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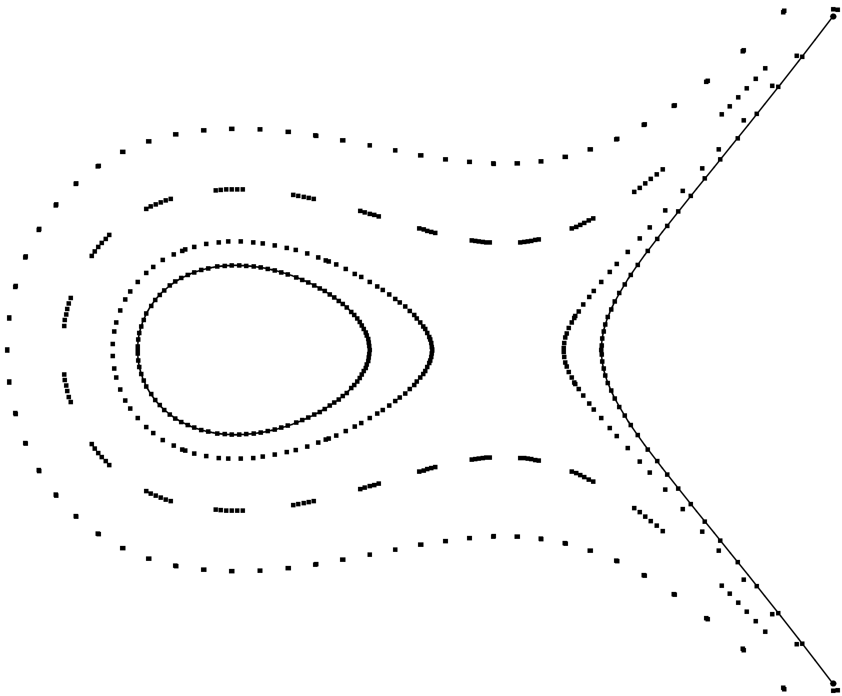

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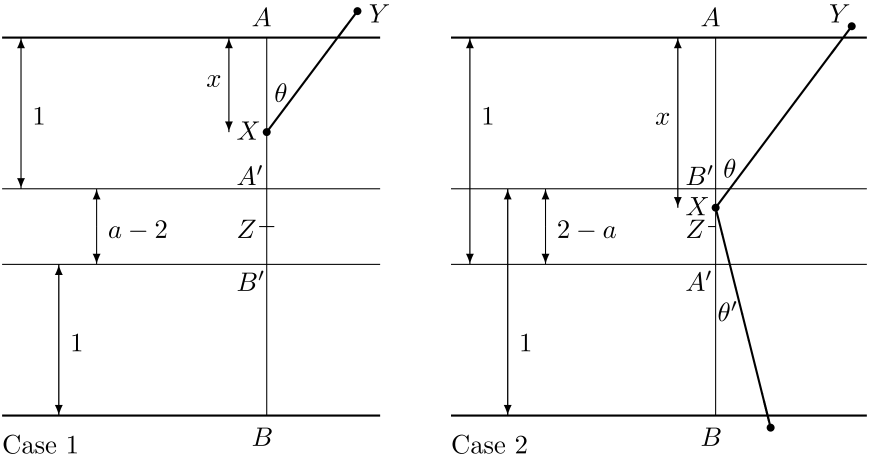
Solution 215.5 – Pins

> What is the probability of a pin of unit length crossing a crack if dropped at random on to a floor consisting of (I) infinitely long parallel floorboards of width a, or (II) square blocks of side a?

Steve Moon



Part I We consider three solutions: $a \geq 2$, $1 \leq a \leq 2$ and $a \leq 1$, although we shall see that the first two cases give the same result.

Case 1 ($a \geq 2$, left-hand diagram, above) Let the ends of the pin be X, Y. The upper and lower lines represent the edges of a floorboard, line AB is perpendicular to these edges, A' and B' are at distance 1 from A and B respectively, Z is the mid point of AB and $\theta = \angle AXY$. The distance AX is x.

By symmetry we can assume that pin end X lands somewhere on AZ and that $0 \leq \theta \leq \pi$. When X falls on AZ, the pin cannot cross the lower edge, and it can cross the upper edge only if it falls on AA', that is, when $0 \leq x \leq 1$. The probability that $0 \leq x \leq 1$ is $2/a$. Also if X falls on AA', the probability that the pin crosses the upper edge is $(\cos^{-1} x)/\pi$.

Now we calculate the mean value of $(\cos^{-1} x)/\pi$ on $[0, 1]$:

$$\overline{\theta}([0,1]) \;=\; \frac{1}{1-0}\int_0^1 \frac{\cos^{-1} x}{\pi}\,dx \;=\; \frac{1}{\pi}.$$

So for $a \geq 2$, the probability of the pin crossing an edge is

$$P(a \geq 2) \;=\; \frac{2}{a} \cdot \frac{1}{\pi} \;=\; \frac{2}{a\pi}.$$

Case 2 ($1 \leq a \leq 2$, right-hand diagram, above) Here, if X lies on AB', the pin can only cross the upper edge. However, if X lies on $B'Z$, for some θ the pin crosses the upper edge and for some θ' it crosses the lower edge in which case $0 \leq \theta' \leq \cos^{-1}(a-x)$.

First we calculate the probability of the pin crossing the upper edge, with X on AZ, $0 \leq x \leq a/2$. The probability of X on AZ is obviously 1, and for any x, $0 \leq x \leq a/2$ the probability of the pin crossing the upper edge is $(\cos^{-1} x)/\pi$. The mean value of θ on $[0, a/2]$ is

$$\overline{\theta}([0,a/2]) \;=\; \frac{1}{a/2-0}\int_0^{a/2} \frac{\cos^{-1} x}{\pi}\,dx \;=\; \frac{2}{a\pi}\left(\frac{a}{2}\cos^{-1}\frac{a}{2} - \sqrt{1-\frac{a^2}{4}} + 1\right)$$

and so for $0 \leq x \leq a/2$ the probability of the pin crossing the upper edge is

$$\frac{2}{a\pi}\left(1 + \frac{a}{2}\cos^{-1}\frac{a}{2} - \sqrt{1-\frac{a^2}{4}}\right). \tag{1}$$

Now we calculate the probability of the pin crossing the lower edge for X on $B'Z$, $a-1 \leq x \leq a/2$. The probability of X on $B'Z$ is $(2-a)/a$, and for $a-1 \leq x \leq a/2$ the probability of the pin crossing the lower edge is $\cos^{-1}(a-x)/\pi$. The mean value of θ' on $[a-1, a/2]$ is

$$\begin{aligned}\overline{\theta'}([a-1,a/2]) &= \frac{1}{a/2-(a-1)}\int_{a-1}^{a/2} \frac{\cos^{-1}(a-x)}{\pi}\,dx \\ &= \frac{2}{(2-a)\pi}\left(-\frac{a}{2}\cos^{-1}\frac{a}{2} + \sqrt{1-\frac{a^2}{4}}\right)\end{aligned}$$

