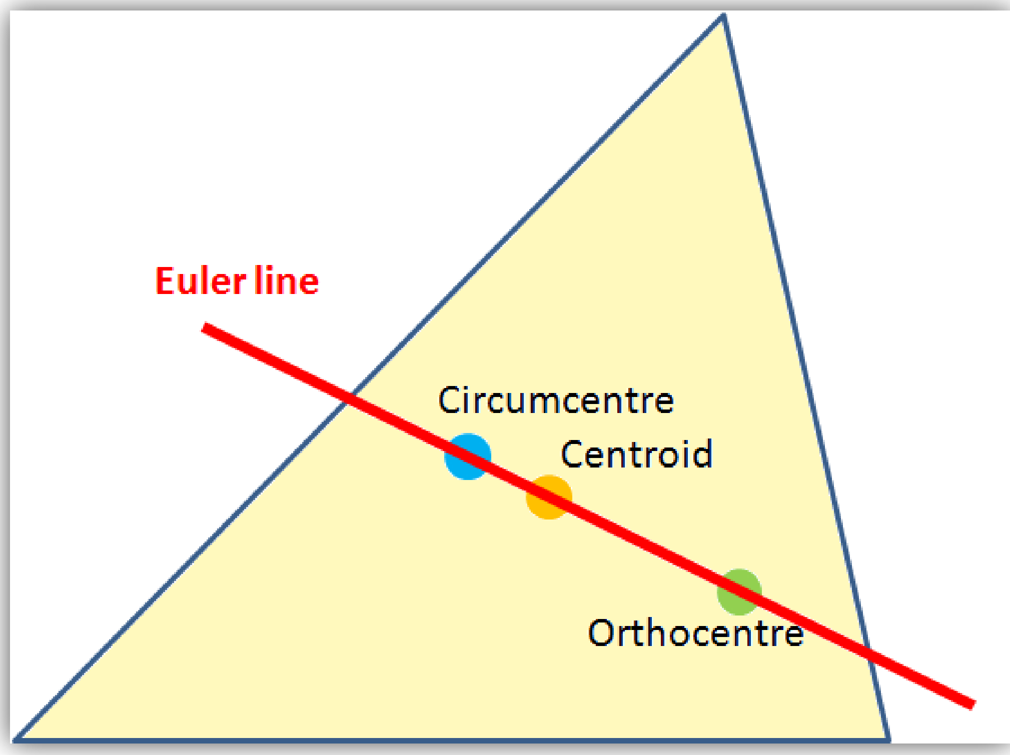
MATHEMATICS ENHANCEMENT COURSE

MATHEMATICAL PROOFS

EULER LINE



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> ***"Proofs are to mathematics what spelling (or even calligraphy) is to poetry. Mathematical works do consist of proofs, just as poems do consist of characters." (Arnold, 2000, p.408)***

Proof is the fundamental and the spirit of mathematics (CadwalladerOlsker, 2011, p.35). According to the MEP Pupil Text: Mathematical Proof (2011, p.5), Mathematical proof is a convincing demonstration with a chain of reasons that establishes the truth of a particular statement clear of any doubt.

The Greek mathematician Euclid, in his book 'The Elements', written about 300BC, covered many mathematical proofs, especially in the field of geometry. This book was used as the main textbook for teaching mathematics until the early 20th century. According to the MEP Pupil Text: Mathematical Proof (2011, p.5), proofs in 'The Element' begin with a clear statement of any initial assumptions and they then explain by means of the logical operations that if those assumptions are true then the desired conclusion must also be true. Mrs. Glosser's Math Goodies (1998-2011) argues that the statements are the sentences that are either true or false but not both, for example, 8 is an even integer and 4 is an odd integer. These are two statements where the first one is true, and the second is false. According to the MEP Pupil Text: Mathematical Proof (2011, p.5), the statement that is proved is called a theorem, for example, Pythagoras' theorem. Once a theorem is proved, it can be used as a stepping stone to prove further statements or in a proof of another theorem.

