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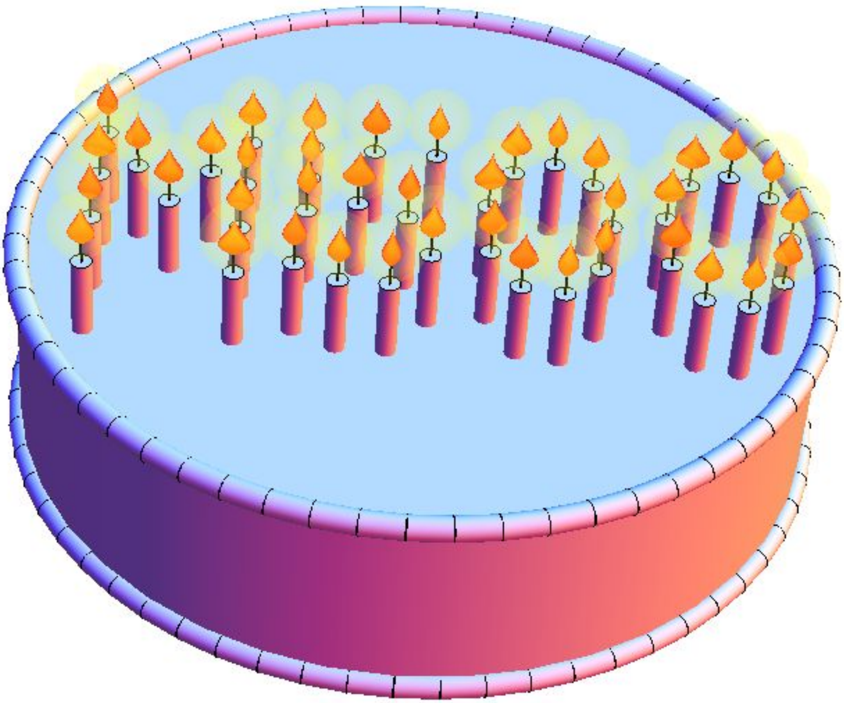
M500 250



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

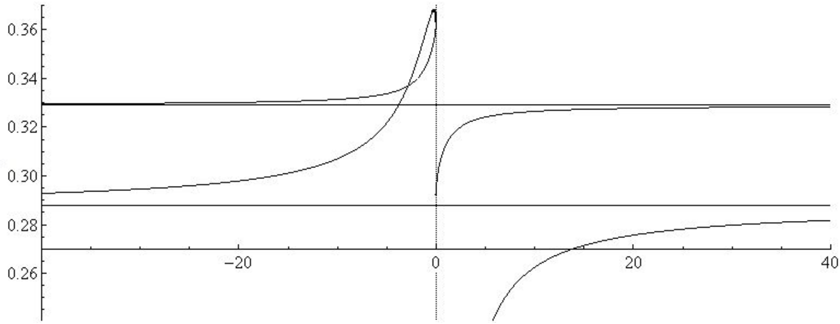
The May Weekend is a residential Friday to Sunday event. In 2013 it will provide revision and examination preparation for students taking undergraduate module examinations in June and study support for postgraduate modules starting in February. For full details and a booking form see m500.org.uk/may.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details see m500.org.uk/winter.htm.

Advice to authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file to tony@m500.org.uk.

Solution 246.6 – Loop

The picture shows the curve $\big((\sin t)(\tan t), (\log t)/t\big)$ as t goes from 1.4 to 2π. What is the area enclosed by the little loop?



Dick Boardman

Write $x(t) = (\sin t)(\tan t)$ and $y(t) = (\log t)/t$. From the graph it is clear that the loop touches the y-axis when $t = \pi$, since at this point we have $x(\pi) = 0$ and $y(\pi) \approx 0.36$. The interval we are interested in is therefore $\pi/2 \le t \le 3\pi/2$.

To find the crossover point we require two values of t, t_1 and t_2, such that $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$. From the symmetry of $x(t)$ we know that $\pi - t_1 = t_2 - \pi$. Therefore the values we want are $t_1 = \pi - z$ and $t_2 = \pi + z$, where z is a solution of $y(\pi - \tau) = y(\pi + \tau)$. We expect $x(\pi - z)$ to be approximately -3. After some experimentation we find that the root we want is in the vicinity of $\tau = 1.25$, the point at which $x(\pi - \tau) \approx -2.9$. Exact evaluation of z seems to be hopeless. Numerical computation gives $z \approx 1.250488957983$.

For the area inside the loop, we integrate $y(t)dx/dt$ over the intervals $[\pi - z, \pi]$ and $[\pi + z, \pi]$ and subtract. Numerical integration gives

$$
\begin{aligned}
I_1 &= \int_{\pi-z}^{\pi} \frac{\log t}{t} \sin t \left(1 + \sec^2 t\right) dt \approx 1.012279341542, \\
I_2 &= \int_{\pi+z}^{\pi} \frac{\log t}{t} \sin t \left(1 + \sec^2 t\right) dt \approx 0.985162038076,
\end{aligned}
$$

and the area inside the loop is $I_1 - I_2 \approx 0.027117303466$.