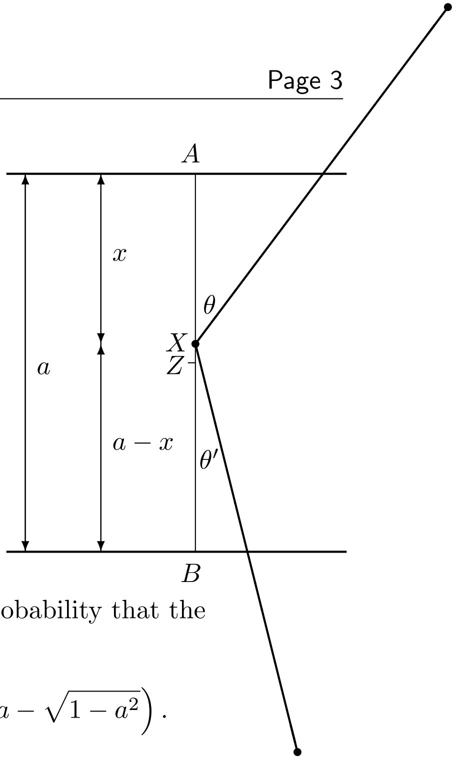
and so for $a-1 \leq x \leq a/2$ the probability of the pin crossing the lower edge is

$$\frac{2}{a\pi}\left(\sqrt{1-\frac{a^2}{4}} - \frac{a}{2}\cos^{-1}\frac{a}{2}\right). \tag{2}$$

Adding (1) and (2) and being surprised to see the similarity to the result from Case 1, we obtain

$$P(1 \leq a \leq 2) \;=\; \frac{2}{a\pi}.$$

Case 3 $(a \leq 1)$ There is some range of θ and θ' which enables the pin to cross either edge for any position if X on AZ $(0 \leq x \leq a/2)$. So using the method from the second calculation of Case 2,



$\overline{\theta}([0, a/2])$

$$= \frac{2}{a\pi} \int_0^{a/2} \left(\cos^{-1} x + \cos^{-1}(a - x)\right) dx$$
$$= \frac{2}{a\pi} \left(1 + a \cos^{-1} a - \sqrt{1 - a^2}\right).$$

Since the probability of X on AZ is 1, the probability that the pin crosses an edge is

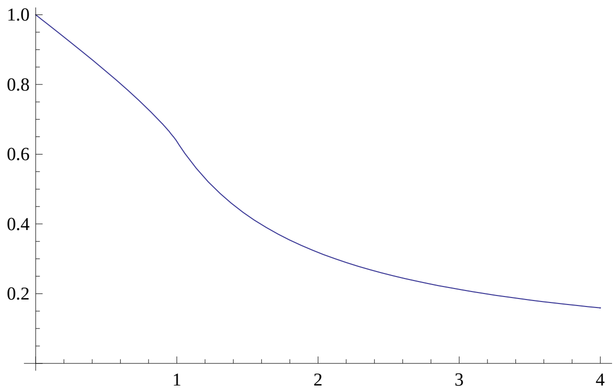
$$P(a \leq 1) = \frac{2}{a\pi} \left(1 + a \cos^{-1} a - \sqrt{1 - a^2}\right).$$

If $a = 1$, we have $P(a \leq 1) = 2/\pi$, consistent with the result from Case 2. Also, expanding as a Taylor series about $a = 0$ gives

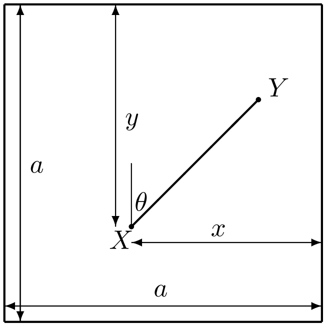
$$P(a \leq 1) = 1 - a/\pi + O(a^3)$$

and so $P(a \leq 1) \to 1$ when $a \to 0$, as expected.

Including the result $P(a \geq 1) = 2/(a\pi)$ from Cases 1 and 2 gives a probability function that looks like this.



Part II After creating a lot of waste paper, I can't readily make the same method work for this; so I have attempted it using multiple integrals. (I suppose this would work for Part I, with the $a \geq 1$ and $a \leq 1$ results arising more naturally.) Anyway, for the end X of the pin we have, as shown on the right, three independent variables:



$$0 \leq x \leq a, \quad 0 \leq y \leq a, \quad 0 \leq \theta \leq \pi.$$

In this coordinate system any outcome (x, y, θ) can be represented as a point in a cuboid of volume $a^2\pi$. The pin will not cross an edge for points given by

$$\int_0^{\pi/2} \int_{\cos\theta}^a \int_{\sin\theta}^a dx\, dy\, d\theta + \int_{\pi/2}^{\pi} \int_0^{a-\cos(\pi-\theta)} \int_{\sin\theta}^a dx\, dy\, d\theta \;=\; a^2\pi + 1 - 4a.$$

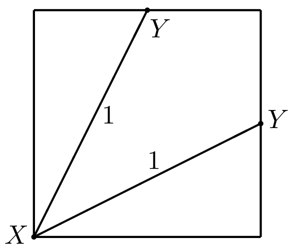
Therefore the probability of not crossing an edge is $(a^2\pi + 1 - 4a)/(\pi a^2)$. Hence the probability of crossing an edge is

$$1 - \frac{a^2\pi + 1 - 4a}{\pi a^2} \;=\; \frac{4a-1}{\pi a^2}.$$

If $a \gg 1$, this tends to $4/(\pi a)$, or twice that for floorboards with $a \geq 1$ as the effect of the corners reduces relative to two pairs of parallel sides.

I have so far not made any assumption about a lower bound for a. If we put $a = 1/\sqrt{2}$, the diagonal is 1 and the probability of the pin crossing an edge is 1. However, the formula gives $(4a-1)/(\pi a^2) = 2(\sqrt{2}-1)/\pi \approx 1.16 > 1$, a nonsense. So there is (at least) one more case to cover.

We need to investigate what happens when $1/\sqrt{2} \leq a \leq 1$. the upper bound $a \leq 1$ imposes a restriction on the range of θ, reducing it from $0 \leq \theta \leq \pi$ to $\cos^{-1} a \leq \theta \leq \sin^{-1} a$. (And similarly for $\pi/2 \leq \theta \leq \pi$, although based on earlier results I assume the second integral will equal the first; so I haven't done it explicitly.)



