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M500 249

3	7	3	7	1	1	3	7	3	9	9	3
1	1	9	7	1	1	3	7	7	1	3	3
1	9	7	7	1	9	7	1	1	3	7	3
9	1	3	9	3	9	9	1	1	9	7	9
3	9	9	3	1	9	1	7	7	1	9	3
1	1	3	7	7	9	3	1	7	7	3	3
9	1	7	9	9	9	3	7	3	1	7	3
3	1	3	7	7	3	1	1	1	9	3	1
7	9	1	3	3	7	3	9	1	7	9	9
9	9	1	3	3	9	7	7	7	3	3	1
1	7	9	7	1	1	1	7	3	7	9	3
3	3	3	7	3	9	3	7	3	7	1	3

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The May Weekend is a residential Friday to Sunday event. In 2013 it will provide revision and examination preparation for students taking undergraduate module examinations in June and study support for postgraduate modules starting in February. For full details and a booking form see m500.org.uk/may.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details see m500.org.uk/winter.htm.

Advice to authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation.

Solution 244.5 – Ten primes

1	1	1	7
1	1	5	3
2	2	7	3
9	3	1	1

The numbers in each row, each column and each diagonal are prime. And no two are the same. Is it possible to find a square with the same properties but with all the digits odd?

Mike Lewis

Scope of the Problem

The number of primes satisfying the condition is 125. If four individual primes are selected to form the rows of a potential square the resulting columns and diagonals will be random odd numbers which must be tested for primality. The total number of squares to be tested using this approach would be

$$N = 125 \times 124 \times 123 \times 122 = 232593000.$$

This number and the number of comparisons to be made can be reduced if the number of initial prime rows, columns and diagonals in the potential square can be increased. A method based on a set of rules for the construction of such squares rather than a search through all possible squares is as follows.

Generate three tables

- A list of the primes.
- An array of vectors of primes with common 1st and 4th digits indexed on the 1st and 4th digits of the primes; call these *outers*.
- A similar array of vectors of primes with common 2nd and 3rd digits, the *inners*.

Loop through the following sequence

1 Select a pair of diagonals from the list of primes to form the skeletons of two squares. This ensures that the squares generated are unique.

$$\begin{vmatrix} 1 & & & 1 \\ & 1 & 5 & \\ & 9 & 1 & \\ 7 & & & 7 \end{vmatrix} \qquad \begin{vmatrix} 1 & & & 1 \\ & 5 & 1 & \\ & 1 & 9 & \\ 7 & & & 7 \end{vmatrix}$$

Any potential square in which the top right hand cell contains 5 can be rejected.

2 Concentrating on the first skeleton, select a set of columns from the inners that will fit with the central digits of each column and similarly for the outer digits of the outer columns. Eliminate the diagonals from the sets, should they be present. If any set is empty then go to step 7. This gives

$$\begin{aligned}&\text{Col1} \in \{1777,\ 1997\},\\ &\text{Col2} \in \{1193,\ 3191,\ 5197,\ 7193,\ 9199\},\\ &\text{Col3} \in \{1193,\ 3191,\ 5197,\ 7193,\ 9199\},\\ &\text{Col4} \in \{1777,\ 1997\}.\end{aligned}$$

These four sets can be applied to the second skeleton by interchanging them, equivalent to interchanging the diagonals.

3 Form a set of rows from the outers in the same way as for the columns:

$$\begin{aligned}&\text{Row2} \in \{1151,\ 1153,\ 5153,\ 7151,\ 7159,\ 9151,\ 9157\},\\ &\text{Row3} \in \{1913,\ 3911,\ 3917,\ 3919,\ 7919\}.\end{aligned}$$

4 Run through the combinations of Row2, Row3, Col1 and Col4 elements to find those that fit the skeleton. The result is a single partial skeleton square: below, left.

$$\begin{vmatrix} 1 & & & 1 \\ 7 & 1 & 5 & 9 \\ 7 & 9 & 1 & 9 \\ 7 & & & 7 \end{vmatrix} \qquad \begin{vmatrix} 1 & 5 & 3 & 1 \\ & 1 & 5 & \\ & 9 & 1 & \\ 7 & 7 & 1 & 7 \end{vmatrix} \quad \begin{vmatrix} 1 & 5 & 1 & 1 \\ & 1 & 5 & \\ & 9 & 1 & \\ 7 & 7 & 1 & 7 \end{vmatrix}$$

5 Repeat the procedure for columns 2 and 3 and rows 1 and 4. The result is two skeleton squares: above, right. The right hand skeleton is eliminated since it contains a repeated prime.