Solved in a similar manner by **Steve Moon**.

Solution 247.1 – 39/163

Applying the 'last resort' method—that is, write $\alpha = 1/\lceil 1/\alpha \rceil + \beta$ and repeat with β if necessary—as in Sebastian Hayes's article, The Ancient Egyptian number system [M500 **247**, 1–5], gives

$$\begin{aligned}\frac{39}{163} &= \frac{1}{5} + \frac{1}{26} + \frac{1}{1247} + \frac{1}{2935993} \\ &\quad + \frac{1}{11082924787499} + \frac{1}{28660618430582834379078750 4} \\ &\quad + \frac{1}{123214657323519667859049566141092194172466586933037520}.\end{aligned}$$

Find a simpler Egyptian fraction expansion of 39/163.

Dick Boardman

May I offer

$$\frac{39}{163} = \frac{1}{11} + \frac{1}{13} + \frac{1}{14} + \frac{1}{326326} = \frac{1}{7} + \frac{1}{11} + \frac{1}{182} + \frac{1}{326326},$$

but I can't find any set of fewer than four fractions.

Vincent Lynch

39/163 = 1/5 + 1/27 + 1/450 + 1/220050.

Tony Forbes

There is nothing special about 39/163; it just happened to fill the page in M500 **247**. For a more systematic analysis, here are the first a/b (ordered $b = 2, 3, \ldots,\ a = 2, 3, \ldots, b-1$) with 'last resort' expansion length n.

n	a/b	last fraction
2	2/3	1/6
3	4/5	1/20
4	8/11	1/4070
5	16/17	1/32640
6	27/29	1/1003066152
8	44/53	1/1458470173998990524806872692984177836808420
9	65/131	$1/(2.929007088348530525717072791634 \cdots \times 10^{112})$
11	11/199	$1/(2.039986670246850822853427080268 \cdots \times 10^{1347})$
12	221/398	$1/(2.039986670246850822853427080268 \cdots \times 10^{1347})$
13	641/796	$1/(2.039986670246850822853427080268 \cdots \times 10^{1347})$
14	958/1819	$1/(1.592227275436832438091464380382 \cdots \times 10^{7172})$
15	1197/2273	$1/(2.152341706044747611206947160254 \cdots \times 10^{14582})$

David Wild

Let $(a, b, \dots)$ represent $1/a + 1/b + \dots$. Then, according to my computer, 39/163 can be written as (5, 26, 1248, 1017120), (5, 26, 1495, 7498), (5, 27, 450, 220050), (6, 14, 856, 2930088), (6, 14, 858, 326326), (6, 14, 861, 140343), (6, 14, 868, 60636), (6, 14, 966, 7498), (6, 14, 978, 6846), (6, 14, 1141, 3423), (7, 11, 182, 326326), (8, 9, 326, 11736) or (11, 13, 14, 326326). If we favour smaller numbers then we should choose (6, 14, 1141, 3423), or (6, 14, 978, 6846) if we also want to use only even numbers.

In the associated article Sebastian Hayes speculates on the number of terms required to express a proper fraction. I shall describe how to find the greatest proper fraction that can be expressed in four or fewer terms. If a proper fraction can be expressed in fewer than four terms then we can always find an additional term which will increase the value. Therefore we only need to consider all possible values (a_1, a_2, a_3, a_4) where $2 < a_1 < a_2 < a_3 < a_4$.

Initially we will look for a set of contiguous integers such that $(n, n+1, n+2, n+3) < 1$ and n is a minimum. The value we find is $(3, 4, 5, 6) = 57/60$. This is greater than any other set of terms with $a_1 \geq 3$. Therefore any potentially larger values must be of the form $(2, a_2, a_3, a_4) = (2) + (a_2, a_3, a_4)$.

We now essentially repeat the process above where we just look at the last three terms. We find a set of contiguous integers such that $(n, n+1, n+2) < 1 - 1/2$. Here we find $(6, 7, 8) = 73/168$. This is greater than any other set of terms with $a_2 \geq 6$; $(2, 6, 7, 8) = 157/168$. Therefore the maximum value of a_2 we now need to consider is 5. The minimum value of a_2 we can use must be at least $2 + 1 = 3$ and as $(2, 3) = 5/6 < 1$, it is 3. Therefore any potentially larger values must be of the form $(2, 3, a_3, a_4)$, $(2, 4, a_3, a_4)$, or $(2, 5, a_3, a_4)$.

We repeat the process for these three candidates and eventually end up with the following potentially largest values (2, 3, 7, 43) = 1805/1806, (2, 3, 8, 25) = 599/600, (2, 3, 9, 19) = 341/342, (2, 3, 10, 16) = 239/240, (2, 3, 11, 14) = 230/231, (2, 3, 12, 13) = 155/156, (2, 4, 5, 21) = 419/420, (2, 4, 6, 13) = 155/156, (2, 4, 7, 10) = 139/140, (2, 4, 8, 9) = 71 / 72, (2, 5, 6, 8) = 119/120, (2, 5, 7, 8) = 271/280, (2, 6, 7, 8) = 157/168, and (3, 4, 5, 6) = 19/20. Therefore any proper fraction greater than 1805/1806 cannot be expressed using four terms (or many smaller values, such as 16/17).