Hence the probability of not crossing a line requires the volume integral

$$2\int_{\cos^{-1} a}^{\sin^{-1} a} \int_{\cos\theta}^a \int_{\sin\theta}^a dx\, dy\, d\theta \;=\; 2a^2(\sin^{-1} a - \cos^{-1} a) + 4a\sqrt{1-a^2} - 2a^2 - 1.$$

So the probability of crossing an edge for $1/\sqrt{2} \le a \le 1$ is

$$1 - \frac{2a^2(\sin^{-1} a - \cos^{-1} a) + 4a\sqrt{1-a^2} - 2a^2 - 1}{\pi a^2}$$
$$= \frac{4a^2 \cos^{-1} a - 4a\sqrt{1-a^2} + 2a^2 + 1}{\pi a^2}.$$

To check for sense and consistency, if we put $a = 1$ this reduces to $3/\pi$, agreeing with the result for $a \ge 1$ earlier. If we put $a = 1/\sqrt{2}$ we obtain 1, as required.

To summarize, for square tiles of side a, the probability of a pin of length 1 crossing an edge is

$$\begin{array}{ll} \dfrac{4a-1}{\pi a^2} & \text{for} \quad a \ge 1, \\[2ex] \dfrac{4a^2 \cos^{-1} a - 4a\sqrt{1-a^2} + 2a^2 + 1}{\pi a^2} & \text{for} \quad \dfrac{1}{\sqrt{2}} \le a \le 1, \\[2ex] 1 & \text{for} \quad a \le \dfrac{1}{\sqrt{2}}, \end{array}$$

and for floorboards spaced a apart, the probability is

$$\frac{2}{a\pi} \text{ for } a \ge 1, \qquad \frac{2}{a\pi}\left(1 + a\cos^{-1} a - \sqrt{1-a^2}\right) \text{ for } a \le 1.$$

Problem 255.1 – Elementary trigonometry

This came up while I (TF) was investigating something to do with mutually touching finite cylinders. I must admit total disbelief initially and having to resort to a calculator to 'prove' it. Show that

$$\cot\left(\frac{\pi}{6} - \frac{1}{2}\arccos\frac{11}{14}\right) = 3\sqrt{3}.$$

Lytton Jarman

We are sorry to hear that Lytton Jarman, a regular contributor to early issues of M500, died in September 2013. Our sympathy goes to his widow, Rosemary.

Solution 253.2 – Quadratic

This is like finding Pythagorean triples but slightly different. Solve the quadratic $3x^2 + y^2 = z^2$ for positive integers x, y, z.

Dick Boardman

Consider the product

$$(a^2 + 3b^2)(c^2 + 3d^2) \;=\; (ac - 3bd)^2 + 3(bc + ad)^2.$$

Thus the product of any two numbers of the form $3x^2+y^2$ can be expressed as a number of the same form. Such a set of numbers is called a multiplicative domain. Hence, if we take any number of the form $a^2 + 3b^2$ and square it, we get another number of the same form, which will therefore be a solution to the original equation. Therefore $x = 2ab$, $y = a^2 - 3b^2$ is a solution.

We can extend this by taking a cube, 4th or any power of $a^2 + 3b^2$ and get a number of the same form so that this method gives a solution to $a^2 + 3b^2 = z^n$ All this generalizes, and in fact for fixed r and s, the set of numbers of the form $a^2 + rab + sb^2$ forms a multiplicative domain since

$$(a^2 + rab + sb^2)(c^2 + rcd + sd^2) \;=\; P^2 + rPQ + sQ^2$$

with

$$P \;=\; ac - sbd, \qquad Q = bc + ad + rbd,$$

so that there is a general solution to $a^2 + rab + sb^2 = z^n$. The theory is taken from *Diophantine Analysis* by Robert Carmichael, now available as a free e-book that may be downloaded from Google.

Vincent Lynch

The equation can be written $3x^2 = (z + y)(z - y)$ and then as

$$\frac{3x}{z+y} \;=\; \frac{z-y}{x} \;=\; \frac{\lambda}{\mu}.$$

Using Kramer's rule or ordinary simultaneous algebra we get

$$\frac{x}{2\lambda\mu} \;=\; \frac{z}{\lambda^2 + 3\mu^2} \;=\; \frac{y}{3\mu^2 - \lambda^2}.$$

If we take λ and μ co-prime with only one even, we have the solutions

$$x \;=\; 2\lambda\mu, \quad y \;=\; 3\mu^2 - \lambda^2, \quad z \;=\; \lambda^2 + 3\mu^2.$$

Any multiples of these are also solutions. But unlike the Pythagoras case, z may be even. We can obtain these solutions by having λ and μ both odd. In this case we need to divide by 2 to give

$$x = \lambda\mu, \quad y = \frac{3\mu^2 - \lambda^2}{2}, \quad z = \frac{\lambda^2 + 3\mu^2}{2}.$$

Problem 255.2 – Bomb

Tony Forbes

(i) A bomb is released from position $(0, 0, h)$ by an aircraft travelling at velocity $\mathbf{v}$ relative to the ground. The wind has velocity $\mathbf{w}$. Air resistance may be ignored. Assuming the ground is flat, where will it land and how long will it take to get there?

(ii) As (i) but air resistance is not ignored. The bomb, travelling at velocity $\mathbf{u}$ relative to the air, experiences an acceleration of $-\mathbf{u}|\mathbf{u}|k$ for some small constant k.

Just to give a couple of simple examples, let $\mathbf{v} = (v, 0, 0)$ and $\mathbf{w} = (0, 0, 0)$. Then by a familiar calculation, with no air resistance the bomb lands at $(v\sqrt{2h/g}, 0, 0)$ after time $\sqrt{2h/g}$. With air resistance it's a little more complicated. The landing point and drop time are now (I think)

$$\left(\frac{1}{k}\log\left(1 + \sqrt{k/g}\, v \cosh^{-1} e^{kh}\right), 0, 0\right) \quad \text{and} \quad \frac{\cosh^{-1} e^{kh}}{\sqrt{gk}}$$

respectively.

Problem 255.3 – Points of inflexion

Tony Forbes

A *point of inflexion* occurs at (u, v) on the elliptic curve $y^2 = x^3 + ax^2 + bx + c$ if the tangent at (u, v) meets the curve at a triple point. Show that the x coordinate of a point of inflexion occurs at a root of

$$3x^4 + 4ax^3 + 6bx^2 + 12cx + 4ac - b^2. \qquad (*)$$

Now forget about elliptic curves. Given a quartic of the form $(*)$ with real a, b and c, explain why it cannot have more than one real root u for which $u^3 + au^2 + bu + c \geq 0$ except possibly when the cubic has zero discriminant.