6 Generate the final square(s) by running through the combinations of skeletons rejecting any that contain common primes. In this example, only one square has been generated and is

$$\begin{vmatrix} 1 & 5 & 3 & 1 \\ 7 & 1 & 5 & 9 \\ 7 & 9 & 1 & 9 \\ 7 & 7 & 1 & 7 \end{vmatrix}$$

7 Repeat from step 3 for the second skeleton interchanging the diagonals and columns used for the first skeleton appropriately.

Results and conclusion

A program based on the algorithm was written in Scilab, an open source package similar to MatLab. The program generated 18750 squares in 51 min, 40 secs on a 3GHz, four Pentium machine with each processor running at 100%. A speed-up could have been achieved by the use of two machines running steps 4 to 7 for the interchanged diagonals concurrently.

Dave Wild

When I tried finding solutions which only used odd digits I found over 18,000 solutions. I then looked for the solutions which contained the most primes when the columns, rows and diagonals are also read in reverse order. Twelve of these contained 18 distinct primes. Ignoring the ones which contain the same set of primes one is left with the 6 examples shown below. The primes which cannot be reversed are highlighted.

1	7	3	**3**
3	5	**7**	1
9	**7**	**1**	**9**
9	7	9	1

1	9	7	**9**
9	1	**7**	**3**
3	**3**	5	**9**
3	3	7	**1**

3	1	9	**1**
3	5	1	**1**
7	5	7	**7**
1	**9**	**3**	**1**

3	3	**1**	9
5	**3**	**5**	**1**
7	5	**7**	7
1	9	**1**	3

3	**3**	**3**	**1**
3	3	5	**9**
7	1	7	**7**
1	9	1	**3**

9	1	3	3
3	7	3	3
1	5	9	7
9	**3**	**1**	**1**

Ten solutions contain 19 primes and use either 15 or 17 distinct primes. One example of each is shown below.

1	9	7	**9**
7	1	**7**	7
3	**3**	5	9
3	3	7	1

1	9	1	3
9	1	7	3
3	3	5	9
3	**3**	**3**	**1**

Tony Forbes

What about a 5×5 square? The search space is enormous but we expect a large number of solutions. One way of attack is to make the problem more difficult by imposing further conditions on the twelve primes.

I have difficulty making up my mind whether to read the antidiagonal top-to-bottom or left-to-right. But I can get around this by insisting on the number being prime in both directions. As there is nothing special about the antidiagonal let's make all of the primes reversible. Now 5 (as well as even digits) must be absent from the border of the square. So to make things balanced I think the middle of the square should be 5-free as well. Finally, I want all 24 primes to be distinct, and consequently none of them can be palindromic. Therefore we shall construct our square subject to the following conditions on the numbers that form the rows, columns and diagonals.

(i) All numbers as well as their reverses must be prime.

(ii) Only digits 1, 3, 7, 9 are allowed.

(iii) All primes must be distinct.

With the severity of these restrictions a complete search is feasible but unfortunately it turns out that unless my computer has misbehaved there are no solutions. So I shall relax condition (iii) slightly to allow one or both of the diagonals to be palindromic. Now there is a solution! Moreover, I claim this solution (below, left) is essentially unique, meaning that the only other possible arrays are the seven obtained by hitting this one with the symmetries of the square.

3	1	7	9	9
1	1	9	3	9
1	7	1	1	7
3	3	1	9	9
9	1	1	9	3

1	1	7	3	3	1
1	9	3	1	3	3
7	3	3	7	9	3
7	9	9	3	1	3
7	9	3	3	3	3
3	3	7	3	9	7

1	1	1	1	7	1	1
1	1	1	1	3	3	9
1	1	3	9	7	7	1
3	7	9	7	1	1	7
3	9	7	7	9	7	7
7	9	7	9	1	7	3
3	3	9	9	3	9	7

7	3	9	1	1	3	9	1
9	3	3	7	9	7	1	7
9	7	3	7	1	1	3	3
3	7	1	1	3	9	9	1
7	7	1	7	3	9	3	7
1	3	7	7	7	1	9	9
1	9	3	7	9	1	1	9
9	7	1	9	7	7	9	7

Well, 6 comes after 5. This time we find a genuine solution (above, middle-left), where conditions (i)–(iii) are satisfied without exception. So now we have a full complement of 28 distinct primes. It was not reasonable to do a complete search and therefore I am no longer claiming uniqueness.

Having succeeded with 6, one is tempted to continue until one's computer runs out of energy. For now I shall be content to go just six steps further: 7×7 (above, middle-right), 8×8 (above, right), 9×9 (below, left), 10×10 (below, middle), 11×11 (below, right) and 12×12 solutions exist, again with (i)–(iii) satisfied, yielding 32, 36, 40, 44, 48 and 52 distinct primes respectively. Being quite pleased with the last discovery I decided to put the 12×12 square on the front cover of this magazine.