What we have done is find a finite range of possible values for a_1. For each possible value of a_1 we find a corresponding finite range of values of a_2, and then repeat the process for a_3 and a_4. As we are always dealing with finite sets of values we must be able to find the largest value and are able to select an n such that $1 - 1/n = (n-1)/n$ is greater than this. This value cannot be expressed in the form (a_1, a_2, a_3, a_4). We could follow this procedure for any number of terms so we can always find a proper fraction which requires more than this number of terms.

Solution 247.3 – Balls

> Cannon balls are stacked in the usual square pyramid structure except that some of the top layers might be missing. When is it possible to stack a square number of cannon balls to make a truncated square pyramid?

Chris Pile

Thank you for a new variation on consecutive sums of squares. I dusted off my old notes and found I already had some examples – all that was needed was to obtain a neat formula.

I recall similar problems on cannon balls [M500 **166** and **169**] but these were mainly about tetrahedral stacking. There are many relations between tetrahedral and square pyramids, as noted in past M500s. In particular, four of the 24-layer square pyramids can be reassembled as a 48-layer tetrahedral pyramid – the only one to have a square number of balls (19600) apart from the trivial 2-layer, 4-ball case.

There are many instances of truncated tetrahedral pyramids having a square number of balls – but you did not ask about that!

The number of balls in a square pyramid of n layers is given by

$$\sum_{r=1}^{n} r^2 \;=\; \frac{n(n+1)(2n+1)}{6}.$$

The only case when this total is also a square is $n = 24$, giving $70^2 = 4900$ balls. Consider a truncated pyramid with a base of B^2 balls and the top n layers removed. The number of balls is

$$\frac{B(B+1)(2B+1)}{6} - \frac{n(n+1)(2n+1)}{6} \;=\; \frac{B-n}{6}\left(2(B-n)^2 + 6Bn + 3(B+n) + 1\right).$$

Let $L = B - n$ be the number of layers in the truncated pyramid. Then the number of balls becomes

$$\frac{L\left(2L^2 - 3L + 1\right)}{6} + BL(B-L+1) \;=\; \frac{L(L-1)(2L-1)}{6} + BLT,$$

where $T = B - L + 1$ is the side of the top square of the truncated pyramid. The problem requires this total to be a square.

Here is a list of truncated square pyramids with base layer less than 100^2 and a square number of balls, N^2.

B^2	T^2	L	N^2	B^2	T^2	L	N^2	B^2	T^2	L	N^2
4^2	3^2	2	5^2	43^2	20^2	24	158^2	73^2	25^2	49	357^2
21^2	20^2	2	29^2	48^2	38^2	11	143^2	77^2	28^2	50	385^2
28^2	18^2	11	77^2	48^2	25^2	24	182^2	80^2	22^2	59	413^2
29^2	7^2	23	92^2	50^2	25^2	26	195^2	92^2	60^2	33	440^2
32^2	9^2	24	106^2	56^2	7^2	50	245^2	93^2	44^2	50	495^2
39^2	17^2	23	138^2	59^2	27^2	33	253^2	96^2	38^2	59	531^2
39^2	7^2	33	143^2	67^2	44^2	24	274^2	99^2	76^2	24	430^2

Vincent Lynch

I decided that the number of layers was the best parameter to fix, since trying to blast it with x and y as the largest and smallest sides just gives a rotten cubic. So, I decided to start small, as I used to tell my students who were doing investigations—two layers. I found it led to a Pellian equation, and since we can easily spot a solution, $3, 4, 5$, it means there are an infinite number of solutions. For example I found $696^2 + 697^2 = 985^2$ using only a calculator.

I next wrote a BASIC program to look for solutions in which the base side was 1000 or less. What surprised me was the paucity of small numbers of layers; only 2 and 11 less than 20. And there doesn't seem to be any pattern.

So I went back to algebra and produced two formulae: For $2k$ layers from sides $x - k + 1$ to $x + k$, we must have

$$k(6x^2 + 6x + 2k^2 + 1) \;=\; 3z^2,$$

where z^2 is the total number of balls. For $2k + 1$ layers from sides $x - k$ to $x + k$:

$$(2k + 1)(3x^2 + k^2 + k) \;=\; 3z^2.$$

I've not been able to derive much useful information from these general equations. But by considering particular values of k, I have been able to prove that many values of the number of layers are impossible. For example, using the odd formula with $k = 8$, and 17 layers, gives $17(x^2 + 24) = z^2$. But $x^2 + 24 \equiv 0 \pmod{17}$ has no solution. I have managed to prove that the only possible values of the number of layers less than 23 are 2 and 11.