Solution 252.3 – Quadratic triangles

(See author's Problem statement, below.)

Edward Stansfield

Definitions

Curve C: $y(x) = ax^2 + bx + c$;

discriminant: $D = b^2 - 4ac > 0$;

roots: $y(\alpha) = y(\beta) = 0$, where $0 < \alpha < \beta$.

Tangent lines to C at $x = \alpha$ and $x = \beta$ intersect at point $\gamma = (u, v)$.

Comment: $D > 0 \Rightarrow$ distinct real roots, $\alpha < \beta$.

Problem statement

(1) Show that the tangent lines at $x = \alpha$ and $x = \beta$ have equal and opposite slope with $y'(\beta) = \sqrt{D}$.

(2) Find the area A of triangle T with corners $\{(\alpha, 0), (\beta, 0), (u, v)\}$ in terms of D and a.

(3) Find the perimeter P of triangle T in terms of D and a.

(4) Check the dimensions of the results assuming x and y are lengths.

(5) Deduce that all quadratic curves are symmetric about $x = \frac{1}{2}(\alpha + \beta)$ and uniquely determined by the corners of triangle T.

Proposed solution

(1) Roots α and β are given by

$$\alpha = \frac{-b - \sqrt{D}}{2a} \quad \text{and} \quad \beta = \frac{-b + \sqrt{D}}{2a} \quad \text{(positive square root).}$$

Hence $\alpha + \beta = -b/a$ and $\alpha - \beta = -\sqrt{D}/a$. Therefore

$$\frac{dy}{dx} = 2ax + b \;\Rightarrow\; \left.\frac{dy}{dx}\right|_{x=\alpha} = -\sqrt{D} \;\text{ and }\; \left.\frac{dy}{dx}\right|_{x=\beta} = +\sqrt{D}.$$

Hence

$$y'(\beta) \;=\; \left.\frac{dy}{dx}\right|_{x=\beta} \;=\; -\left.\frac{dy}{dx}\right|_{x=\alpha} = \sqrt{D},$$

as required. □

(2) The tangent at $x = \alpha$ has equation $y = -\sqrt{D}x + p$ and passes through point $(\alpha, 0)$. Hence $p = \alpha\sqrt{D}$.

The tangent at $x = \beta$ has equation $y = \sqrt{D}x + q$ and passes through point $(\beta, 0)$. Hence $q = -\beta\sqrt{D}$.

The two lines $y = (\alpha - x)\sqrt{D}$ and $y = (x - \beta)\sqrt{D}$ meet at the point $\gamma = (u, v)$. The abscissa u is the solution for x of $(\alpha - x)\sqrt{D} = (x - \beta)\sqrt{D}$, namely $u = \frac{1}{2}(\alpha + \beta)$, and the ordinate v is thus $v = \frac{1}{2}(\alpha - \beta)$.

Triangle T therefore has base length $\beta - \alpha$ and height $h = |v| = \frac{1}{2}(\beta - \alpha)\sqrt{D}$. Hence the area is $A = \frac{1}{4}(\beta - \alpha)^2\sqrt{D}$. Since $\beta - \alpha = \sqrt{D}/a$ this gives $A = D^{3/2}/(4a^2)$. □

(3) Let the perimeter of the isosceles triangle T with base $\beta - \alpha$ be denoted by P, and the length of each (non-base) side be g. By Pythagoras's theorem we have that $g^2 + h^2 + \left(\frac{1}{2}(\beta - \alpha)\right)^2$. Substituting for h and $\beta - \alpha$ then gives $g = \sqrt{D(1 + D)}/(2a)$. The perimeter is therefore given by

$$P \;=\; 2g + \beta - \alpha \;=\; \frac{\sqrt{D}}{a}\left(1 + \sqrt{1 + D}\right). \qquad \square$$

(4) Let $[\phi] = L \Leftrightarrow \phi$ is a length, and $[\theta] = 1 \Leftrightarrow \theta$ is dimensionless. Since $y(x) = ax^2 + bx + c$, we must have that

$$[y] = L, \quad [x] = L, \quad [a] = L^{-1}, \quad [b] = 1, \quad [c] = L, \quad [D] = 1.$$

Hence $[A] = [a]^{-2} = L^2$ and $[P] = [a]^{-1} = L$, as expected. □

(5) To verify symmetry about the line $x = \frac{1}{2}(\alpha + \beta) = u$, consider the function $y(u \pm z)$ in the variable z. Then

$$y(u \pm z) \;=\; a(u \pm z)^2 + b(u \pm z) + c \;=\; au^2 \pm 2auz + az^2 + bu \pm bz + c.$$

Since $u = -b/(2a)$ this becomes

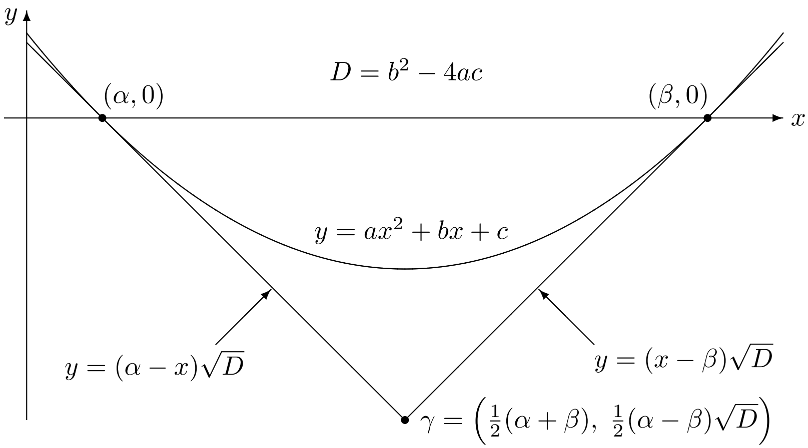
$$y(u \pm z) \;=\; \frac{b^2}{4a} \mp bz + az^2 - \frac{b^2}{2a} \pm bx + c \;=\; az^2 - \frac{b^2}{4a} + c.$$

Hence $y(u+z) = y(u-z)$ and the symmetry of $y(x)$ about $x = u = \frac{1}{2}(\alpha+\beta)$ is confirmed. □

To confirm the uniqueness of $y(x)$ in terms of the corners α, β and γ of the triangle T, it is sufficient to be able to determine the coefficients a, b and c uniquely in terms of α, β and γ. Since

$$\gamma \;=\; (u, v) \;=\; \left(\frac{1}{2}(\alpha + \beta),\; \frac{1}{2}(\alpha - \beta)\sqrt{D}\right),$$

the only parameters we really need are α, β and D. Furthermore, we know that $(\alpha-\beta) = -\sqrt{D}/a$ and $(\alpha+\beta) = -b/a$. These equations give $a = \sqrt{D}/(\beta-\alpha)$ and $b = -a(\alpha+\beta)$ respectively. Finally, from $D = b^2 - 4ac$ we also deduce that $c = (b^2 - D)/(4a)$. Hence a, b and c are uniquely determined from α, β and γ. □



Multiplication

Without doubt elementary mathematics has an important role to play in modern society. Just to give a few examples, see how simple multiplication can solve typical problems that present themselves in everyday life.

