Analytic Number Theory*, p. 239)

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{N^s} \cdot \exp * \frac{K}{L}(n) \;=\; \sum_{n=1}^{\infty} \exp * \frac{K}{N^s \cdot L}(n)$$

where ζ is Riemann's zeta function. Second, the expression for $\exp * K/L(n)$ has a finite number of terms for any n. Finally, the expression on the right is typically extremely complicated when written out in full for any given n, while the left hand (that is, $u(n)$) is of course extremely simple.

The payoff Given our previous remark one could reasonably ask what has been gained? To explain this we need one more result, a transform from functions on the positive (i.e. non-zero) integers to functions on the non-negative integers (including zero). Given L as above it may be that, given m, $L(n_i) = m$ for a set of integers n_i. We call this set $L^{-1}(m)$. We define the transform

$$f \to E_f : \; E_f(m) \;=\; \sum_{n \in L^{-1}(m)} f(n).$$

This transform has some very useful properties, which the reader may wish to check (or see my article in M500 **220**);

$$\begin{aligned} E_{f*g} &= E_f \circ E_g, \\ E_{\nabla f} &= N \cdot E_f \;\equiv\; \hat{\nabla} E_f \end{aligned}$$

where $N(n) = n$ for all $n \geq 0$ and $F \circ G(n) = \sum_{k+l=n} F(k)G(l)$.

Note that whichever L we use to define the derivative matched with the $*$ product, we always get the derivative $\hat{\nabla} F \equiv N \cdot F$ matched with the transformed $\circ$-product. It is simple to check that this derivative and product combination satisfies Leibniz's rule.

Now under this transform we find that E_u is the partition function associated with $\mathbb{L}$. For example, if the prime numbers are ordered $2 = p_1$, $3 = p_2$ and so on we can define $\mathbb{L}(p_i) = i$. In this case E_u is the unrestricted partition function and $E_u(n)$ is the number of ways of writing n as a sum of integers. If instead $\mathbb{L}(p) = p$ then $E_u(n)$ is the number of prime partitions of n.

The transformed differential equation is $\hat{\nabla} E_u = c \circ E_u$ where $c(n)$ is the sum of all the divisors of n of the form $\mathbb{L}(p)$ counted once. Thus if $\mathbb{L}(p) = p$

then $c(n)$ is the sum of the prime divisors of n. The solution (introducing the notation $\exp\circ$ and the function $\delta(0)=1, \delta(n)=0$ for $n>0$) is

$$E_u \;=\; \exp\circ\frac{c}{N} \;=\; \delta+\frac{c}{N}+\frac{1}{2!}\frac{c}{N}\circ\frac{c}{N}+\cdots.$$

Concluding comments If we define $Z(s)=\sum_{n=1}^{\infty} e^{-sL(n)}$, we find that we can also write $Z(s)=\sum_{m=0}^{\infty} E_u(m)e^{-sm}$. That is, $Z(s)$ is a generating function for the partition function associated with L. Substituting the result $E_u=\exp\circ\frac{c}{N}$ and summing we find that

$$Z(s) \;=\; \exp\left(-\sum_p \log\left(1-e^{-sL(p)}\right)\right)$$

and that

$$-\sum_p \log\left(1-e^{-sL(p)}\right) \;=\; \sum_{n=1}^{\infty}\frac{c}{N}(n)e^{-sn}.$$

This gives a common framework for exploring a wide range of partition types, and in particular the asymptotic expansions of E_u.

Problem 248.1 – Two theorems

What's wrong with the following?

Theorem 1 $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1.$

Proof Since $\sin 0=0$ we use l'Hôpital's rule:

$$\lim_{x\to 0}\frac{\sin x}{x} \;=\; \lim_{x\to 0}\frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} \;=\; \lim_{x\to 0}\frac{\cos x}{1} \;=\; 1.$$

Theorem 2 $\displaystyle\frac{d(\sin x)}{dx}=\cos x.$

Proof Using the definition of the derivative we have

$$\begin{aligned}\frac{d(\sin x)}{dx} &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{(\sin x)(\cos h)+(\cos x)(\sin h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{(\cos x)(\sin h)}{h} \;=\; (\cos x)\lim_{h\to 0}\frac{\sin h}{h} \;=\; \cos x.\end{aligned}$$

Polynacci sequences

Ron Potkin

The Fibonacci sequence begins with the numbers 0 and 1, and subsequent numbers are the sum of the preceding two. The first few numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, ... We usually express the sequence as $S_n = S_{n-1} + S_{n-2}$.

But we can elaborate on this by introducing the variables a and b and expressing it as $S_n = aS_{n-1} + bS_{n-2}$. So if $a = 1$ and $b = 1$ we obtain the Fibonacci sequence and by changing the values of a and b we can generate an infinite number of Fibonacci-like (polynacci) sequences. For example, if $a = 1$ and $b = 2$ then $S_n = S_{n-1} + 2S_{n-2}$ and we obtain the sequence 0, 1, 1, 3, 5, 11, 21, ...

We are not limited to adding two numbers together: we can go further and beginning with 0, 0, 1, add the last three numbers together or using 0, 0, 0, 1 add the last four and so on. This is referred to as the order of the sequence.