More balls

Chris Pile

Re: Problem 247.3

While attempting the above-mentioned problem I was pleased to find that the number of balls in a 7-layer square pyramid is 140 as this happens to be my house number. (I am thinking of using this pyramid as a finial on my gate post!)

The volume of a regular tetrahedron is half the volume of a regular square pyramid with the same side length. A 7-layer triangular pyramid has 84 balls, suggesting a more efficient packing. The ratio between the number of balls in a square pyramid and a triangular pyramid approaches 2 as the number of layers increases. By my calculation, the volume occupied by the number of balls in a 7-layer pyramid compared with the circumscribed polyhedron is about 62 percent for both square and triangular.

Twelve balls can be arranged to touch a central ball. Such a 13-ball cluster is contained in a 5-layer triangular pyramid with three balls appearing on each face of the pyramid. This cluster represents the closest packing scheme; the centres of the 12 balls are at the vertices of a cuboctahedron.

My local 'Poundland' store is selling 16 spherical Christmas baubles (about 1.5 inches in diameter) packed in a $2 \times 2 \times 4$ arrangement, while 25 baubles, about half the diameter, are randomly packed into a cylindrical box. Using the packing scheme as in the square pyramid it is possible to pack 244 unit-diameter balls in a $6 \times 6 \times 6$ cubical box, eight layers alternately 6×6 and 5×5, instead of 216 balls in six layers of 6×6. Is it better to use a systematic packing arrangement instead of just tipping the balls into a box and shaking? How many unit diameter balls can be packed into an n^3 box? Is a cubical box the best shaped container?

Exercise for reader. Calculate the exact height of the stack of eight layers alternately 6×6 and 5×5 that fits into the $6 \times 6 \times 6$ cubical box. — TF

Problem 250.1 – Quadratic sum

Tony Forbes

Show that

$$\sum_{n=0}^{\infty} \frac{1}{an^2 + n + 1} \;=\; \gamma - \log a + O(a)$$

as $a \to 0$. Here, $\gamma \approx 0.5772156649$ is Euler's constant.

Solution 236.1 – Relationships

> How many distinct sets of relationships can you have involving six people if there are two types of entanglement, platonic and romantic.

Tony Forbes

Recall that the problem was inspired by Caitlin Moran's review of ITV's *Married Single Other* (*The Times*, 6 March 2010). Also recall that I could not agree with answers 36 and 15 offered by *Times* correspondents and that I thought it would be a good idea to collect M500 readers' thoughts on the matter before I attempted to explain why the correct answer is 755. I have not received very many responses from readers, so I might as well go ahead with my solution.

Sets of relationships are like partitions, and the number of partitions of 6 (that is, the number of ways of writing 6 as a sum of positive integers) is 11, as represented in the first column of the table, below. But people are not indistinguishable; so there are a number of ways in which a specific partition can occur. This is the second column. A brief explanation is appropriate. The first entry corresponds to one big relationship involving all six people and in the second entry there is one lonely person, who can be chosen in 6 ways. More complicated is the 10th row, where in the partitioning of the 6 people into 3 sets we can choose the first set in 15 ways, the second in 6 ways and the third in one way—that's 90. But the three sets have the same size and order is not relevant; so we divide by 3! to get 15. The power of 2 in the third column accounts for the two types of entanglement. I am assuming one cannot be in a relationship with oneself. Column 4 is obviously column 2 times column 3.

XXXXXX	1	2	2
XXXXX X	6	2	12
XXXX X X	15	2	30
XXXX XX	15	4	60
XXX X X X	20	2	40
XXX XX X	60	4	240
XXX XXX	10	4	40
XX X X X X	15	2	30
XX XX X X	45	4	180
XX XX XX	15	8	120
X X X X X X	1	1	1
	203		755

Solution 247.2 – Integral

For $n > 1$, show that $\displaystyle\int_0^\infty \frac{dx}{x^n+1} = \frac{\pi}{n\,(\sin \pi/n)}$.

Tommy Moorhouse

First we set out some notation and identities involving nth roots of unity. Let $\zeta = e^{2\pi i/n}$ be an nth root of unity so that $\zeta^n = 1$, and let $\epsilon = e^{i\pi/n}$ so that $\epsilon^n = -1$, and indeed $(\zeta^m\epsilon)^n = -1$ where $m = 0, 1, \cdots n-1$. The set of complex numbers $\zeta^m\epsilon$ is the compete list of nth roots of -1. We have

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} \;=\; 0. \tag{1}$$

To prove this let

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} \;=\; X,$$

say, and note that $\zeta \neq 0$ and $\zeta \neq 1$. Multiply both sides by ζ. The left hand side is the same (the terms simply get permuted) and we have $\zeta X = X$. Since $\zeta \neq 1$ we must have $X = 0$.

Now we also have

$$1 + \frac{1}{\zeta} + \frac{1}{\zeta^2} + \cdots + \frac{1}{\zeta^{n-1}} \;=\; 0, \tag{2}$$

which follows from (1). We also note that

$$e^{-\pi i/n} \;=\; \epsilon^{-1} \;=\; \zeta^{n-1}\epsilon, \tag{3}$$

which is clear on writing out the exponentials.