(i) If a computer programmer takes three weeks to write a computer program, how long would it take three programmers to write the same program? Answer 9 weeks (3×3).

(ii) If it takes on average 10 seconds for a person to cross a busy road, how long on average would it take a group of 6 people to cross the same road under the same traffic conditions? Answer 60 seconds (6×10).

(iii) If a man takes one hour to dig a hole, how long would it take 5 men to dig the same hole? Answer 5 hours (5×1).

Can readers find more examples?

(i) Eight weeks deciding how to divide up the work, 1 week to do it. (ii) Personal experience—try it and see. (iii) Old Gas Board joke. One digs, the other four are there to observe, supervise and provide distraction.

Solution 253.6 – Four sums

Prove the following

$$\begin{aligned}
\frac{1}{1\cdot 4}+\frac{1}{6\cdot 9}+\frac{1}{11\cdot 14}+\frac{1}{16\cdot 19}+\dots &= \frac{\pi}{15}\sqrt{1+\frac{2}{\sqrt{5}}},\\
\frac{1}{2\cdot 3}+\frac{1}{7\cdot 8}+\frac{1}{12\cdot 13}+\frac{1}{17\cdot 18}+\dots &= \frac{\pi}{5}\sqrt{1-\frac{2}{\sqrt{5}}},\\
\frac{1}{1\cdot 11}+\frac{1}{13\cdot 23}+\frac{1}{25\cdot 35}+\frac{1}{37\cdot 47}+\dots &= \frac{(2+\sqrt{3})\pi}{120},\\
\frac{1}{5\cdot 7}+\frac{1}{17\cdot 19}+\frac{1}{29\cdot 31}+\frac{1}{41\cdot 43}+\dots &= \frac{(2-\sqrt{3})\pi}{24}.
\end{aligned}$$

Tommy Moorhouse

Introduction

We start with some general results. These can be applied to many other sums of this type, and there are generalizations to other problems.

Lemma 1 If $\operatorname{Re} a < 1$ then

$$\sum_{k=1}^{\infty}\frac{1}{k^2-a^2} = \frac{-1}{2a^2}\left(\frac{2\pi a i}{e^{2\pi a i}-1}+i\pi a-1\right).$$

Proof We write

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{k^2-a^2} &= \sum_{k=1}^{\infty}\frac{1}{k^2}\frac{1}{1-a^2/k^2}\\
&= \sum_{k=1}^{\infty}\frac{1}{k^2}\left(1+\frac{a^2}{k^2}+\left(\frac{a^2}{k^2}\right)^2+\dots+\left(\frac{a^2}{k^2}\right)^m+\dots\right)\\
&= \zeta(2)+a^2\zeta(4)+\dots+a^{2m}\zeta(2m+2)+\cdots.
\end{aligned}$$

The ζ functions can be expressed in terms of the Bernoulli numbers:

$$\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}B_{2m}}{2(2m)!}.$$

Writing out the sum and using the definition of the Bernoulli numbers

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n$$

we obtain the desired result with $z=2\pi i a$.

Lemma 2 With a as in Lemma 1

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 - a^2} = \frac{\pi}{4a} \tan(\pi a/2).$$

Proof Note that the sum is over odd integers, and can be written as

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 - a^2} = \sum_{k=1}^{\infty} \frac{1}{k^2 - a^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2 - a^2}.$$

The second term on the right is just

$$\frac{1}{2^2} \sum_{k=1}^{\infty} \frac{1}{k^2 - (a/2)^2},$$

and if we write

$$S(a) = \sum_{k=1}^{\infty} \frac{1}{k^2 - a^2},$$

the sum over odd integers is just $S(a) - S(a/2)/4$. Writing out the sums using Lemma 1 and noting that

$$\tan(x) = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}$$

we obtain the result.

The sums

All four sums (and many more) can be reduced to the form set out in Lemma 2. The first sum, S_1 may be written

$$\sum_{n=0}^{\infty} \frac{1}{(5n+1)(5n+4)},$$

which in turn may be written as

$$\sum_{n=0}^{\infty} \frac{1}{25(n+1/2)^2 - (3/2)^2}.$$

This can be recast in the form of Lemma 2 as

$$\frac{4}{25} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 - (3/5)^2}$$

so that $a = 3/5$. From Lemma 2 we have

$$S_1 = \frac{4}{25} \cdot \frac{\pi}{4 \cdot 3/5} \tan\left(\frac{3\pi}{10}\right) = \frac{\pi}{15} \tan\left(\frac{3\pi}{10}\right) = \frac{\pi}{15}\sqrt{1 + \frac{2}{\sqrt{5}}}.$$

Similarly the second sum, S_2 say, is

$$\sum_{n=0}^{\infty} \frac{1}{(5n+2)(5n+3)},$$

which we rewrite as

$$\sum_{n=0}^{\infty} \frac{1}{25(n+1/2)^2 - (1/2)^2}.$$

Now we follow the steps above to find $a = 1/5$ and

$$S_2 = \frac{4}{25} \cdot \frac{\pi}{4/5} \tan\left(\frac{\pi}{10}\right) = \frac{\pi}{5} \tan\left(\frac{\pi}{10}\right) = \frac{\pi}{5}\sqrt{1 - \frac{2}{\sqrt{5}}}.$$

The third sum may be deduced using

$$S_3 = \sum_{n=0}^{\infty} \frac{1}{(12n+1)(12n+11)} = \sum_{n=0}^{\infty} \frac{1}{(12(n+1/2)+5)(12(n+1/2)-5)}.$$

We find that $a = 5/6$ and

$$S_3 = \frac{1}{36} \cdot \frac{\pi}{4 \cdot 5/6} \tan\left(\frac{5\pi}{12}\right) = \frac{\pi}{120} \tan\left(\frac{5\pi}{12}\right) = \frac{\pi}{120}(2 + \sqrt{3}).$$

The fourth sum is

$$S_4 = \sum_{n=0}^{\infty} \frac{1}{(12n+5)(12n+7)} = \sum_{n=0}^{\infty} \frac{1}{(12(n+1/2)+1)(12(n+1/2)-1)}.$$

We find $a = 1/6$ and

$$S_4 = \frac{1}{36} \cdot \frac{6\pi}{4} \tan\left(\frac{\pi}{12}\right) = \frac{\pi}{24}(2 - \sqrt{3}).$$

Incidentally, the expressions for the tangents can be found by using polynomials based on

$$\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$$

and, for example,

$$\frac{1}{\tan(5 \cdot \pi/10)} = 0.$$

Rubik's Clock

Tony Forbes

Whist trying to find something in M500 **110** for 'Twenty-five years ago' I came across one of my early contributions to this magazine in which I asked, *What is the length of God's algorithm for Rubik's Clock?* Unfortunately, there were some mistakes in the details and the analysis at the end of the article was sadly inadequate. So here it is again, hopefully with fewer errors.