When writing a proof, the argument may use self-evident facts, which are called mathematical axioms. They are logical statements that are assumed to be true. Therefore a proof of a theorem is a finite sequence of claims which are being derived logically using mathematical axioms, statements and the theorems that have been already proved. An unproven proposition that is assumed to be true is known as a conjecture. According to Krantz (2007) the only way to be certain that the conjecture is true, is by presenting a valid mathematical proof that may contains lots of data and require a deeper understanding of the problem. Therefore being able to write a valid proof indicates your detailed knowledge about the theory and the deeper thinking about, and understanding of, the subject and the problem (Krantz, 2007, p.2).

Euclidean Geometry

Euclidean Geometry is the study of flat space on the basis of axioms and theorems employed by the Greek mathematician Euclid in 300BC, whose 'The Elements' is the earliest known logical discussion of geometry.
In Euclidean geometry it has been proved that unlike squares and circles, triangles have many centres. The incentre, centroid, circumcentre, and orthocentre are four triangle centres known to the ancient Greeks. A fifth centre, found much later, is the Fermat point. Thereafter, points now called nine-point centre, symmedian point, Gergonne point, and Feuerbach point were added to the literature. These centres are generated by different triangular properties and they form a bridge between elementary and advanced Euclidean geometry (Baker, 2006).

Triangular Geometry

Triangular geometry is the study of the properties of triangles. A triangle with its three vertices and three sides is actually a very remarkable object. Crelle (1821) states that, "It is indeed a wonder that so simple a figure as the triangle is so inexhaustible in its properties,"

Each of the four triangle centres known to the ancient Greeks is generated because of four triangular properties, such as:

- medians of a triangle generate the centroid
- altitudes of a triangle generate the orthocentre
- perpendicular bisectors of a triangle generate the circumcentre
- angle bisectors of a triangle generate the incentre

The median of a triangle is the line segment from the vertex to the midpoint of the opposite side of a triangle. There are three medians in each triangle. The interesting property of these three medians is that they are concurrent lines. When two or more line segments have a point in common, they are said to be concurrent lines. This point in common is said to be the point of concurrency. This point of concurrency because of the medians of a triangle is called the centroid of a triangle. The centroid divides each median in the ratio 2:1 as the line segment from the vertex to the centroid is twice the length of the line segment from the centroid to the opposite side of the triangle. It is an interesting fact that the centroid of a triangle is always inside the triangle.

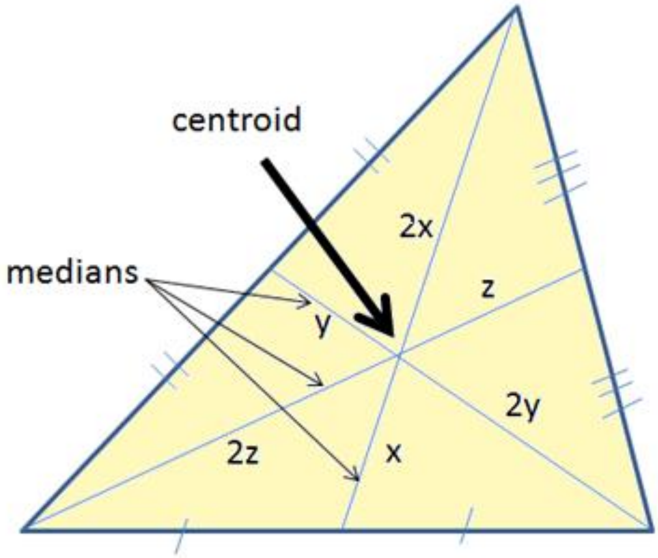


Figure 1
Centroid of a triangle (Adopted by Bowman, 2009)

The altitude of a triangle is the straight line going through the vertex and perpendicular to the line containing the base of the triangle. It is also called the perpendicular height of a triangle. There are three altitudes in each triangle. It is common to assert that the altitudes of a triangle are concurrent, that is passing through a point. This point in common is called the orthocentre of a triangle. An important fact of this centre is that it is not necessarily inside the triangle; it could be outside the triangle as well.