Since $1 + x^n$ vanishes if x is any nth root of -1 (and consequently $1/x$ is also a root) we can write

$$\frac{1}{1+x^n} \;=\; \frac{1}{(1-\epsilon x)(1-\zeta\epsilon x)(1-\zeta^2\epsilon x)\cdots(1-\zeta^{n-1}\epsilon x)}.$$

Now we can expand this as a sum of partial fractions, say

$$\frac{1}{(1-\epsilon x)(1-\zeta\epsilon x)\cdots(1-\zeta^{n-1}\epsilon x)}$$

$$= \frac{a_0}{(1-\epsilon x)} + \frac{a_1}{(1-\zeta\epsilon x)} + \cdots + \frac{a_{n-1}}{(1-\zeta^{n-1}\epsilon x)}.$$

Expanding the right hand side and equating powers of x we find that $a_0 = a_1 = \cdots = a_{n-1} = 1/n$.

Next we integrate term by term to get

$$-\frac{1}{n\epsilon}\left\{\log(1-\epsilon x)+\frac{1}{\zeta}\log(1-\zeta\epsilon x)+\cdots+\frac{1}{\zeta^{n-1}}\log\left(1-\zeta^{n-1}\epsilon x\right)\right\}.$$

At the lower limit of integration, $x=0$, this sum vanishes because $\log(1)=0$. For $x\gg 1$ we write

$$\log(1-\zeta^m\epsilon x) \;=\; \log(-\epsilon x)+m\log(\zeta)+\log\left(1-\frac{1}{\zeta^m\epsilon x}\right)$$

from which it is clear that the sum tends to

$$-\frac{1}{n\epsilon}\left\{\log(-\epsilon x)\left(1+\frac{1}{\zeta}+\frac{1}{\zeta^2}+\cdots+\frac{1}{\zeta^{n-1}}\right)\right.$$
$$\left.+\left(\frac{1}{\zeta}+\frac{2}{\zeta^2}+\frac{3}{\zeta^3}+\cdots+\frac{n-1}{\zeta^{n-1}}\right)\log(\zeta)\right\}.$$

From identity (2) above we see that the term in $(-\epsilon x)$ is zero, and we find

$$\int_0^\infty \frac{dx}{1+x^n} \;=\; -\frac{1}{n\epsilon}\left(\frac{1}{\zeta}+\frac{2}{\zeta^2}+\frac{3}{\zeta^3}+\cdots+\frac{n-1}{\zeta^{n-1}}\right)\log(\zeta).$$

The expression on the right is

$$-\frac{1}{n\epsilon\zeta^{n-1}}\left(\zeta^{n-2}+2\zeta^{n-3}+\cdots+(n-2)\zeta+(n-1)\right)$$

and if we multiply top and bottom of the expression in brackets by $(1-\zeta)$ we get $n/(1-\zeta)$. Since $\log\zeta=2\pi i/n$ and using identity (3) we have

$$\int_0^\infty \frac{dx}{1+x^n} \;=\; -\frac{2\pi i}{n\epsilon(\zeta^{n-1}-1)} \;=\; \frac{\pi}{n}\cdot\frac{2i}{e^{\pi i/n}-e^{-\pi i/n}} \;=\; \frac{\pi}{n\sin(\pi/n)}.$$

Problem 250.2 – Circle

Dick Boardman

Given two points A and B, show that the locus of a moving point P such that the ratio $|AP|/|BP|$ is constant is a (possibly degenerate) circle. Can you provide a Euclidean construction?

Problem 250.3 – Ellipsoid

Tony Forbes

Show that

$$4\pi\left(\frac{(ab)^{8/5}+(bc)^{8/5}+(ca)^{8/5}}{3}\right)^{5/8}$$

is quite a good approximation to the surface area of an ellipsoid with radii a, b and c. For instance, if $a = 10$ and $b = c = 15$, the formula gives 2225.5 whereas the true value is about 2225.0.

Problem 250.4 – Divisor sum

Let k be a positive integer. Denote by $\sigma_k(n)$ the sum of the k^{th} powers of the (positive) divisors of n.

(i) Show that if $k \geq 2$ and $n^k + 1$ divides $\sigma_k(n)$, then n must be prime. For example, $\sigma_2(6) = 1^2 + 2^2 + 3^2 + 6^2 = 50$ is not divisible by $6^2 + 1 = 37$, but $\sigma_2(7)$, which is also equal to 50, is divisible by $7^2 + 1$, and hence 7 is prime.

(ii) Apart from the example above, are there any $k, n \geq 2$ for which $\sigma_k(n) = \sigma_k(n+1)$?

Problem 250.5 – Guess

If you guess the solution to this problem, what's the probability of getting the correct answer?

(a) 0, (b) 0.25, (c) 0.25, (d) 0.5.

Problem 250.6 – Three towns

Dick Boardman

Three towns form an acute-angled triangle. They are connected by roads to a single point in such a way that the total length of road is a minimum. The sides of the triangle and the lengths of the road segments are all integers. Find one or more examples.