The device consists of 14 little clocks that just show the hours 1–12. The aim is to set them all to 12 o'clock by performing a finite sequence of moves. A typical basic operation advances some of the clocks by one hour and leaves all others unchanged. A repeated operation counts as a single move. It turns out that the group $\mathbb{Z}_{12}^{14}$ is generated by 14 basic operations represented by the vectors shown in the following table.

A	$(1,1,1,1,1,1,1,1,1,0,0,0,0,0)$	$abcdA$	$(1,1,1,1,0,0,0,0,0,1,1,1,1,1)$
aB	$(0,1,1,1,1,1,1,1,1,0,0,0,0,0)$	bA	$(1,0,1,1,1,1,1,1,1,0,0,0,0,0)$
cA	$(1,1,0,1,1,1,1,1,1,0,0,0,0,0)$	dA	$(1,1,1,0,1,1,1,1,1,0,0,0,0,0)$
abA	$(1,1,0,0,0,0,0,0,0,1,1,1,0,1)$	abC	$(0,0,1,1,1,0,1,1,1,0,0,0,0,0)$
bcA	$(1,0,0,1,1,1,0,1,1,0,0,0,0,0)$	bcB	$(0,1,1,0,0,0,0,0,0,1,1,1,1,0)$
cdA	$(1,1,0,0,1,1,1,0,1,0,0,0,0,0)$	cdC	$(0,0,1,1,0,0,0,0,0,1,0,1,1,1)$
adA	$(1,0,0,1,0,0,0,0,0,1,1,0,1,1)$	adB	$(0,1,1,0,1,1,1,1,0,0,0,0,0,0)$

Although the mechanical details need not concern us, you can relate my notation to a real Rubik's Clock—they do occasionally turn up in charity shops—as follows. The vector elements correspond to the 14 clocks, which I shall call $(A,B,C,D,0,1,2,3,4,5,6,7,8,9)$. The clocks on the front are labelled

B	1	A
2	0	4
C	3	D

with 5, 6, 7, 8, 9 on the back of 0, 1, 2, 3, 4 respectively.

The buttons next to the corner clocks are called a, b, c, d, and a move is created by setting them up or down and then manually rotating one of the corner clocks through an integer number of hours. Gearing invisible to the naked eye causes other clocks to move by the same amount. The labels next to the vectors in the table indicate which buttons are *down* and which corner clock is advanced. And when I say 'advanced' I mean that clocks on the front go clockwise but those on the rear go anticlockwise. You might find it helpful to view the back of the thing in a mirror.

With a little experimentation you can discover expressions that move just a single clock other than 0 and 5. For example, $A - aB$ advances just A by one hour, and $bA - abC - A + aB$ affects only clock 1.

This means that you can always restore the system with 14 moves. Given any state of the clocks, first determine how many of A and $abcdA$ you need to restore clocks 0 and 5. Then work out which sequences of moves you now need to restore each non-central clock by itself. Add everything together modulo 12 and you will have a sequence of at most 14 moves that restores all the clocks. To show that it really does work, you can verify that this sequence restores the nine clocks visible in the picture and leaves the other five unchanged:

$$5aB + 2abC + 11adB + 10bA + 3bcA + 6cA + 11cdA + 10dA + 6A.$$

Can the upper bound of 14 be improved?

Solution 253.5 – Integral

Compute $\displaystyle\int \frac{d\theta}{\sin^6\theta + \cos^6\theta}$.

Vincent Lynch

We have

$$(\sin^2\theta + \cos^2\theta)^3 \;=\; 1 \;=\; \sin^6\theta + 3\sin^4\theta\cos^2\theta + 3\sin^2\theta\cos^4\theta + \cos^6\theta.$$

Then

$$\begin{aligned}\sin^6\theta + \cos^6\theta \;&=\; 1 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta) \;=\; 1 - \frac{3}{4}\sin^2 2\theta \\ &=\; 1 - \frac{3}{4}\cdot\frac{1-\cos 4\theta}{2} \;=\; \frac{5 + 3\cos 4\theta}{8},\end{aligned}$$

so that

$$I \;=\; \int \frac{d\theta}{\sin^6\theta + \cos^6\theta} \;=\; 8\int \frac{d\theta}{5 + 3\cos 4\theta}.$$

Substitute $\tan 2\theta = t$; then $\cos 4\theta = \dfrac{1-t^2}{1+t^2}$ and $d\theta = \dfrac{dt}{2(1+t^2)}$. So after a little work we have

$$I \;=\; 2\int \frac{dt}{t^2+4} \;=\; \arctan\frac{t}{2} + C \;=\; \arctan\frac{\tan 2\theta}{2} + C.$$

Solution 253.7 – Quintic roots

Show that the 27th powers of the roots of $x^5 + ax + b$ sum to $90(ab)^3$.

Stuart Walmsley

Let the roots of a quintic be $r_1, r_2, \ldots, r_5$ and let S_n denote the sum of the nth powers of the roots; that is

$$S_n = r_1^n + r_2^n + \cdots + r_5^n.$$

The general quintic can be written

$$y = x^5 - p_1x^4 + p_2x^3 - p_3x^2 + p_4x - p_5, \tag{1}$$

where p_1 is the sum of the roots, p_2 is the sum of all distinct products of two different roots and in general p_n is the sum of all distinct products of n different roots.

For the polynomial under consideration,

$$y = x^5 + ax + b, \tag{2}$$

so that

$$p_1 = p_2 = p_3 = 0, \quad p_4 = a, \quad p_5 = -b.$$

The problem in hand is then to prove $S_{27} = 90(ab)^3$.