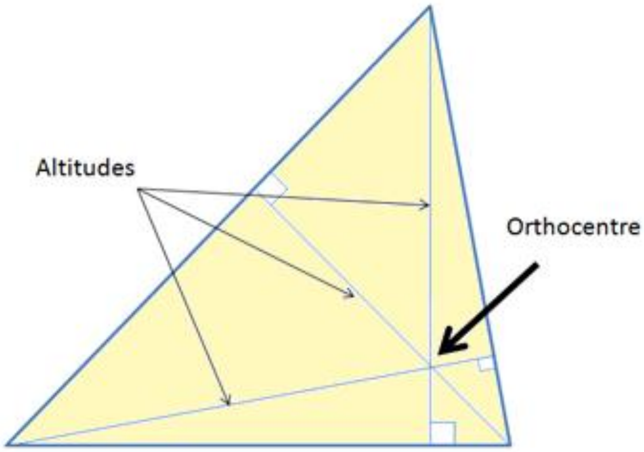


Figure 2
Orthocentre inside a triangle
(Adopted by Bowman, 2009)

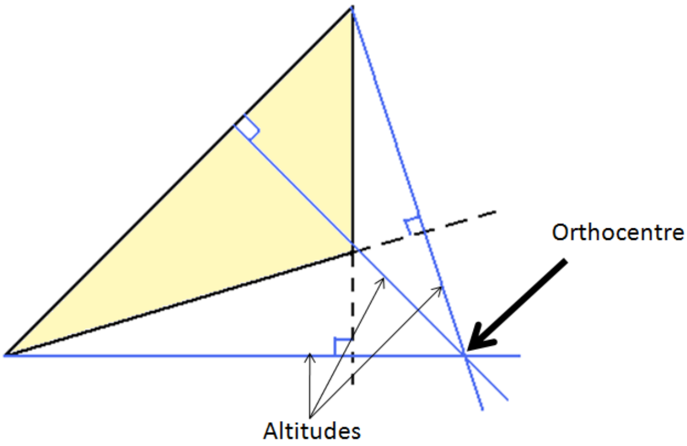


Figure 3
Orthocentre outside a triangle
(Adopted by Bowman, 2009)

The perpendicular bisector of a triangle is the straight line passing through the midpoint of a side of a triangle and is perpendicular to it. Same as the medians and the altitudes of a triangle, the perpendicular bisectors of the sides of a triangle are always concurrent too. This point of concurrency is said to be the circumcentre of the triangle. This is also the centre of the circumscribe circle of the triangle. Similar to the orthocentre, the circumcentre is not necessarily inside the triangle.

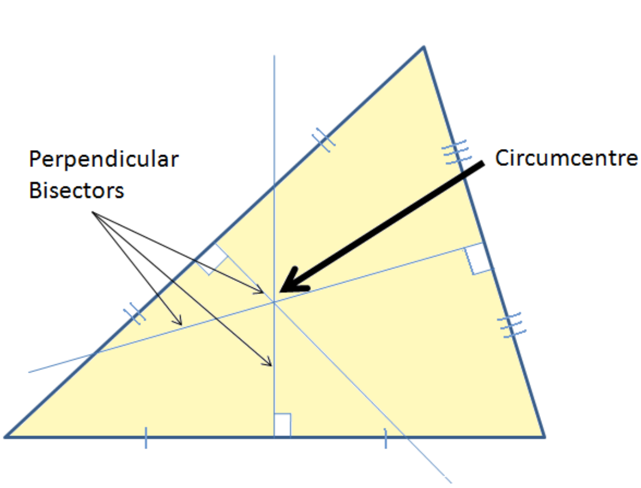


Figure 4
Circumcentre inside a triangle
(Adopted by Bowman, 2009)