The expressions are related to polynomials. Thus $S_n = aS_{n-1} + bS_{n-2}$ is equivalent to $a + bx = x^2$ and S_n divided by S_{n-1} approaches one of its roots (provided of course that they are not complex!). There was an explanation of this in my article (M500 **200**) entitled 'Fibonacci and all that.'

Reciprocal of 89

It is well known that the reciprocal of 89 is the sum of the Fibonacci sequence where each number is moved one decimal point to the right. I examined this ...

0	0.0
1	0.01
1	0.001
2	0.0002
3	0.00003
5	0.000005
8	0.0000008
13	0.00000013
subtotal	0.01123593

... and sure enough even with just a few terms we find that the reciprocal of 0.01123593 is closing in on 89.

Order of 2

This raises the obvious question: does the same apply to any polynacci sequence? We can immediately see that if $a = 1$ and $b = 0$ then we obtain 0.011111, which is the reciprocal of 90. And if $a = b = 0$ we obtain 0.01, the reciprocal of 100.

Let's set $a = 4$ and $b = 7$, so that $S_n = 4S_{n-1} + 7S_{n-2}$.

0	0.0	23	0.0023	3404	0.0003404
1	0.01	120	0.00120	18103	0.00018103
4	0.004	641	0.000641	subtotal	0.01866243

The subtotal is approaching the recurring decimal 0.0188679245283 the reciprocal of 53 showing that the reciprocal of 89 is not unique; in fact, it would have been surprising if it had been.

Table A below shows some of the 100 integers that occur for $a = 0$ to 9 and $b = 0$ to 9. Every integer from 100 to 1 appears and is represented by $100 - 10a - b$.

Table A

	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$	$b=6$	$b=7$	$b=8$	$b=9$
$a=0$	100	99	98	97	96	95	94	93	92	91
$a=1$	90	89	88	87	86	85	84	83	82	81
etc.										
$a=8$	20	19	18	17	16	15	14	13	12	11
$a=9$	10	9	8	7	6	5	4	3	2	1

Order of 3

What happens if we introduce a third variable, c? Remember that this time we start the sequence with 0, 0, 1 so we can see that the decimal will be approximately one-tenth and its reciprocal will be ten times higher than the integers in Table A. So lets look at a sequence where $a = 1$, $b = 2$ and $c = 3$. This time the expression is $S_n = S_{n-1} + 2S_{n-2} + 3S_{n-3}$.

0	0.0	8	0.000008
0	0.00	17	0.0000017
1	0.001	42	0.00000042
1	0.0001	100	0.000000100
3	0.00003	subtotal	0.001140220

This is approaching 0.001140250855..., the reciprocal of 877.

This next table shows a further 100 integers; all slightly less than ten times Table A, as expected. The integer is given by $1000 - 100a - 10b - c$. This format is true if more variables are included.

Table B variable $c = 3$

	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$	$b=6$	$b=7$	$b=8$	$b=9$
$a=0$	997	987	977	967	957	947	937	927	917	907
$a=1$	897	887	877	867	857	847	837	827	817	807
etc.										
$a=8$	197	187	177	167	157	147	137	127	117	107
$a=9$	97	87	77	67	57	47	37	27	17	7

So, let's suppose that we need the reciprocal of, say, 2447. (You'd have to be pretty desperate to use this method!) The number lies between 10^3 and 10^4 so it will be derived from a 4th order sequence. Deducting 2447 from 10000 gives us its complement 7553 and we have the expression $S_n = 7S_{n-1} + 5S_{n-2} + 5S_{n-3} + 3S_{n-4}$.

Notice that the numbers go in pairs. So, for example, $S_n = 2S_{n-1} + 4S_{n-2} + 4S_{n-3} + 7S_{n-4}$ will result in the reciprocal of 7553. Every number has its complement: 11 and 89, 47 and 53, 123 and 877, and so on, but examine the reciprocal of 9911. The complement should be 89 but we must express it as 0089 so that $S_n = 0S_{n-1} + 0S_{n-2} + 8S_{n-3} + 9S_{n-4}$.

So far I have only shown examples with integers, but we must not forget that the sequences apply to all rational numbers. For example, if $a = 1$ and $b = 1.5$ we will obtain 88.5.

Finally, why do some sequences close in on their target much faster that others? There is a proof of 1/89 and a more general proof at the web site www.mathpages.com/home/kmath108.htm entitled 'Fibonacci, 1/89 and all that' but, somehow, in view of its simplicity, I feel that there must be a more elementary explanation.

Problem 248.2 – Necklaces

A *necklace* is a piece of string on to which red and green gemstones indistinguishable except for colour are threaded, together with a mechanism for temporarily bringing the ends together to form a closed loop. Let $N(n, r)$ denote the number of necklaces made from n gemstones of which r are red and $n - r$ are green. What is $N(n, r)$? Assume the usual symmetries, so that for example $N(5, 3) = 6$, the same as the number of isomers of TNT.

Solution 243.3 – Odd sequence

Alexander Sharkovsky defined an ordering on the positive integers by virtue of the fact that each may be uniquely specified in the form $2^r p$ where r is a non-negative integer and p is a positive odd number. Sharkovsky's famous theorem on limit cycles in iterated functions is based on this ordering, which is usually specified informally thus:

$$3, 5, 7, \ldots, 2{\cdot}3, 2{\cdot}5, 2{\cdot}7, \ldots, 2^2{\cdot}3, 2^2{\cdot}5, 2^2{\cdot}7, \ldots,\ \ldots, 2^3, 2^2, 2^1, 2^0.$$

Give a precise definition of this ordering.