9	9	9	7	1	7	3	1	1
3	3	7	3	7	7	9	7	7
9	9	1	3	9	1	7	7	3
9	3	7	9	9	3	1	9	3
7	9	7	9	3	1	9	3	1
1	3	9	7	9	9	9	1	7
3	9	3	1	3	9	1	9	9
3	9	7	7	1	9	1	9	3
9	7	3	9	7	9	1	1	9

1	3	3	9	7	1	9	1	9	3
9	9	9	3	9	3	7	9	1	3
3	7	1	3	7	3	3	9	7	1
7	3	1	9	7	9	3	9	1	7
7	7	1	7	7	9	1	9	9	7
1	1	9	3	9	1	7	7	1	7
9	1	3	9	9	7	9	7	3	1
3	3	7	9	1	9	1	1	9	1
3	7	9	9	1	1	3	9	7	1
1	3	9	1	1	9	9	7	3	1

1	7	9	7	9	3	3	9	3	7	7
1	9	1	3	7	9	7	7	7	1	1
3	1	9	1	9	7	7	3	3	3	9
1	7	1	1	9	7	1	7	7	1	7
7	7	1	1	7	9	3	7	9	1	7
1	1	9	1	9	3	3	9	3	9	1
1	3	7	3	7	1	9	9	3	3	1
3	3	7	9	9	9	3	3	7	9	9
9	1	1	1	1	9	7	1	9	1	9
3	7	1	1	9	1	1	7	3	9	1
1	7	7	3	7	1	3	3	7	9	7

My technique for finding these things is similar to Mike Lewis's method except that I start with columns. Call numbers that satisfy (i)–(iii) *special primes* and denote by $s(d)$ the number of d-digit special primes.

d	1	2	3	4	5	6	7	8	9	10	11	12	13
$s(d)$	0	8	10	22	84	264	716	2210	6742	22086	72296	238230	821120

For $d \geq 6$, $s(d)$ seems to be large enough for a $d \times d$ square of special primes to exist. Since I am interested only in producing a single example, I find that a method which works quite well is to choose the first $d-3$ columns at random and then try to fill in the rows by a systematic search using a table that maps a $(d-3)$-digit number n to the set of d-digit special primes of the form $1000n + m$. The last three columns are checked when the rows have been completed successfully.

I have included an entry for $d = 13$ in the $s(d)$ table to remind me that a search for a 13×13 square of special primes should be feasible. However, I shall leave it for someone else to try.

Problem 249.1 – Hypersphere

Tony Forbes

Show that the volume of the unit n-dimensional hypersphere is given by

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-\left(x_1^2+x_2^2+\dots+x_n^2\right)^{n/2}} dx_1 dx_2 \dots dx_n.$$

For instance, we see that the area of the 2-dimensional disc is π, the same as the volume under the bell-shaped surface $e^{-x^2-y^2}$. See Problem 248.4 for the 3-dimensional case.

Problem 249.2 – Estimate

Tony Forbes

For positive integer q and real $x > 1$, let

$$Q_q(x) \;=\; \frac{\log q}{q^x - 1} \qquad \text{and} \qquad T_q(x) \;=\; \sum_{n=q+1}^{\infty} Q_q(x).$$

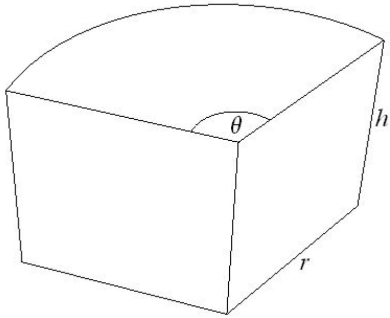
Show that

$$Q_q(q) \;\sim\; (e-1) T_q(q) \qquad \text{as } q \to \infty.$$

Hence or otherwise prove that $Q_q(q) > T_q(q)$ for $q \geq 6$.

Solution 246.2 – (It's a piece of) Cake

A slice of cake, a sector of a cylinder of radius r, height h and subtending an angle of θ, has total surface area 1. Determine r, h and θ to maximize its volume.



Ken Greatrix

The total surface area is $A = \theta r^2 + 2rh + rh\theta = 1$ and the volume is $V = \theta r^2 h/2$. Determine h from A, $h = (1 - \theta r^2)/(r(2 + \theta))$, and substitute this value into V:

$$V = \frac{\theta r^2}{2} \cdot \frac{1 - \theta r^2}{r(2+\theta)} = \frac{\theta r}{2} \cdot \frac{1 - \theta r^2}{2 + \theta} = \frac{\theta r - \theta^2 r^3}{4 + 2\theta}. \qquad (*)$$