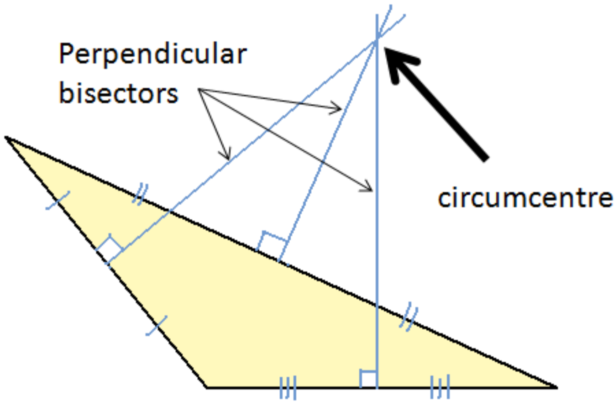


Figure 5
Circumcentre outside a triangle
(Adopted by Bowman, 2009)

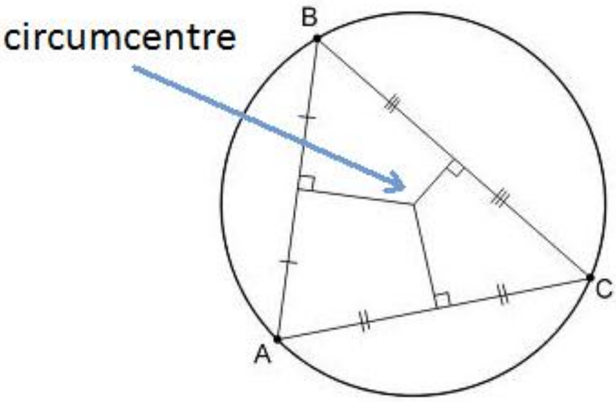


Figure 6
The circumcentre is the centre of the circumscribe circle of the triangle (Verniana, 2009)

The angle bisector of a triangle is the line segment going through the vertex and divides the angle into two equal parts. In a triangle, there are three such lines. Again the angle bisectors of a triangle pass through a same point which means they are concurrent and this point of concurrency is said to be the incentre of the triangle. Incentre always be positioned inside the triangle. It is also the centre of the incircle, the circle inscribed in the triangle.

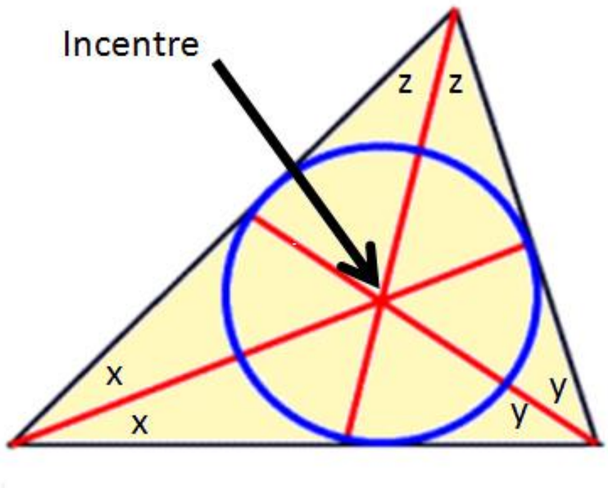


Figure 7
Incentre of a triangle (Adopted by Bowman, 2009)

While these four triangle centres were known to the ancient Greeks, they somehow missed the simple relationship between three of them which are the centroid, orthocentre and circumcentre. This relationship was discovered by the Swiss mathematician Leonhard Euler in the 18th century (Baker, 2006).