Reinhardt Messerschmidt

Let $<$ denote the standard order on the positive integers, and let $\prec$ denote Sharkovsky's order. A precise definition of $\prec$ is: $2^{r_1}p_1 \prec 2^{r_2}p_2$ if and only if one of the following is true:

(i) $r_1 < r_2$ and $p_1 > 1$ and $p_2 > 1$;

(ii) $r_1 = r_2$ and $1 < p_1 < p_2$;

(iii) $r_1 > r_2$ and $p_1 = p_2 = 1$.

The order $\prec$ is not a well-ordering; i.e. there is at least one nonempty set of positive integers that does not have a least element, for example $\{2^r p \mid p = 1\}$. Neither does it have the least upper bound property, i.e. there is at least one nonempty set of positive integers that has an upper bound but does not have a least upper bound. For example, the set of all upper bounds of $\{2^r p \mid p > 1\}$ is $\{2^r p \mid p = 1\}$, which does not have a least element.

Solution 242.2 – Quintic

Show that the real root of the cubic $x^3 - x - 1$ is also a root of the quintic $x^5 - x^4 - 1$.

Tommy Moorhouse

Suppose α is a solution of $x^3 - x - 1 = 0$. Then $\alpha^3 = 1 + \alpha$. Now substitute α into $x^5 - x^4 - 1 = 0$. We find

$$\alpha^5 - \alpha^4 - 1 = \alpha^3(\alpha^2 - \alpha) - 1 = \alpha(1 + \alpha)(\alpha - 1) - 1 = \alpha^3 - \alpha - 1 = 0.$$

Further observations on 242.2 Quintic

Tommy Moorhouse

I was curious about how one arrived at the expression for the real solution of the cubic $x^3 - x - 1 = 0$ in Problem 242.2 and realized that this is an interesting example of trading a cubic equation (which we might not be able to solve easily) for two quadratic equations which, with luck, we can deal with. The method below can be adapted for other cubic equations.

Suppose α is a solution of $x^3 - x - 1 = 0$. Write $\alpha = \gamma + \zeta$, where γ and ζ are positive real numbers, and substitute into $x^3 - x - 1 = 0$. Expanding we have

$$\gamma^3 + 3\gamma^2\zeta + 3\gamma\zeta^2 + \zeta^3 - \gamma - \zeta - 1 = 0.$$

If we can choose $\gamma^3 + \zeta^3 = 1$ we can reduce the equation to a quadratic, in γ, say. A little trial confirms that we can satisfy this condition, and we have

$$3\zeta\gamma^2 + (3\zeta^2 - 1)\gamma - \zeta = 0.$$

The solutions, using the quadratic formula, are $\gamma = 1/3\zeta$ and $\gamma = -\zeta$. We reject the latter and substitute into $\gamma^3 + \zeta^3 = 1$ to find

$$\frac{1}{27\zeta^3} + \zeta^3 = 1.$$

This gives us a quadratic in ζ^3 which we solve by the usual formula to get (noting that ζ and γ can be interchanged without changing the conclusion)

$$\zeta^3 = \frac{1}{2} + \frac{1}{18}\sqrt{69}$$

and

$$\gamma^3 = \frac{1}{2} - \frac{1}{18}\sqrt{69}.$$

The complex roots can be expressed in terms of α. Let $w = \alpha + \epsilon$ be a complex solution of $x^3 - x - 1 = 0$. The conjugate of w is the other solution. Substitute into $x^3 - x - 1 = 0$ to find

$$\alpha^3 + 3\alpha^2\epsilon + 3\alpha\epsilon^2 + \epsilon^3 - \alpha - \epsilon - 1 = 0.$$

We can eliminate α using the original equation, and a factor of ϵ can be extracted from the remainder (because α is a solution of the cubic), giving

$$\epsilon^2 + 3\alpha\epsilon + (3\alpha^2 - 1) = 0.$$

Solving for ϵ and substituting back into w gives the two complex solutions in terms of the real solution.

Solution 244.2 – A quick number wonder

Show that

$$\begin{aligned}\sqrt{1\times 2\times 3\times 4+1} &= 2\times 3-1,\\ \sqrt{2\times 3\times 4\times 5+1} &= 3\times 4-1,\\ \sqrt{3\times 4\times 5\times 6+1} &= 4\times 5-1,\ \ldots.\end{aligned}$$

Stewart Robertson

The pattern will continue to hold. We can prove this by showing that

$$\sqrt{n(n+1)(n+2)(n+3)+1} \;=\; (n+1)(n+2)-1 \quad \text{for all } n\in\mathbb{N}.$$

We can do this directly with a little manipulation as follows:

$$\begin{aligned}\big((n+1)(n+2)-1\big)^2 &= \big((n+1)(n+2)\big)^2-2(n+1)(n+2)+1\\ &= (n+1)(n+2)\big((n+1)(n+2)-2\big)+1\\ &= (n+1)(n+2)(n^2+3n)+1\\ &= n(n+1)(n+2)(n+3)+1.\end{aligned}$$