Partial differentiation of V with respect to r and equating to 0 for a maximum, minimum or a point of inflexion gives

$$\frac{\partial V}{\partial r} = \frac{\theta - 3\theta^2 r^2}{4 + 2\theta} = 0 \;\Rightarrow\; 1 - 3\theta r^2 = 0 \;\Rightarrow\; r = \sqrt{\frac{1}{3\theta}},$$

and if $r = \sqrt{1/3\theta}$, the second derivative $\partial^2 V/\partial r^2$ is negative (indicating a maximum) when $\theta > 1/12$. Partial differentiation of V gives

$$\frac{\partial V}{\partial \theta} = \frac{(4 + 2\theta)(r - 2\theta r^3) - 2(\theta r - \theta^2 r^3)}{(4 + 2\theta)^2}.$$

At a maximum point the numerator of this expression equates to 0:

$$(4 + 2\theta)(r - 2\theta r^3) - 2(\theta r - \theta^2 r^3) = 0,$$

which simplifies to $4 - 8\theta r^2 - 2\theta^2 r^2 = 0$. From above, $r^2 = 1/(3\theta)$; hence

$$4 - \frac{8\theta}{3\theta} - \frac{2\theta^2}{3\theta} = 0 \;\Rightarrow\; \theta = 2 \text{ radians}.$$

Since this value of θ is greater than the above value of $1/12$, this corresponds to a maximum volume. Thus

$$\theta = 2 \text{ radians}, \qquad r = h = \sqrt{\frac{1}{6}} \qquad \text{and} \qquad V = \frac{1}{6\sqrt{6}}.$$

Given these calculated values, the moral of the story is in two parts. First, measure your cake's unit in feet so that you get a decent piece. Secondly, don't be the fourth person sharing this cake.

($*$) A while after completing the above, I realized that from this stage, I could have referred to *MST204* (Open University, 1982), unit 25, section 4 and used the $AC - B^2$ criterion to decide where a maximum point occurs.

I also had similar solutions by **Steve Moon**, **myself** and everyone to whom I have personally shown this problem. The value of θ is very surprising, especially to someone who has been conditioned into believing that radians always come in rational multiples of π. Even more surprising is Vincent Lynch's contribution, below, and the avoidance of all that messy differentiation. I am now wondering whether the same kind of trickery can be used to attack other problems that have appeared in M500. — **TF**

Vincent Lynch

I always enjoy a maximization problem which can be solved without calculus.

The surface area is $r\theta h + 2rh + r^2\theta = 1$.

And the volume is $V = \dfrac{1}{2}r^2\theta h$.

Now consider the three quantities $r\theta h$, $2rh$ and $r^2\theta$. Their arithmetic mean is $1/3$. Hence their geometric mean is at most $1/3$, with equality when $r\theta h = 2rh = r^2\theta$. So

$$\left(2r^4h^2\theta^2\right)^{1/3} = \frac{1}{3}$$

when the maximum occurs. Thus $8V_{\max}^2 = 2r^4h^2\theta^2 = 1/27$, giving

$$V_{\max} = \frac{\sqrt{6}}{36} \quad \text{when} \quad \theta = 2 \quad \text{and} \quad r = h = \frac{\sqrt{6}}{6}.$$

Solution 244.3 – Two sums

Show that

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+2n)^2} = \frac{4\pi^2-33}{48} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{(n^3+3n^2+2n)^2} = \frac{4\pi^2-39}{16}.$$

Steve Moon

First, express the sum in terms of its partial fraction decomposition:

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+2n)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2(n+2)^2} = \frac{1}{4}\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{(n+2)^2} + \frac{1}{n+2} - \frac{1}{n}\right).$$

Recalling the properties of the Riemann zeta-function, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; \zeta(2) \;=\; \frac{\pi^2}{6};$$

so

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)^2} \;=\; \zeta(2) - 1 - \frac{1}{4} \;=\; \frac{\pi^2}{6} - \frac{5}{4}.$$

Also

$$\sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n}\right) \;=\; \left(\frac{1}{3} - 1\right) + \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{5} - \frac{1}{3}\right) + \ldots \;=\; -\frac{3}{2}.$$

Putting these results together:

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+2n)^2} \;=\; \frac{1}{4}\left(\frac{\pi^2}{3} - \frac{5}{4} - \frac{3}{2}\right) \;=\; \frac{4\pi^2-33}{48}.$$

Using the same method, accepting that the decomposition into partial fractions is much more laborious,

$$\sum_{n=1}^{\infty} \frac{1}{(n^3+3n^2+2n)^2} \;=\; \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2(n+2)^2}$$

$$=\; \sum_{n=1}^{\infty} \left(\frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2} + \frac{E}{n+2} + \frac{F}{(n+2)^2}\right).$$

Multiplying out the numerator after adding these six terms and equating coefficients yields

$$A = -\frac{3}{4}, \quad B = \frac{1}{4}, \quad C = 0, \quad D = 1, \quad E = \frac{3}{4}, \quad F = \frac{1}{4},$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{(n^3+3n^2+2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\left(\left(\frac{-3}{n}+\frac{3}{n+2}\right)+\frac{1}{n^2}+\frac{4}{(n+1)^2}+\frac{1}{(n+2)^2}\right).$$