The S_n are related by recurrence relations known as Newton formulae, one subset of which is readily derived. Substituting one of the roots r_j into the polynomial (1) gives

$$r_j^5 - p_1r_j^4 + p_2r_j^3 - p_3r_j^2 + p_4r_j - p_5 = 0. \tag{3}$$

Summing over j and rearranging gives

$$S_5 = p_1S_4 - p_2S_3 + p_3S_2 - p_4S_1 + 5p_5,$$

which is one of the Newton formulae. This may be extended to higher values of n. Multiply (3) by r_j^{n-5} and sum over j to give

$$S_n = p_1S_{n-1} - p_2S_{n-2} + p_3S_{n-3} - p_4S_{n-4} + p_5S_{n-5}, \qquad n > 5. \tag{4}$$

The derivation of the formulae for lower values of n is much more complicated. Here, their form is given directly, emphasizing their regularity. In an appendix, an indication is given of a derivation.

As n is reduced one by one, the number of terms is reduced one by one, keeping a characteristic pattern.

$$\begin{aligned} S_5 &= p_1S_4 - p_2S_3 + p_3S_2 - p_4S_1 + 5p_5, \\ S_4 &= p_1S_3 - p_2S_2 + p_3S_1 - 4p_4, \\ S_3 &= p_1S_2 - p_2S_1 + 3p_3, \\ S_2 &= p_1S_1 - 2p_2, \\ S_1 &= p_1. \end{aligned} \tag{5}$$

The terms with higher values of n fit this pattern if p_n for $n > 5$ is taken to be zero. Then, for example,

$$\begin{aligned} S_6 &= p_1S_5 - p_2S_4 + p_3S_3 - p_4S_2 + p_5S_1 \quad (-\ 6p_6), \\ S_7 &= p_1S_6 - p_2S_5 + p_3S_4 - p_4S_3 + p_5S_2 \quad (-\ p_6S_1 + 7p_7) \end{aligned} \tag{6}$$

and the terms in brackets are zero.

When the formulae in (5) and (6) are applied to the specific polynomial (2), the results are as follows.

$$\begin{array}{ccccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 \\ 0 & 0 & 0 & -4a & -5b & 0 & 0 \end{array} \tag{7}$$

Higher members are then found by using (4) adapted to polynomial (2):

$$\begin{aligned} S_n &= p_1S_{n-1} - p_2S_{n-2} + p_3S_{n-3} - p_4S_{n-4} + p_5S_{n-5}, && n > 5, \\ -S_n &= aS_{n-4} + bS_{n-5}, && n > 5. \end{aligned}$$

The value for S_{27} can be built up as a Pascal triangle by repeated use of (7), each line being equivalent to $-S_{27}$.

$$\begin{gathered} -S_{27} \\ aS_{23} + bS_{22} \\ -(a^2S_{19} + 2abS_{18} + b^2S_{17}) \\ a^3S_{15} + 3a^2bS_{14} + 3ab^2S_{13} + b^3S_{12} \\ -(a^4S_{11} + 4a^3bS_{10} + 6a^2b^2S_9 + 4ab^3S_8 + a^5S_7) \\ a^5S_7 + 5a^4S_6 + 10a^3b^2S_5 + 10a^2b^3S_4 + 5ab^4S_3 + a^5S_2 \end{gathered}$$

So finally

$$S_{27} = a^5S_7 + 5a^4S_6 + 10a^3b^2S_5 + 10a^2b^3S_4 + 5ab^4S_3 + a^5S_2.$$

Using the values of S_n, this gives

$$S_{27} = -\ 10a^2b^2(aS_5 + bS_4) = 10a^2b^2(5ab + 4ab) = 90(ab)^3,$$

which is the required result.

Appendix

An indication is given of the derivation of the Newton formula for a quintic with $n \le 5$. There are two forms:

$$\begin{aligned} y &= x^5 - p_1x^4 + p_2x^3 - p_3x^2 + p_4x - p_5 \\ &= (x-r_1)(x-r_2)(x-r_3)(x-r_4)(x-r_5). \end{aligned}$$

For the factored form,

$$\frac{dy}{dx} = y\left(\frac{1}{x-r_1} + \frac{1}{x-r_2} + \frac{1}{x-r_3} + \frac{1}{x-r_4} + \frac{1}{x-r_5}\right).$$

Since $y = 0$ when $x = r_j$,

$$y = (x^5 - r_j^5) - p_1(x^4 - r_j^4) + p_2(x^3 - r_j^3) - p_3(x^2 - r_j^2) + p_4(x - r_j).$$

Then

$$\begin{aligned} \frac{y}{x-r_j} &= x^4 + r_jx^3 + r_j^2x^2 + r_j^3x + r_j^4 - p_1(x^3 + r_jx^2 + r_j^2x + r_j^3) \\ &\quad + p_2(x^2 + r_jx + r_j^2) - p_3(x + r_j) + p_4 \\ &= x^4 + (r_j - p_1)x^3 + (r_j^2 - p_1r_j + p_2)x^2 \\ &\quad + (r_j^3 - p_1r_j^2 + p_2r_j - p_3)x + (r_j^4 - p_1r_j^3 + p_2r_j^2 - p_3r_j + p_4). \end{aligned}$$

Summing over j gives dy/dx,

$$\begin{aligned} \frac{dy}{dx} &= 5x^4 + (S_1 - 5p_1)x^3 + (S_2 - p_1S_1 + 5p_2)x^2 \\ &\quad + (S_3 - p_1S_2 + p_2S_1 - 5p_3)x + (S_4 - p_1S_3 + p_2S_2 - p_3S_1 + 5p_4). \end{aligned}$$

This can be compared with

$$\frac{dy}{dx} = 5x^4 - 4p_1x^3 + 3p_2x^2 - 2p_3x + p_4,$$

leading to

$$\begin{aligned} S_1 - 5p_1 &= -4p_1, \\ S_2 - p_1S_1 + 5p_2 &= 3p_2, \\ S_3 - p_1S_2 + p_2S_1 - 5p_3 &= -2p_3, \\ S_4 - p_1S_3 + p_2S_2 - p_3S_1 + 5p_4 &= p_4. \end{aligned}$$

The Newton formulae then follow:

$$\begin{aligned} S_1 &= p_1, \\ S_2 &= p_1S_1 - 2p_2, \\ S_3 &= p_1S_2 - p_2S_1 + 3p_3, \\ S_4 &= p_1S_3 - p_2S_2 + p_3S_1 - 4p_4. \end{aligned}$$