Mike Lewis

Is this the extent of the answer? Nothing in the solution suggests that n must be a non-negative integer. For example let $n=1/2$:

$$\sqrt{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}+\frac{2}{2}} \;=\; \frac{3}{2}\cdot\frac{5}{2}-\frac{2}{2}$$

and get rid of the denominators: $\sqrt{1\cdot 3\cdot 5\cdot 7+16}=3\cdot 5-4$. By inspection a more general set of sequences can be produced for fractions:

$$\sqrt{\left(\frac{n}{m}-1\right)\frac{n}{m}\left(\frac{n}{m}+1\right)\left(\frac{n}{m}+2\right)+1} \;=\; \frac{n}{m}\left(\frac{n}{m}+1\right)-1.$$

And a general set of sequences for integers that follows from the above is

$$\sqrt{(n-m)n(n+m)(n+2m)+1} \;=\; n(n+m)-1.$$

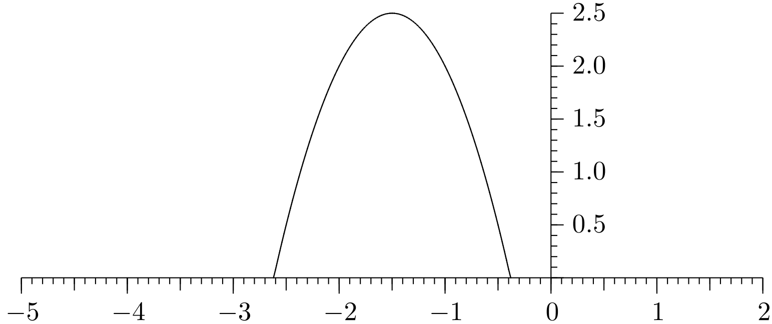
Can the last sequence be used for irrationals? The answer must be yes since nothing in the manipulation requires either n or m to be integers or rational fractions.

A popular problem! We also had contributions similar to that of Stewart Robertson from **Reinhardt Messerschmidt**, **Patrick Walker**, **Basil Thompson**, **Edward Stansfield**, and finally **Ken Greatrix**, who has a little more to say ....

Ken Greatrix

What about negative numbers? Let's see what happens when we plot

$$f(x) \;=\; \sqrt{x(x+1)(x+2)(x+3)+1} - \big((x+1)(x+2)-1\big).$$



What's going on? The problem is that the quadratic $(x+1)(x+2)-1$ is negative in the interval (α, β), where α and β are its roots:

$$\alpha \;=\; \frac{-3-\sqrt{5}}{2} \;\approx\; -2.61803, \qquad \beta \;=\; \frac{-3+\sqrt{5}}{2} \;\approx\; -0.381966.$$

So to get an equality between the two main terms of $f(x)$ we should take the negative square root. Thus

$$-\sqrt{x(x+1)(x+2)(x+3)+1} \;=\; (x+1)(x+2)-1, \qquad x \in (\alpha, \beta).$$

Outside (α, β) we have $f(x) = 0$, as expected.

Problem 248.3 – Integer triangles

Tony Forbes

A triangle has integer area and consecutive integer sides. Apart from $(3, 4, 5)$, is it the case that exactly one height must also be an integer?

Even Hypotenuse would have trouble working out these angles.

Sid Waddell (1940–2012), darts commentator

Solution 245.2 – Intersecting cylinders

> Determine the the volume of the intersection of the cylinders $x^2 + y^2 \leq 1$ and $x^2 + z^2 \leq 1$.

Richard Gould

I was not around M500 at the time of Problem 192 (the 3-cylinder problem) but it was one that my son brought home from boarding school one Easter, claiming that his physics teacher was fascinated by it and was saving it for his approaching retirement. I hadn't thought about the 2-cylinder problem until now but my approach is similar and avoids multiple integrals.

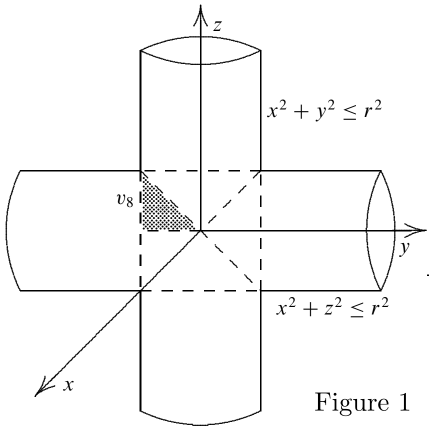


Figure 1

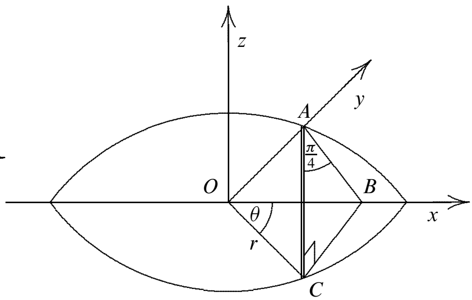


Figure 2

In Figure 1 the volume of interest is made up of eight copies of the shaded region v_8. This, together with the region immediately below it, is shaped rather like the segment of an orange but with the curved surface cylindrical rather than spherical. The region v_8 is bounded by the planes $z = 0$ and $y = z$ and the surface $x^2 + y^2 = r^2$, with $-r \leq x \leq r$, $z \geq 0$ and $y \leq 0$. This is shown in larger scale in Figure 2.