The Euler Line

In the 18th century, the Swiss mathematician Leonhard Euler (1707–1783) accidentally noticed that three of the many centres of a triangle are always collinear, that is, they always lie on a straight line. He discovered this property while he was looking for another way of constructing a triangle using the classical triangle centres which were found by the ancient Greeks. This line has come to be named after him - the Euler line. The three interesting centres that have this surprising property are the triangle's centroid, circumcentre and orthocentre (Math open reference, 2003).

The Euler Line Theorem

The Orthocentre, the Circumcentre, and the Centroid of any triangle are collinear. Furthermore, Centroid is between Orthocentre and Circumcentre (unless the triangle is an equilateral triangle, in which case the three points coincide) and the distance between Centroid and the Orthocentre is twice as the distance between Centroid and the Circumcentre. (Baker, 2006)

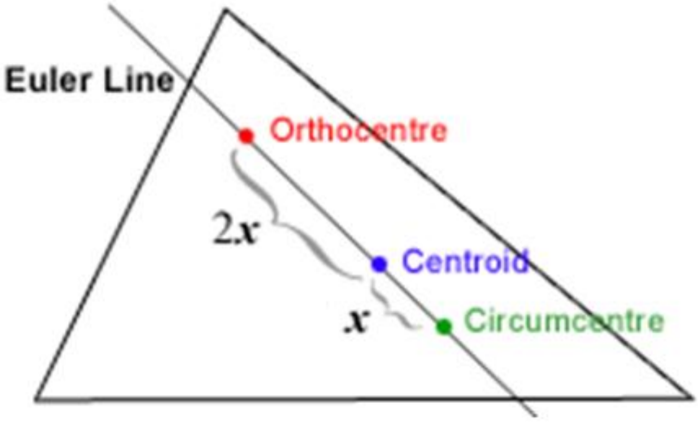
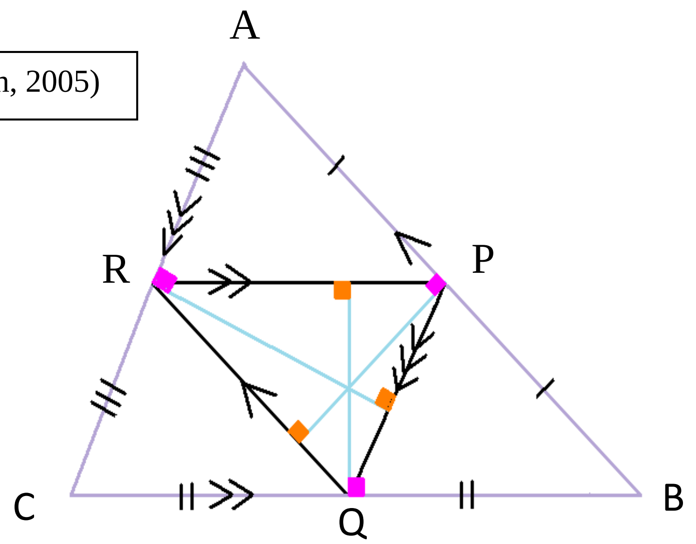


Figure 8
Euler Line (not dated)

Prove the existence of the Euler Line to a year 10 class

Proving the existence of the Euler line means to prove that the orthocentre, centroid and the circumcentre of any triangle are always collinear.

Diagram 1 (Adopted by Polymath, 2005)