Vincent Lynch

Notation: s_k denotes the sum of the kth powers of the roots; σ_k denotes the symmetric function of order k. For example,

$$\sigma_2 \;=\; \alpha\beta + \alpha\gamma + \alpha\delta + \alpha\epsilon + \beta\gamma + \beta\delta + \beta\epsilon + \gamma\delta + \gamma\epsilon + \delta\epsilon$$

in this case. We can start by writing down the values of the symmetric functions:

$$\sigma_1 = \sigma_2 = \sigma_3 = 0, \quad \sigma_4 = a, \quad \sigma_5 = -b.$$

Then by using Newton's formula, we can calculate the first few values of s_k:

$$s_1 = \sigma_1 = 0, \quad s_2 = \sigma_1^2 - 2\sigma_2 = 0, \quad s_3 = \sigma_1 s_2 - \sigma_2 s_1 + 3\sigma_3 = 0,$$

$$s_4 = \sigma_1 s_3 - \sigma_2 s_2 + \sigma_3 s_1 - 4\sigma_4 = -4a,$$

$$s_5 = \sigma_1 s_4 - \sigma_2 s_3 + \sigma_3 s_2 - \sigma_4 s_1 + 5\sigma_5 = -5b.$$

This last use shows how the formula works. Now, we substitute a root into the equation $\alpha^5 = -a\alpha - b$ and multiply by α^{n-5} to give $\alpha^n = -a\alpha^{n-4} - b\alpha^{n-5}$. And then sum with the other four similar expressions for the other roots to give

$$s_n \;=\; -\,a s_{n-4} - b s_{n-5}.$$

Now we just need to use this iteration formula a rather large number of times to give

$$s_6 = 0, \quad s_7 = 0, \quad s_8 = 4a^2, \quad s_9 = 9ab, \quad s_{10} = 5b^2, \quad s_{11} = 0, \quad s_{12} = -4a^3,$$

$$s_{13} = -13a^2b, \quad s_{14} = -14ab^2, \quad s_{15} = -5b^3, \quad s_{16} = 4a^4, \quad s_{17} = 17a^3b,$$

$$s_{18} = 27a^2b^2, \quad s_{19} = 19ab^3, \quad s_{20} = -4a^5 + 5b^4, \quad s_{21} = -21a^4b,$$

$$s_{22} = -44a^3b^2, \quad s_{23} = -46a^2b^3, \quad s_{24} = 4a^6 - 24ab^4,$$

$$s_{25} = 25a^5b - 5b^5, \quad s_{26} = 65a^4b^2,$$

and finally $s_{27} = 90a^3b^3 = 90(ab)^3$. I would love to have found a quick method for this. Perhaps someone will.

TF writes. I don't know of any easier way to get this result. I was just curious to see a simple expression appearing every so often. From the above we see that $s_9 = 9ab$, $s_{18} = 27(ab)^2$, $s_{27} = 90(ab)^3$, and just when you are thinking s_{9n} is going to be a constant times $(ab)^n$ we find that $s_{36} = 315a^4b^4 - 4a^9$ and $s_{45} = 1134a^5b^5 - 5b^9 - 45a^{10}b$. Oh well.

Solution 251.6 – Integer

Show that for positive integer n, $D(n)$ is an integer, where

$$D(n) \;=\; \frac{\left(10^{9n10^{n-1}-1}-1\right)\left(100^n-10^n+10\right)+9}{\left(10^n-1\right)^2}. \qquad (1)$$

Tony Forbes

This is one of those situations where I started with the answer and went on to derive a hideous formula after a considerable amount of work. After trying a few values of n,

$$\begin{aligned} D(1) &= 123456789, \\ D(2) &= 101112131415161718192021\ldots 90919293949596979899, \end{aligned}$$

you can probably guess what $D(n)$ will look like in general. However, I cannot see any easy way of proving this directly. So I put it in M500 to give readers a chance to tell me that I had overlooked something obvious.

Turning the problem around, I shall write down the numbers in order from 10^{n-1} to $10^n - 1$ without leaving any gaps and then try to compute the resulting $n(10^n - 10^{n-1})$-digit number, $E(n)$, say. Observing that the units digit of k is in position $n(10^n - 1 - k)$ from the right-hand end of $E(n)$, we have

$$E(n) \;=\; \sum_{k=10^{n-1}}^{10^n-1} k10^{n(10^n-1-k)} \;=\; 10^{n(10^n-1)} \sum_{k=10^{n-1}}^{10^n-1} k10^{-nk}.$$

By differentiating $\sum_{k=0}^{a} 10^{-kn} = (10^n - 10^{-an})/(10^n - 1)$ we obtain

$$\sum_{k=0}^{a} k10^{-kn} \;=\; \frac{10^n - 10^{-an}\big(10^n + a(10^n-1)\big)}{(10^n-1)^2}. \qquad (2)$$

Unfortunately I can't think of anything better to do than substitute $a = 10^n - 1$ and $a = 10^{n-1} - 1$ in (2) and subtract. Thus

$$\begin{aligned} E(n) \;=\; \frac{10^{n(10^n-1)}}{(10^n-1)^2}\Big(&-10^{-n(10^n-1)}\big(10^n + (10^n-1)^2\big) \\ &+ 10^{-n(10^{n-1}-1)}\big(10^n + (10^{n-1}-1)(10^n-1)\big)\Big), \end{aligned}$$

which eventually simplifies to (1) as the M500 reader can verify for himself. (The masculine form is appropriate. I am told that no woman in her right mind would waste time on such a futile exercise.)

M500 Mathematics Revision Weekend 2014

The M500 Revision Weekend 2014 will be held at

Yarnfield Park Training and Conference Centre, Yarnfield, Staffordshire ST15 0NL

between 16th and 18th May 2014.

For bookings received before 16th April, the standard cost, including accommodation (with en suite facilities) and all meals from dinner on Friday evening to lunch on Sunday, is £290. The standard non-residential cost, which includes Saturday and Sunday lunch, is £155. An additional administration fee of £20 will be applied to all bookings received after this date.

There will be an early booking period to the end of February with a discount of £25 for members and £15 for non-members. Members may make a reservation with a £25 deposit, with the balance payable at the end of February. Non-members must pay in full at the time of application and all applications received after the 28th February must be paid in full before the booking is confirmed. Members will be entitled to a reduced discount of £10 for all applications received between 1st March and 15th April.

A shuttle bus service will be provided from Stone station on Friday afternoon, with a return service to Stone station after the final teaching session on Sunday. This will be free of charge, but seats will be allocated for each service and must be requested before 1st May. There is free on-site parking for those travelling by private transport.

For full details and an application form see the Society web site at www.m500.org.uk/may.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for undergraduate and postgraduate mathematics OU modules, with the exception of M347, subject to the availability of tutors and sufficient applications.

Cambridge group theorist Simon Phillips Norton enjoys this very interesting property: $SIMON \times P = NORTON$ has a unique solution in distinct decimal digits. Does any M500 reader have a similar property?

If your first name is longer than your surname, try division ($\div$). You don't have to use all ten digits. Only valid decimal representations should be allowed; so in our example we must insist that $N \neq 0$.

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