We see that triangle ABC is a right-angled isosceles triangle so that $AC = BC = r \sin\theta$ and the area of ABC is $\frac{1}{2}r^2 \sin^2\theta$. Since $OB = r\cos\theta$, the element of volume is a prism of thickness $\delta(r\cos\theta) = -r\sin\theta\delta\theta$. The minus sign arises because increasing θ corresponds to decreasing OB, but since we are integrating in this direction we can safely ignore it. The element of volume is thus given by $\delta v = \frac{1}{2}r^3 \sin^3\theta\,\delta\theta$ and

$$v_8 \;=\; \frac{r^3}{2}\int_0^{\pi} \sin^3\theta\,d\theta \;=\; \frac{r^3}{2}\int_0^{\pi} (1-\cos^2\theta)\sin\theta\,d\theta \;=\; \frac{2r^3}{3}.$$

So the volume of intersection of the two cylinders is $16r^3/3$.

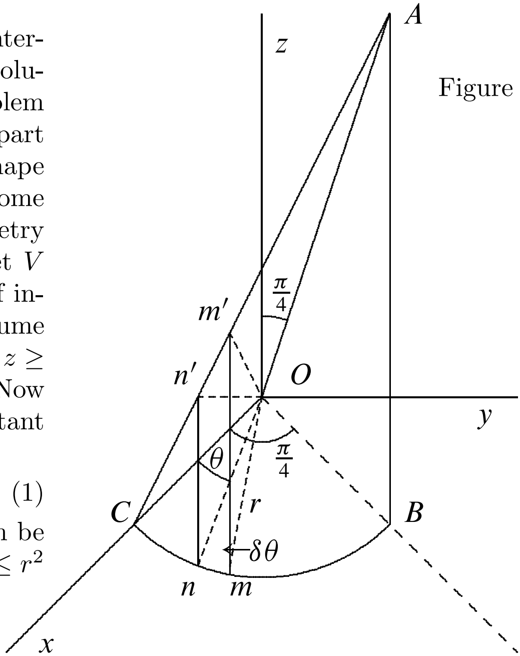


Figure 3

In case it may be of interest I will also include my solution to the 3-cylinder problem here. The most difficult part of this was envisaging the shape of the resulting solid, but some preliminary appeals to symmetry greatly simplify matters. Let V be the volume of the solid of interest, and let v_8 be the volume of the octant defined by $x,\ y,\ z \geq 0$. By symmetry, $V = 8v_8$. Now consider the part of this octant for which

$$x \geq y \geq z. \qquad (1)$$

Let the volume of this region be v_6. For this region, $x^2 + y^2 \leq r^2$ and inequality (1) ensure that $y^2 + z^2 \leq r^2$ and $z^2 + x^2 \leq r^2$; so this solid is bounded by the cylindrical surface $x^2 + y^2 = r^2$, and the planes $z = 0$, $x = y$, and $y = z$, as shown in Figure 3. Since there are six ways to set up inequality (1), we have, again by symmetry, $v_8 = 6v_6$, and $V = 48v_6$.

In Figure 3 (greatly exaggerated in the z-direction for clarity) CAB is part of the surface $x^2 + y^2 = r^2$, COA is part of the plane $y = z$ and BOA is part of the plane $x = y$. Consider the pyramidal element of volume δv defined by $Onmm'n'$. Since n' lies in the plane $y = z$, we have $nn'{=}r \sin\theta$. Thus

$$\delta v \;=\; \frac{1}{3} \times r\sin\theta \times r\delta\theta \times r \;=\; \frac{1}{3} r^3 \sin\theta\,\delta\theta.$$

Therefore

$$v_6 \;=\; \frac{r^3}{3}\int_0^{\frac{\pi}{4}} \sin\theta\,d\theta \;=\; \frac{r^3}{3}\Big[-\cos\theta\Big]_0^{\frac{\pi}{4}} \;=\; \frac{r^3}{3}\left(1 - \frac{\sqrt{2}}{2}\right).$$

So

$$V \;=\; 48v_6 \;=\; 16r^3\left(1 - \frac{\sqrt{2}}{2}\right).$$

I wonder if that physics master ever found out that the solution to his problem boiled down to just the integral of $\sin x$!

Solution 245.6 – Quintic

Solve $x^4 + x^5 = e^6$.

Vincent Lynch

What a superb problem for April. It is M500's tribute to Martin Gardner. I used to take *Scientific American* in the 70s, and when his six hoaxes were published in April 1975, I was taken in by them, though I certainly didn't believe the disproof of the four colour theorem. This could have been 'Vermont schoolboy [my grandson] discovers identity connecting π and e.'

I used Newton's method to solve the equation. Taking logs to base e, the equation to be solved is

$$f(x) \;=\; 4\ln(x) + \ln(1+x) - 6 \;=\; 0.$$

And

$$x_{n+1} \;=\; x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x) = \frac{4}{x} + \frac{1}{1+x}.$$

So into my calculator I put `3 ENTER` (the value of x_0). Then

`ANS - (4ln(ANS) + ln(1+ANS)-6)/(4/ANS + 1/(1+ANS)) ENTER`.