As before,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty}\frac{4}{(n+1)^2} = 4\left(\frac{\pi^2}{6}-1\right), \quad \sum_{n=1}^{\infty}\frac{1}{(n+2)^2} = \frac{\pi^2}{6}-\frac{5}{4},$$

$$\sum_{n=1}^{\infty} 3\left(\frac{-1}{n}+\frac{1}{n+2}\right) = -3\left(1+\frac{1}{2}\right) = -\frac{9}{2},$$

and so

$$\sum_{n=1}^{\infty} \frac{1}{(n^3+3n^2+2n)^2} = \frac{1}{4}\left(\frac{\pi^2}{6}+\frac{4\pi^2}{6}-4+\frac{\pi^2}{6}-\frac{5}{4}-\frac{9}{2}\right) = \frac{4\pi^2-39}{16}.$$

Coconut

A scientist and a mathematician are each required to perform a simple task: *retrieve a coconut, extract the milk and enjoy a refreshing drink.*

The scientist goes first. His coconut is attached to the top of the trunk of a very tall palm tree. Watched by the mathematician the scientist thinks for a few seconds and then climbs the tree. After a bit of a struggle (with some risk of a catastrophic fall) he breaks the coconut off and climbs down with it. He searches around the area and eventually finds some rocks. He bashes the coconut against them to remove the outer husk and then with a particularly jagged rock he pierces the inner shell to release the milk.

Next it's the mathematician's turn. But his coconut is sitting on the table in front of him together with a large machete. The mathematician ponders for several minutes. A difficult problem. Suddenly he smiles to indicate illumination and understanding. He takes the coconut, climbs up the scientist's tree, attaches the coconut to the top of the trunk and climbs down. The solution is now obvious.

Letters

Jos Leys

Dear Tony,

I would like to draw the attention of M500 readers to the work of Jos Leys, which can be seen on the internet. Jos uses mathematics to draw pictures. Whether it is art or not is arguable; however, the choice of objects and the use of lighting and colour give me pleasure in a way that a visit to the National Gallery does not. There is also a series of articles which I intend to study. These are difficult. (For example, create an object in four dimensions, use a stereographic projection to produce a 3-dimensional image and then ray-trace to produce a 2-dimensional image.) The articles are in French, but Google Chrome translates them. He also creates some very strange objects (a Menger sponge football, based on a Sierpinski triangle, which has zero volume). I really do think that anyone interested in maths will get enjoyment from a visit to his web site.

Regards,

Dick Boardman

Postman probability

Yesterday at my sister's house I came across a book called *What Are the Odds* by Tim Glynne–Jones. It's a book on probability aimed at the masses. Most of what he says is OK. But in his 'Postman' page he says:

> Statistics from the Royal Mail in the UK show that 5,000 postmen are bitten by dogs each year—that's 1 in 10. So any postman spending 10 or more years in the job can expect to be bitten at least once.

That seems a bit naughty. It would suggest to the layman that the 1 in 10 chance implies that if you do it 10 times you get one result. Which of course is not the case. It is the case that if the postman serves 10 years then his chance of no bites is down to about one-third, and it goes under 50 per cent around year 7. So he's probably expecting a bite after that if he hasn't already got one. So Glynne-Jones is sort of right in what he says. But it's not the way to say it.

Jeremy Humphries

Arabic numbers

Dear Eddie,

Many thanks for M500 **247**. I was surprised to hear that my attempt at a solution to 243.8 – Pentadecagon was the only one submitted, although it only works when the Moon is in the seventh house and Jupiter aligns with Mars.

I read Sebastian Hayes's article on the ancient Egyptian number system with interest until my tiny brain gave out in the middle of page 3. He remarks, 'The Egyptian system is ... arranged in ascending, rather than descending, order of size by our reckoning since the Egyptians, like the Arabs still do, write from left to right.' This oversimplifies things, since Arabs write numerals just as we do, highest on the left, and also speak their numbers beginning at the top. This apparent clash is due to their having borrowed their numbering system from the Indians, who write from left to right. Later, we re-borrowed it from the Arabs, and thus removed the directional mismatch. For example, we say 'one thousand five hundred and six', and write it as 1506. Arabs write it the same way, using their forms of the numerals, ١٥٠٦. And they say, using the same order as we do (but, of course, writing it from right to left), ألف وخمسمائة وستة, which can be transliterated as 'alif wa-khamsamān'a wa-sita' and means 'thousand and-fivehundred and-six'.