QUESTION: How many notes are there in an octave?

MUSICIAN: Eight of course. It's from the Latin: octo, eight.

MATHEMATICIAN: Wrong! The correct answer is seven.

Solution 204.7 – Arctangent identities

Assuming that A, B, C, M, N are non-zero integers, show that

$$N \arctan \frac{1}{B} + M \arctan \frac{1}{C} \;=\; \arctan \frac{1}{A}$$

if and only if $(B+i)^N (C+i)^M (A-i)$ is real.

Unfortunately something has gone wrong here! For example, put $A = B = C = 1$, $M = -1$ and $N = 6$. If, as is customary, we consistently use the arctan function with range $(-\pi, \pi)$, then $6 \arctan 1 - \arctan 1 - \arctan 1 = \pi \neq 0$ whereas $(1+i)^6 (1+i)^{-1} (1-i) = -8$, which looks very real. I leave it for the reader to determine what the problem meant to say. In the meantime let's change 'if and only if' to 'only if'. — TF

Steve Moon

Using $\tan^{-1}(1/\theta) = \cot^{-1}\theta$, define a, b, c by $A = \cot a$, $B = \cot b$, $C = \cot c$. So we seek to show that $Nb + Nc - a = 0$ only if $(B+i)^N (C+i)^M (A-i)$ is real. Now use the identity

$$\cos\theta + i \sin\theta = e^{i\theta} \;\Rightarrow\; \cot\theta + i = \frac{e^{i\theta}}{\sin\theta}, \qquad \sin\theta \neq 0.$$

Thus

$$B + i = \frac{e^{ib}}{\sin b} \;\Rightarrow\; (B+i)^N = \frac{e^{iNb}}{(\sin b)^N}, \tag{1}$$

$$C + i = \frac{e^{ic}}{\sin c} \;\Rightarrow\; (C+i)^M = \frac{e^{iMc}}{(\sin c)^M}, \tag{2}$$

$$\cot(-a) + i = \frac{e^{-ia}}{\sin(-a)} \;\Rightarrow\; -\cot a + i = \frac{-e^{-ia}}{\sin a} \;\Rightarrow\; A - i = \frac{e^{-ia}}{\sin a}, \tag{3}$$

and on multiplying (1), (2) and (3),

$$(B+i)^N (C+i)^M (A-i) \;=\; \frac{e^{i(Nb+Mc-a)}}{(\sin b)^N (\sin c)^M (\sin a)},$$

which is real when $Nb + Mc - a = 0$, as required.

Archimedes discovered that the volume of an object can be deduced from the volume of water it displaces. — TV science programme

Problem 250.7 – Bernoulli numbers

Tony Forbes

Recall that the Bernoulli numbers, B_n, are defined for non-negative integers n as the coefficients of $x^n/n!$ in the Taylor expansion of $x/(e^x - 1)$:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}. \tag{1}$$

Using l'Hôpital's rule we see that $B_0 = \lim_{x \to 0} x/(e^x - 1) = 1$. Also observe that $x/(e^x - 1) + x/2$ is an even function of x, and therefore the coefficients of odd powers of x in its Taylor expansion will be zero. Thus $B_1 = -1/2$ and $B_n = 0$ for odd $n \geq 3$. Multiplying (1) by $(e^x - 1)/x$,

$$1 = \frac{x}{e^x - 1} \frac{e^x - 1}{x} = \left(\sum_{n=0}^{\infty} \frac{B_n x^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} \right),$$

and equating coefficients of x^n,

$$0^n = \sum_{k=0}^{n} \frac{B_k}{k!\,(n-k+1)!},$$

gives

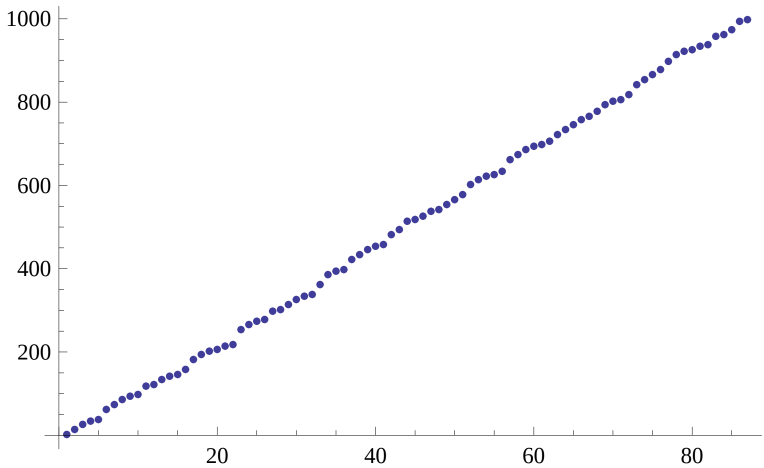
$$B_n = 0^n - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1},$$

a nice recursive formula from which one can quite easily calculate values of B_n for even n.

n	0	1	2	4	6	8	10	12	14	16	18	20
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$

That's enough background—now for the problem. When is B_n a fraction with denominator 6?