The output: 3.13847.... Pressing `ENTER` again: 3.14159.... Could it be π? Pressing `ENTER` again:

$$3.141592683\ldots.$$

Pressing `ENTER` again gave no change. Groan. It's not π; $\pi = 3.141592653\ldots$. So $\pi^4 + \pi^5$ is not equal to e^6 after all.

My son and grandson both came over from Vermont for my graduation ceremony in Harrogate. Here is a photo taken at the venue.

Problem 248.4 – Integral

Compute

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2+z^2)^{3/2}}\,dx\,dy\,dz.$$

Problem 248.5 – Handicapped coin tossing

Tony Forbes

Toss a coin 100 times, score +1 for a head and −1 for a tail and add them up. As is well known the final score has approximately normal distribution with mean 0 and standard deviation $\sqrt{100} = 10$.

Do it again but this time with a special 'variable probability' coin. We start as usual with Pr(head) = 1/2 for the first trial but thereafter the probability of getting a head is the proportion of *tails* appearing in the previous trials. Presumably some kind of handicapping process is operating. How is the distribution of the score affected?

To see how it works, let $p_n = \Pr(n^{\text{th}}$ result is a head). We start with $p_1 = 1/2$; and $p_2 = 0$ if the first result was a head, 1 if it was a tail. Then $p_3 = 1/2$ unconditionally but p_4 will be 1/3 if the third result was a head, 2/3 otherwise. Thereafter things get more complicated.

In an experiment I collected the scores from performing the 100 coin tossings 100000 times and they do in fact appear to be normally distributed. The standard deviation is somewhat smaller, 5.76 approximately, and it would be nice to see what the exact figure should be.

Solution 245.4 – GCSE question

$$\text{Compute} \quad \sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k-1}} \quad \text{and} \quad \sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k}-\sqrt{k-1}}.$$

Basil Thompson

Multiply top and bottom by $\sqrt{k}-\sqrt{k-1}$,

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k-1}} \;=\; \sum_{k=1}^{n} \frac{\sqrt{k}-\sqrt{k-1}}{k-(k-1)} \;=\; \sum_{k=1}^{n} (\sqrt{k}-\sqrt{k-1}) \;=\; \sqrt{n},$$

and for the other one, multiply top and bottom by $\sqrt{k}+\sqrt{k-1}$,

$$\sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k}-\sqrt{k-1}} \;=\; \sum_{k=1}^{n} \frac{(-1)^k(\sqrt{k}+\sqrt{k-1})}{1} \;=\; \begin{cases} \sqrt{n} & n \text{ even}, \\ -\sqrt{n} & n \text{ odd}. \end{cases}$$

Solution 245.5 – Numbers

> You and your opponent play a game. You start by choosing a positive integer X_0. Thereafter your opponent and you take turns to choose positive integers X_1, X_2, ..., such that X_0,X_1, X_2, ... are distinct and $X_{n+1} = X_n - 2$, $X_n - 1$ or $X_n + 1$. If not possible, the player whose turn it is loses. Assuming you both play perfectly, classify the starting numbers X_0 as either (i) you win, (ii) you lose, or (iii) draw (infinitely long game).

Reinhardt Messerschmidt

We will use the following two lemmas:

Lemma 1 *If, at any stage of the game, I play $2n$ for some positive integer n, and no smaller integer has been played, then my opponent wins.*

Proof We will use induction on n. The base case is clear: if I play 2 and 1 has not been played, then my opponent plays 1 and wins.

For the inductive case, suppose $m > 1$ is such that Lemma 1 holds for all $n < m$. We have to show that it holds for m, so suppose that at some stage of the game I play $2m$ and that no smaller integer has been played. My opponent then plays $2m - 1$.

If I respond with $2m - 2$, then my opponent wins by the inductive hypothesis.

If I respond with $2m - 3$, then my opponent plays $2m - 2$. If $m = 2$, then I have run out of plays. If $m > 2$, then I can only play $2m - 4$, and my opponent wins by the inductive hypothesis. □

Lemma 2 If X_0 is odd, then the game is a draw.

Proof If $X_0 = 1$, then the only admissible continuation of the game is $2, 3, 4, \ldots$.

Suppose $X_0 = 2m + 1$ for some positive integer m.

If $X_1 = 2m$, then I win by Lemma 1.

If $X_1 = 2m - 1$, then I respond with $2m$. If $m = 1$, then my opponent has run out of plays. If $m > 1$, then he can only play $2m - 2$, and I win by Lemma 1.

If $X_1 = 2m + 2$ and I respond with $2m$, then my opponent wins by Lemma 1. However, if I respond with $2m + 3$, then the only admissible continuation of the game is $2m + 4, 2m + 5, 2m + 6, \ldots$. This is the path that my opponent and I will follow. □

It follows from these two lemmas that I lose if X_0 is even, and the game is a draw if X_0 is odd.

Solution 245.1 – Birthday dinner

> Four people dine out four times a year, each time to celebrate the birthday of one of their number. At these events the birthday person's meal is free, the entire cost being met equitably by the other three members of the group. This year the third and fourth meals are combined into one. How should the bill be settled?