Best wishes,

Ralph Hancock

Problem 249.3 – Continued fraction

S. Ramanujan

For $|q| < 1$, let $R(q)$ denote the continued fraction defined by

$$R(q) \;=\; \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ldots}}}}.$$

Then one has the familiar result $R(e^{-2\pi i}) = 1/\phi$, where $\phi = (\sqrt{5}+1)/2 = 1.618033988\ldots$ is the golden ratio. Now show that

$$R(e^{-2\pi}) \;=\; 5^{1/4}\sqrt{\phi} - \phi \;\approx\; 0.2840790438.$$

The CDF of the Standard Normal Distribution

Ken Greatrix

The formula for the PDF of the Standard Normal Distribution is given by

$$\phi(x) \;=\; \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}.$$

During one of my computer projects, I needed to calculate the CDF on a continual basis, meaning that it wasn't convenient to keep stopping the program to input values from the tables. Simple, or so I thought! I just need to expand the above formula, integrate and then write it into my program with sufficient terms to provide the required accuracy. Expanding the above formula, we get

$$\phi(x) \;=\; \frac{1}{\sqrt{2\pi}}\left(1+\frac{-x^2}{2}+\left(\frac{-x^2}{2}\right)^2\frac{1}{2!}+\left(\frac{-x^2}{2}\right)^3\frac{1}{3!}+\cdots+\left(\frac{-x^2}{2}\right)^k\frac{1}{k!}+\ldots\right).$$

So that when integrated, it becomes

$$\Phi(x) = \frac{1}{2}+\frac{1}{\sqrt{2\pi}}\left(x+\frac{-x^2}{2}\cdot\frac{x}{3}+\left(\frac{-x^2}{2}\right)^2\frac{1}{2!}\cdot\frac{x}{5}+\cdots+\left(\frac{-x^2}{2}\right)^k\frac{1}{k!}\cdot\frac{x}{2k+1}+\ldots\right).$$

(The term $\frac{1}{2}$ is a constant of integration so that the CDF ranges from 0 to 1 for values $-\infty < x < \infty$.)

Unfortunately, it didn't work. No matter how I presented it to the program, it kept crashing because values were somehow being overloaded. Even more unfortunate is that I never found out why (or rather, I didn't try to find out why!).

I wondered if it was possible to rewrite the integral to make a more robust format. Perhaps it could be expressed as a series of terms, each term being an exponential series in itself. So I continued thus. Since x is a common factor in all the terms of the series in the brackets,

$$\Phi(x) \;=\; \frac{1}{2}+\frac{x}{\sqrt{2\pi}}\left(1+\frac{-x^2}{2}\cdot\frac{1}{3}+\left(\frac{-x^2}{2}\right)^2\frac{1}{2!}\cdot\frac{1}{5}+\cdots+\left(\frac{-x^2}{2}\right)^k\frac{1}{k!}\cdot\frac{1}{2k+1}+\cdots\right).$$

I then separated the single series in the larger brackets into a sum of two other series—these are

$$\left(1+\frac{-x^2}{2}+\left(\frac{-x^2}{2}\right)^2\frac{1}{2!}+\left(\frac{-x^2}{2}\right)^3\frac{1}{3!}+\cdots+\left(\frac{-x^2}{2}\right)^k\frac{1}{k!}+\ldots\right),$$

which is merely $e^{-\frac{1}{2}x^2}$, and

$$\left(-\frac{-x^2}{2}\cdot\frac{2}{3}-\left(\frac{-x^2}{2}\right)^2\frac{1}{2!}\cdot\frac{4}{5}-\left(\frac{-x^2}{2}\right)^3\frac{1}{3!}\cdot\frac{6}{7}-\cdots-\left(\frac{-x^2}{2}\right)^k\frac{1}{k!}\cdot\frac{2k}{2k+1}-\cdots\right).$$

In this second bracket $-(-x^2/2)\cdot 2/3 = x^2/3$ is then removed as a common factor:

$$\frac{x^2}{3}\left(1+\frac{3}{5}\cdot\frac{-x^2}{2}+\frac{3}{7}\left(\frac{-x^2}{2}\right)^2\frac{1}{2!}+\cdots+\frac{3}{2k+1}\left(\frac{-x^2}{2}\right)^{k-1}\frac{1}{(k-1)!}+\cdots\right).$$

If we now only consider the right-hand bracket, this again can be expressed as the sum of two separate series:

$$\left(1+\frac{-x^2}{2}+\left(\frac{-x^2}{2}\right)^2\frac{1}{2!}+\left(\frac{-x^2}{2}\right)^3\frac{1}{3!}+\cdots+\left(\frac{-x^2}{2}\right)^k\frac{1}{k!}+\cdots\right),$$

(which is again $e^{-\frac{1}{2}x^2}$), and

$$\left(-\frac{-x^2}{2}\cdot\frac{2}{5}-\left(\frac{-x^2}{2}\right)^2\frac{1}{2!}\cdot\frac{4}{7}-\left(\frac{-x^2}{2}\right)^3\frac{1}{3!}\cdot\frac{6}{9}-\cdots-\left(\frac{-x^2}{2}\right)^k\frac{1}{k!}\cdot\frac{2k}{2k+3}-\cdots\right),$$

where similarly $-(-x^2/2)\cdot 2/5$ is removed as a common factor.