To try to help you spot a pattern I have computed a list of all such $n < 1000$: 2, 14, 26, 34, 38, 62, 74, 86, 94, 98, 118, 122, 134, 142, 146, 158, 182, 194, 202, 206, 214, 218, 254, 266, 274, 278, 298, 302, 314, 326, 334, 338, 362, 386, 394, 398, 422, 434, 446, 454, 458, 482, 494, 514, 518, 526, 538, 542, 554, 566, 578, 602, 614, 622, 626, 634, 662, 674, 686, 694, 698, 706, 722, 734, 746, 758, 766, 778, 794, 802, 806, 818, 842, 854, 866, 878, 898, 914, 922, 926, 934, 938, 958, 962, 974, 994, 998. Plotting these values (by which I mean that a point occurs at $(x,\ y)$ if y is the x^{th} number in the list) produces a graph which looks almost as if it could be linear. The slope is about 11.8.



I am curious; so I shall ask two more questions. Does this linearity property hold for arbitrarily large n? And if it does, what is the exact limiting value of the slope?

Problem 250.8 – Roman numerals

Tony Forbes

The number of decimal digits in the positive integer n is given by the formula $\left\lfloor \dfrac{\log n}{\log 10} \right\rfloor + 1$. Find a similar kind of formula for the number of letters in the Roman numeral representation of n.

Problem 250.9 – Product

Show that

$$\prod_{n=2}^{\infty} \frac{n^3+1}{n^3-1} \;=\; \frac{3}{2}.$$

Letters

In our time

I'm just listening to *In Our Time* (R4, Melvyn Bragg). It's about Simone Weil (sister of the more famous André, as we would say). One of the people on the programme remarked that Weil was nicknamed 'The Martian' by her tutor Émile Chartier, because of the size of her intellect. The speaker went on to speculate that this was because of the Mekon, who had a huge brain.

But hang on. Weil's dates are 1909 to 1943, and the Mekon first appeared in 1950. Furthermore, the Mekon was from Mekonta, North Venus, as any Dan Dare aficionado knows.

There was a programme a couple of months back on 'The Cell'. Melvyn remarked: 'The DNA in the human body would stretch to the Moon and back 8000 times. Despite its unimaginable length . . . ' Well, for something 'unimaginable', I reckon he did a pretty good job of imagining it there.

That same programme had Steve Jones, who almost referred to 'infinitely long mitochondria'. But, being a scientist, he realized what he was saying and stopped himself mid-word, substituting 'very long', or something similar.

Jeremy Humphries

Pi

Dear Eddie,

Many thanks for M500 **248**. I was intrigued by Tony's $5 \times 5 \times 5$ Rubik cube until I found that it is old hat and you can now buy $7 \times 7 \times 7$ ones, and that the redesign of the internal mechanism that made this possible is theoretically good up to $11 \times 11 \times 11$; though probably such a cube would have to be made out of something more rigid and more accurately formed than moulded plastic. At some stage the physical cube will become unworkable, but you could go digital and have virtual cubes with any number of pieces.

Also interested in Vincent Lynch's Solution 245.6 – Quintic, in which the result comes out as very slightly more than π. Some time ago I read that cosmologists had done one of their usual fudges to make their sums come out right, and this one was to suppose that the fabric of space–time has a double curvature like that of a Pringles crisp, which has a convex curvature along its north–south axis, and a concave curvature along its east–west axis. If you lower a circular slice of raw potato on to the heated former that curves it into that shape, its perimeter will be stretched and its centre compressed.

So I wondered whether an inhabitant of pringlespace who drew a circle on the pringleplane would actually find that π was greater than the value that we consider correct. I suspect that this depends on whether one is viewing the circle from inside or outside the pringleverse. If π were indeed greater, its value would depend on the size of the circle that was drawn, something that pringlites would presumably consider normal.

There are, of course, stories about π being taken as 3. One of these is in the Bible, I Kings 7 23–26, where a 'sea' — a large circular bath made of cast bronze in which the High Priest washed himself before officiating in the Temple — is said to have been 'ten cubits from one brim to the other' and 'a line of thirty cubits did compass it round about'. The other is a much mythologised account of an American State decreeing in the 19th century that π should be considered as 3 to make calculation easier.

As everyone and his dog now knows, gravitation distorts the fabric of space–time, and this is conventionally represented by a drawing of a rubber sheet into which massive bodies are shown as sinking. And I wondered whether the great density of the ancient Israelites and the 19th century Americans could cause a distortion that would produce a sagging concave shape in which, when a circle was drawn, the relationship of radius to circumference really did represent π as 3.

However, neither of these stories will wash. The 'sea' in the Bible was 'an hand breadth thick, and the brim thereof was wrought like the brim of a cup'. If you assume a cubit to be 18 inches and a hand breadth to be 4 inches, and that the stated diameter included the thickness of the rim, but the circumference was measured around the inner edge of the bowl, you get a ratio of circumference to diameter of $3.13953\ldots : 1$, near enough to π for practical purposes such as temple building.