We had essentially similar answers from Tamsin Forbes, Carrie Rutherford and Basil Thompson. Suppose the three meals costs the same, £1, say. Let diners be X_1, X_2, Y_1 and Y_2, where X_i are the two that have already enjoyed their birthday dinners. Suppose X_i and Y_i are to pay £x each and £y each respectively for the third meal. Then $2x + 2y = 1$ and for overall fairness, $1/3 + x = 2/3 + y$. Hence $x = 5/12,\ y = 1/12$. I (TF) believe this is what actually happened.

But this is one of those problems where any answer can be wrong. In the above settlement Y_i are complaining that they didn't get free meals. One answer is to drop the assumption that the third meal should cost the same as the other two. If the diners go to a slightly cheaper restaurant and spend only $66\frac{2}{3}$p, the equations become $2x+2y = 2/3$ and $1/3+x = 2/3+y$ with solution $x = 1/3$, $y = 0$.

However, Y_i are still complaining because their free meals were inferior to those of X_i. So they go instead to an establishment that charges £2. The equations are now $2x + 2y = 2$ and $1/3 + x = 2/3 + y$ with solution $x = 2/3$, $y = 1/3$. Everyone is happy, which is hardly surprising because the situation is equivalent to two separate £1 meals.

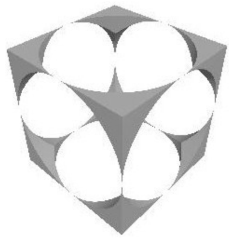
If on the other hand £2 is too dear, they might, as Carrie suggests, agree that economy overrides fairness and go to the same restaurant as before but simply treat the £1 meal as two separate 50p meals. Then it's half the previous solution: $x = 1/3$, $y = 1/6$.

Problem 248.6 – Bus stop

Tony Forbes

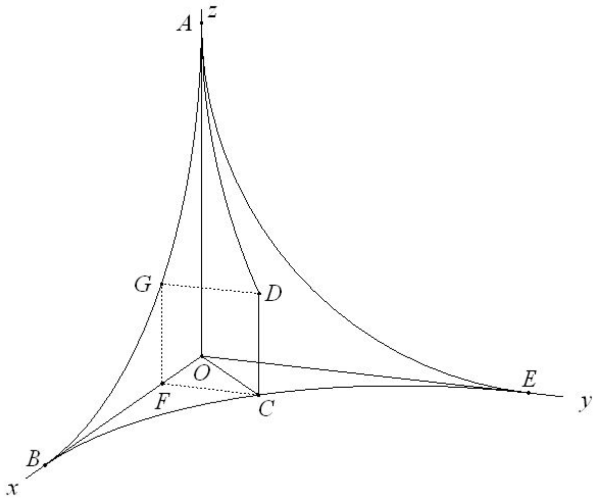
Buses arrive at a bus stop according to the Poisson process with arrival rate β. People arrive at the same bus stop also according to the Poisson process but with arrival rate α. You arrive at the same bus stop and see n people (other than yourself) waiting. How long would you expect to wait for the bus. Assume for simplicity that only one bus route is served by the stop.

Solution 242.6 – Three cylinders



Start with a $1\,\mathrm{m}^3$ cube. Take out three mutually orthogonal cylinders of length $1\,\mathrm{m}$ and diameter $1\,\mathrm{m}$. What is the volume that remains? The cylinders should of course fit snugly inside the cube along its main axes, as suggested by the picture on the right.

Steve Moon



Let one of the cube vertices be at the origin, O. The curves AB, BE and EA are cross-section quadrants of circles through the cylinders in the xz, xy and yz planes respectively, with centres $(\frac{1}{2}, 0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(0, \frac{1}{2}, \frac{1}{2})$ respectively. Also we see that the coordinates of C and F are respectively $(\alpha, \alpha, 0)$ and $(\alpha, 0, 0)$, where $\alpha = (2 - \sqrt{2})/4$.

By the symmetry of the cube, the residue volume we seek is $V = 16$ times the volume of $OABCD$. Let V_1 be the volume of $OAGFCD$ and let V_2 be the volume of $BGFCD$. Writing $\beta(x) = \frac{1}{2} - \sqrt{\frac{1}{4} - (x - \frac{1}{2})^2}$, we have

$$
\begin{aligned}
V_1 &= \int_0^{\alpha} \int_0^{x} \int_0^{\beta(x)} dz\, dy\, dx = \frac{12 - 5\sqrt{2}}{96} - \frac{\pi}{64}, \\
V_2 &= \int_{\alpha}^{\frac{1}{2}} \int_0^{\beta(x)} \int_0^{\beta(x)} dz\, dy\, dx = \frac{11\sqrt{2} - 6}{96} - \frac{\pi}{32}
\end{aligned}
$$

and therefore $V = 16(V_1 + V_2) = 1 + \sqrt{2} - 3\pi/4$.

Appeal to former OU mathematics students

Daniel Weinbren

I'm writing the history of the Open University and I'd like to learn more about the learning of students and former students from those people themselves. Personal accounts of how you create and maintain projects and learn from one another will help me to get to grips with how a higher education institution can engage with self-directed, informal learning.