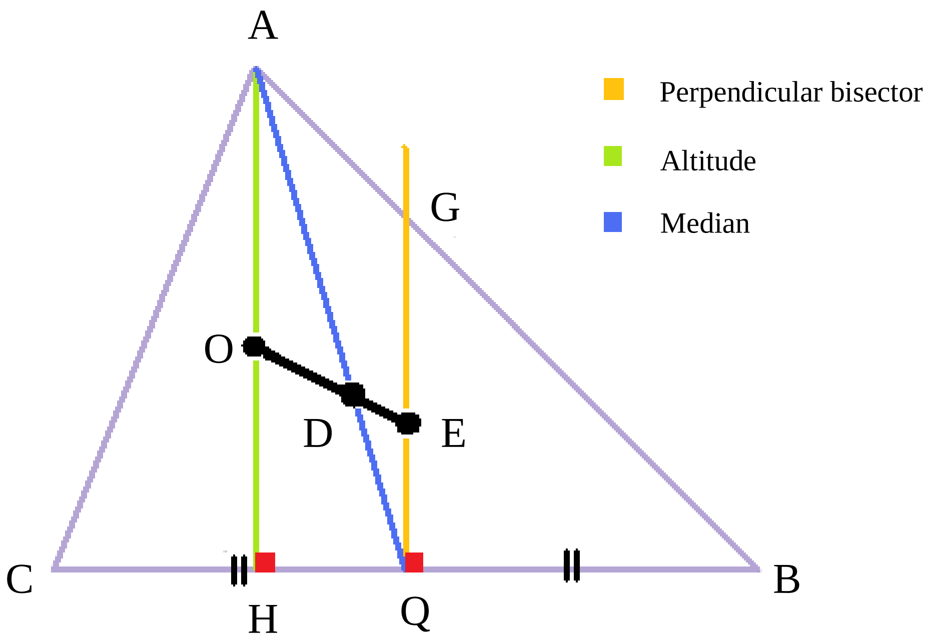


Lemma 1:

- Since the segments connecting the midpoints (P,Q,R) of the sides of the triangle ABC are half the length of the parallel third side (*Theorem 56, Midpoint Theorem – 'The Elements')*, the triangle formed by all three of those segments (PQR triangle) is similar to the ABC triangle in a 2:1 ratio.

- The lines that serve as the perpendicular bisectors of the sides of the ABC triangle are clearly altitudes of the PQR triangle (because AB//RQ, BC//PR, AC//PQ). Therefore, the circumcentre of the ABC triangle coincides with the orthocentre of the PQR triangle.

Diagram 2 (Adopted by Polymath, 2005)



Lemma 2:

In the above diagram, AH is an altitude containing the orthocentre (O) of the ABC triangle. AQ is a median containing the centroid (D) of the ABC triangle. QG is the perpendicular bisector of BC, and it contains the circumcentre (E) of the ABC triangle. Connecting D to O and to E gives 2 segments that we want to prove to belong to the same line. This can be proved by proving that angle ADO is congruent to angle QDE, which could only happen if O, D, E were collinear, since O and E must fall on opposite sides of the median.

To prove those angles congruent, by showing that the triangles ADO and QDE are similar:

1. Since the altitude and the perpendicular bisector are clearly parallel, angle OAD and angle DQE are congruent.
2. AD and DQ stand in a ratio of 2:1, since that's one of the basic properties of a centroid.
3. If you consider the PQR triangle of ABC triangle in the diagram 1, it is clear that AO and QE are precisely corresponding segments of the ABC and PQR triangles (since E is the orthocentre of the PQR triangle and O is the circumcentre of the ABC triangle). Thus by lemma 1, AO:QE = 2:1.

This proves the triangles are similar (SSA – side,side,angle), which means the desired angles (Angles ODA & QDE) are congruent, so the points O, D, E are collinear and the Euler line exists. (Polymath, 2005)

<u>Some Properties of the Euler line</u>

- The centroid is one third of the way from the circumcentre to the orthocentre

This is what really makes the Euler line so famous. Even when you vary the shape of the triangle ABC, the relative distances between the points circumcentre (O), orthocentre (H) and centroid (G) of the triangle remain the same as OG:OH= 1:3.