The above process is then continued and finally the CDF is expressed as (with a few further rearrangements done for convenience):

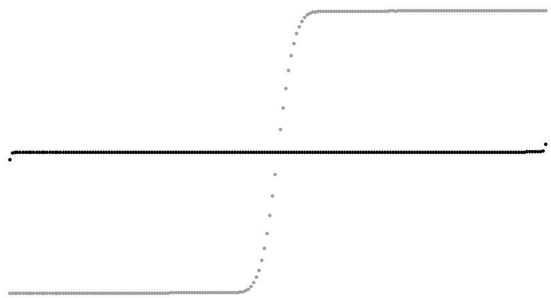
$$\begin{aligned}\Phi(x) \;=\; &\frac{1}{2}+\frac{x}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}x^2}+\frac{x^2}{3}\left(e^{-\frac{1}{2}x^2}+\frac{x^2}{5}\left(e^{-\frac{1}{2}x^2}+\frac{x^2}{7}\left(e^{-\frac{1}{2}x^2}\right.\right.\right.\right.\\ &\left.\left.\left.\left.+\frac{x^2}{9}\left(e^{-\frac{1}{2}x^2}+\cdots+\frac{x^2}{2k+1}\left(e^{-\frac{1}{2}x^2}+\frac{x^2}{2k+3}\left(e^{-\frac{1}{2}x^2}+\cdots\right)\right)\right)\right)\right)\right)\right).\end{aligned}$$

(Although stated without proof, I am certain that this expression can be proven by induction.)

For purposes of convenience within my program, I started counting $k = 0$ from the term $x^2/3$ in the first bracket, so that the denominator in this term and similar terms throughout the series is the value $2k+3$ for $k = 0, 1, 2, \ldots$. For most practical purposes $-4 < x < +4$ (i.e. x has a limiting value), so that $x^2/(2k+3) \to 0$ as $k \to \infty$. So for a high value of k, the multiplying term $x^2/(2k+3)$ reduces the remainder of the series to almost 0. This means that in my computer program, I can start with a high value of k and use an iterative formula, decreasing k until $k=0$. Then I finally add $e^{-\frac{1}{2}x^2}$, multiply by $x/\sqrt{2\pi}$ and then add $\frac{1}{2}$ to evaluate the required CDF.

Having done this, I next needed to find out how robust it is under extreme conditions. I wanted to know how big a starting value of k I should choose for extreme values of x without a significant error. To do this I called the subroutine twice with different k-values for a range of x-values from -20 to $+20$ and compared the results. I used a starting index-value of $k = 200,000$ as a basis for comparison and I didn't notice any significant errors until the second index-value was below $k = 300$. To display the error, I multiplied the difference by 10^9 before showing this on a graph.

My results can be seen on this graph. The grey dotted line is the CDF, the black dots showing the error. For this display, I used a value of $k = 280$ to indicate that an error just creeps in.



I find this second start-value of k somewhat surprising, I expected a much higher value before errors would be noted in the comparisons. Since ± 20 would never be used in practice, I decided to use a value of $k = 500$ in my program. This choice was nominal, but it ensures a reasonable accuracy without undue processing time.

The resulting series is not what I expected at the outset, but it is a more robust version. All in all I'd say I'm extremely satisfied with the outcome—I don't have any program-crash with this version. However this creates a dilemma. Can anyone explain why this rewritten version doesn't cause errors (i.e. crashes in the program) whereas the simply integrated version does?

Problem 249.4 – Radium

Tony Forbes

Imagine you have constructed a sphere, 10 cm radius, of pure radium-226. You then leave it alone and return to it about 1600 years later. What would you expect to see?

Recall that Ra-226 is radioactive, decaying to radon-222 with half-life 1601 years, and Rn-222 decays to polonium-218 with half-life 3.8235 days. There are further steps, with Po-218 decaying to lead-214 and astatine-218, Pb-214 to bismuth-214, At-218 to Bi-214 and more radon, this time Rn-218, half-life 0.036 seconds, and so on until the stable element Pb-206 is reached. As a bonus the process also creates five helium-4 atoms. After the stated time the sphere will have lost about half of its radium, but amongst the decay products are noble gases. Will they work their way out and escape gracefully from the surface of the sphere, or will there be sufficient accumulation in the centre to build up pressure and blow the thing apart?

For goodness sake, do not try this at home. One curie of radioactivity is sufficient to cause health problems. But with a specific gravity of 5.5 the sphere will weigh over 23 kg. That's 23 kCi.