To take part in this project please email me at the below address with your phone number. An interviewer, Ronald McIntyre, will ring you to establish a mutually convenient time when you can both be undisturbed for about 60 minutes. They will then ring back and record your answers to questions. There will be a few questions about your background to give us some context. It will be useful to know, or example, what qualifications you had prior to starting at the OU. You will then be asked about what you've done in regard to maths since you completed your OU maths courses (modules). You will be asked what you feel you gained from having studied through the OU and about how you benefit by remaining in contact with former students. If you helped administer M500 I'd be pleased to hear from you. A third section of questions will be about your connections to other organisations and people (e.g. have you ever served as an active member of another group or charity?). You will also be asked if there is anybody who you think we should also interview.

This project has been funded by the Society for Research into Higher Education which exists to stimulate and co-ordinate research into all aspects of Higher Education. For more about the Project, or me, see my blog at http://www.open.ac.uk/blogs/History-of-the-OU/ and the website http://www8.open.ac.uk/researchprojects/historyofou/.

I hope that you are able to take part because listening to those who have built, maintained and developed groups such as M500 could help me gain a sense of why such groups have succeeded. I will of course share my conclusions with you at the end of the project and, if you request, your contribution can be anonymous.

If you would like to be part of this project please contact me, Dr Daniel Weinbren, by post, via the Faculty of Arts, Wilson A, The Open University, Walton Hall, MK7 6AA or by email: d.weinbren@open.ac.uk. I am really looking forward to hearing from you.

Letters

Tongue-twisters and cakes

Dear Eddie,

Many thanks for M500 **246**. Another pretty tough issue, and I sympathized with the person who wrote and said that she missed the fun in the old ones, such as tongue-twisters. Mind you, English tongue-twisters pale into insignificance beside Polish ones, such as *W Szczebrzeszynie chrzaszcz brzmi w trzcinie*. In Szczebrzeszyn a beetle buzzes in the reeds.

It has been made into a brief but impenetrable poem, which you can hear pronounced at http://en.wikipedia.org/wiki/Chrz%C4%85szcz – click on the loudspeaker icon next to 'Polish original'.

But even ordinary Polish words are taxing enough. I was idly glancing at a dictionary, in which the first headword on the page was *powietrze*, and the example of its use was *rozrzedzone górskie powietrze*, meaning 'rarefied mountain air'. The first word was so simply unbelievable that I had to put the phrase into the Google translator to find out how on earth it is pronounced. Having listened several times to a sound like that of someone using an insufficiently damp chamois leather to clean a car windscreen, I am little the wiser.

An odd thing struck me about the cake problem, 246.2. [Find the parameters to maximize the volume of a slice of cake (a sector of a cylinder) of given surface area.] Cake goes stale because moisture evaporates from its surface. So a piece of cake that has a small surface area in relation to its volume will go stale more slowly than one with a large surface area in relation to its volume. Tony's slice of cake has the maximum volume for its surface area. Therefore it will go stale more slowly than the whole unsliced cake. But this is clearly nonsense.

Best wishes,

Ralph Hancock

Finland, too, has its fair share of these things. For instance, there is

Käki söi keksiä keskellä keskioksaa.

The cuckoo ate to come up in the middle of the average branch. And one is reminded of a contribution from Colin Davies in M500 **148**:

Kokoo kokoon koko kokko. Koko kokkoko? Koko kokko.

Gather together all the bonfire. All the bonfire? All the bonfire. — TF

Flour

Yesterday Rose was ordering some eco-friendly laundry bleach from an online ethical supermarket. She asked me if I wanted anything, so I thought of some bread flour, and had a look. They had Doves Farm Organic Wholewheat Strong Flour 1.5kg. Prices offered were:

List price £1.90 Our price £1.65 You save £0.25 (13%)

Case price £10.97 (List price £9.03) (5 packs) £2.19 per pack, you save -15%.

I was tempted to go for the case of five, saving minus 15%, but in the end I decided to add flour to my ASDA online list, where Allinson Very Strong Wholemeal is £1.50.

Jeremy Humphries

M500 Winter Weekend 2013

Join with fellow mathematicians for a weekend of fun. If you want a fantastic weekend and are interested in things mathematical, then this is for you, accessible to anyone who has studied mathematics—even if you are just starting. The **thirty-second M500 Society Winter Weekend** will be held at

Florence Boot Hall, **Nottingham University**

Friday 4th – Sunday 6th January 2013.

The theme is to be decided. Cost: £195 to M500 members, £200 to non-members. You can obtain a booking form from the M500 site.

http://www.m500.org.uk/winter/booking.pdf

If you have no access to the internet, send a stamped addressed envelope to

Diana Maxwell

Please note that the address has changed from last year.

We will have the usual extras. On Friday we will be running a pub quiz with Valuable Prizes, and for the ceilidh on Saturday night we urge you to bring your favourite musical instrument (and your voice). Hope to see you there.

Perfection is achieved not when there is nothing more to add, but when there is nothing left to take away.

Antoine de Saint-Exupery (*Terre des Hommes*, 1939)

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Front cover: the Editor's $5 \times 5 \times 5$ Rubik cube. And here is a 50-move sequence which produces that pattern.

$$d^2u^2(L,[R'BR,bf'])(B,[lr',F'LF])ud'lr'ud'rl'(FBU^2R^2,[d,l_*][d_*,l])$$