And yes, there really was an attempt in the United States to make π a nicer number. It happened in Indiana in 1897, when a bill was introduced into the state's House of Representatives proposing that π should be fixed as one of three more rational numbers: 16/5, or 3.2; the area of a square whose side is a quarter the circumference of the circle, which gives a value of 4; the ratio of a length of a 90 degree arc to the length of a segment connecting the arc's two endpoints, which comes out as the $\sqrt{2} \times 16/7$, which is about 3.23 and actually no more convenient than the correct value of π.

Fortunately C. A. Waldo, a professor of mathematics at Purdue University, happened to be in town and was summoned to give his opinion on the proposals. He was against them, and the bill failed. According to a local newspaper, 'Although the bill was not acted on favorably no one who spoke against it intimated that there was anything wrong with the theories it advances. All of the Senators who spoke on the bill admitted that they were ignorant of the merits of the proposition. It was simply regarded as not being a subject for legislation.'

Best wishes,

Ralph Hancock

Finnish tongue twisters

Dear Tony,

Käki söi keksiä keskellä keksioksaa seems to me to mean 'The cuckoo was eating a biscuit on the centre of the central branch' [rather than 'The cuckoo ate to come up in the middle of the average branch'].

If it can be eaten, then Keksi means a biscuit. However it can also mean a gadget like a boathook: a long pole with a spike on the end often used to direct floating logs towards a saw mill. Either way, keksiä is the partitive case of keksi, and creates the meaning that 'the cuckoo was eating' rather than 'the cuckoo ate.' I wondered how the translator came up with the expression 'to come up', until I realized that the verb 'keksiä' means 'to invent', and hence 'to come up with something.'

For another tongue twister: Yksikseskö istuskellet itsekseskö yskiskellet? is a rather contrived way of asking 'Are you sitting alone and coughing by yourself?'

I have been involved with Finland and its fascinating language since I first went there in 1951. Some years ago, when in Finland, I met a professor of mathematics at Swansea University. I think his name was Gerald Gould; he was over 80 then, and his speciality and research subject was Game Theory. He had gone to a maths conference in Helsinki some years previously to our meeting, and had noticed the mathematical precision of the grammar of the Finnish language. So he decided to learn the language, and he learned it very well. I met him several times later at Finnish language courses.

I would describe Finnish grammar as being very much like the grammar of Latin, but with only one declension (albeit with many root changes like 3rd declension Latin), only one conjugation, no genders at all, very few exceptions, and a spelling/ pronunciation system that is absolutely precise.

Colin Davies

Forty years of M500

TF

Good grief, it really has been that long. M500 began life as the *Solent M202 Newsletter*. No. 1 was published in 1973; according to founding editor **Marion Stubbs**: At 2300 hrs., precisely, 16 February, 1973, Southampton, England, 51° N, 1°25′ W (born out of despair)—24 copies, together with an application form to join the Solent OU Mathematics Self-Help Scheme, dashed off in four hours flat for an M202 tutorial. It was an instant success. M202, *Topics in Pure Mathematics*, was perhaps the most difficult course the Faculty had to offer in the early years of the OU, and the *Newsletter* was just the kind of thing that geographically isolated students needed.

When issue 6 went out in July, 1973, readers were invited to supply a new title as it was no longer restricted to the Solent, nor to M202. **Peter Weir** suggested 'M500' for reasons: (1) Why not? (2) A top-level course in communications. Full credit. (3) It's an overview of OU maths. (4) Why not? [*sic*] (6) [*sic*] I thought of it. By Issue 7, the first to bear the name 'M500', readership had risen to about 200, and later it rose to over 500 when the Faculty allowed M500 to be advertised in the maths stop presses. The distinctive logo first appeared on M500 **59**.

The editorship has been remarkably stable throughout the last 40 years. Marion did everything for the first few issues. Eddie Kent joined her from Number 25 and thereafter Eddie edited while Marion published. Jeremy Humphries was recruited as Problems Editor and later took over from Eddie at M500 **68**. Seventeen years later Jeremy handed the job to me at M500 **161**. Coincidentally, issue 250 also marks my 15th anniversary.

"Out of all the [undergraduate mathematics] magazines I've seen, you're the best," was the enthusiastic comment of an eminent mathematician. It's because you the readers are the contributors. If you look at other similar publications, you will often notice a pretty obvious division: authors are superior, omni-cognate beings, readers are mere mortals. However, there's no such class distinction in M500. Readers and writers operate on equal terms.

We will continue to flourish if you keep up the good work. You have done very well to keep M500 going for such a long time against fierce competition. Many, many thanks. You have demonstrated that the entire resources of the Internet—all those blogs, twits, forums, newsgroups and such like—can never provide an adequate substitute for the regular appearance of a real paper journal dropping through your letter-box. But do keep the articles coming. As usual, be as informal as you like and write to us about anything to do with mathematics and at any level.

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Front cover: Birthday cake.