Problem 249.5 – Three circles

Dick Boardman

Given a triangle, construct three circles inside it which touch each other and each circle touches two sides. Can this be done with ruler and compasses?

Problem 249.6 – Polynomial sum

Let $P(x)$ be a polynomial in x of degree at least 2. Show that

$$\sum_{n=0}^{\infty} \frac{1}{P(n)} \;=\; -\sum_{\{x:P(x)=0\}} \frac{\Gamma'(-x)}{\Gamma(-x)P'(x)},$$

assuming you can devise some imaginative scheme for dealing with the case where $P(x)$ has a multiple root.

Find Russian phrases (or even whole sentences) that make sense in English when viewed in a mirror. For example, МАЯ ТОТ ТОН НО (may one ton but) ⇒ OH HOT TOT RAM (addressing the sheep that is partial to mulled wine).

Solution 244.4 – Another product

Show that $\displaystyle\prod_{k=1}^{\infty} \frac{k^2}{k^2+1} = \Gamma(1-i)\Gamma(1+i) = |i!|^2 = \frac{\pi}{\sinh \pi} \approx 0.272029.$

Steve Moon

First consider the expression of $(\sin x)/x$ as an infinite product of factors, each generating a zero of $(\sin x)/x$:

$$\begin{aligned}\frac{\sin x}{x} &= \left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\cdots \\ &= \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right) = \prod_{k=1}^{\infty}\frac{k^2\pi^2-x^2}{k^2\pi^2}.\end{aligned}$$

Substitute $x = iz$ and take the reciprocals of both sides and recall that $\sin iz = i \sin z$:

$$\prod_{k=1}^{\infty}\frac{k^2}{k^2+(z/\pi)^2} = \prod_{k=1}^{\infty}\frac{k^2\pi^2}{k^2\pi^2+z^2} = \frac{iz}{\sin iz} = \frac{z}{\sinh z}. \tag{1}$$

Now set $z = \pi$:

$$\prod_{k=1}^{\infty}\frac{k^2}{k^2+1^2} = \frac{\pi}{\sinh \pi} \approx 0.272029.$$

Equation (1) provides a way to calculate any infinite product of the form $\prod_{k=1}^{\infty} k^2/(k^2+a^2)$, where $a = z/\pi$, giving $a\pi/(\sinh a\pi)$.

The remaining elements of the problem can be derived using the functional equations of the Gamma function, which are

$$\text{(a)}\quad \Gamma(z+1) = z\Gamma(z), \qquad \text{and} \qquad \text{(b)}\quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Starting from

$$\prod_{k=1}^{\infty}\frac{k^2}{k^2+1} = \frac{\pi}{\sinh \pi} = \frac{i\pi}{\sinh i\pi},$$

and using (b) and (a) with $z = i$, we obtain

$$\prod_{k=1}^{\infty}\frac{k^2}{k^2+1} = i\Gamma(i)\Gamma(1-i) = \Gamma(1+i)\Gamma(1-i) = \Gamma(1+i)\overline{\Gamma(1+i)} = |i!|^2$$

on interpreting $i!$ as $\Gamma(i+1)$ and remembering that $\Gamma(\overline{z}) = \overline{\Gamma(z)}$.

M500 Mathematics Revision/MSc Weekend 2013

The **M500 Revision Weekend / MSc Study Weekend 2013** will be held at

Yarnfield Park Training and Conference Centre,
Yarnfield, Staffordshire ST15 0NL

between

17th and 19th May 2013.

The standard cost, including accommodation (with en suite facilities) and all meals from dinner on Friday evening to lunch on Sunday is £290, with an early booking discount of £25 if booked and paid in full before 1st March or £310 after 16th April. The standard cost for non-residents, including Saturday and Sunday lunch, is £155, with the same early booking discount / late booking fee applied. For full details and an application form see the Society's web site at www.m500.org.uk or send a stamped, addressed envelope to:

M500 Society

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most mathematics-based OU modules plus a limited number of science modules, subject to the availability of tutors and sufficient applications.

Problem 249.7 – Syllables

Tony Forbes

A long time ago I offered M500 readers a rather complicated algorithm that converts a positive integer n to its English representation, $E(n)$, say, as in, for example, $E(1203)$ = 'one thousand, two hundred and three' [M500 **177**, 26–27]. Here we are asking for something that should in theory be a lot easier. Construct a simple algorithm for computing the number of syllables in $E(n)$. If that's too difficult, then try doing it in the andless system, as used by Americans, so that the above becomes $A(1203)$ = 'one thousand, two hundred, three.'

Contents — M500 249 – December 2012

Front cover: A 12×12 square of decimal digits. The 12-digit numbers formed by rows, columns and diagonals are prime in both directions, and the 52 primes are distinct. See page 4.