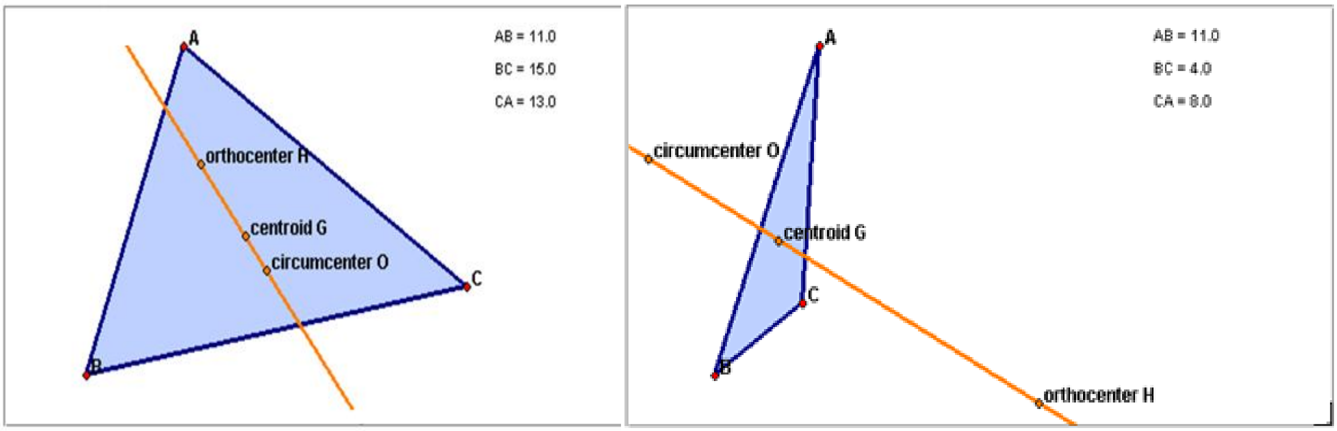


Figure 9
Screen shot to show the relative distances between the points the circumcentre, orthocentre and centroid (Properties of the Euler line, 2003)

- The circumcentre, orthocentre, and the centroid of a triangle coincide if and only if the triangle is equilateral (triangle in which all three sides are equal)
- **The circumcentre, orthocentre, and the centroid of a triangle coincide if and only if the triangle is equilateral (equilateral triangle means a triangle which all three sides are equal)**

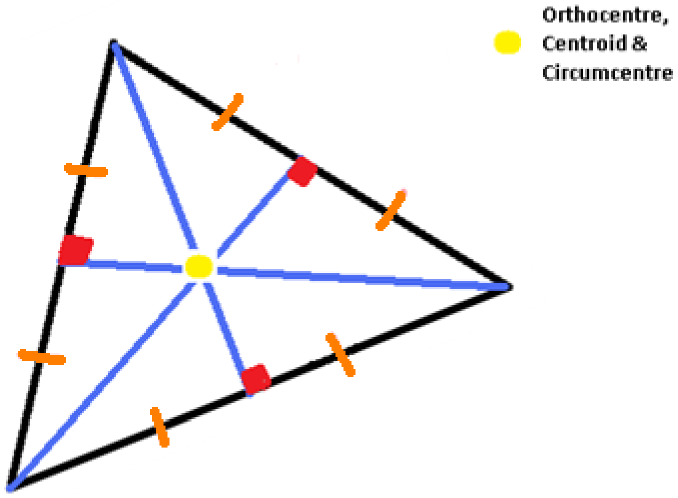


Figure 10
In an equilateral triangle centroid, orthocentre and circumcentre coincide
(Adopted by properties of the Euler line, 2003)

This is because in an equilateral triangle medains, altitudes and perpendicular bisectors of the triangle are all same. Therefore the three triangle centres that are generated from those properties coincide (lie at the same point).

Conclusion

By the end of 19th century Euclidean geometry was considered as one of the valuable achievement of mathematics, and the Euler line was one of its precious findings. The story behind Euler's unexpected discovery, the Euler line, is comparable to Columbus's "discovery" of America because both of these discoveries were made while looking for something else. Euler was trying to find a new way to construct a triangle, using the most interesting triangular centres, known to the ancient Greeks. Even though Euler knew what he discovered, he did not realize how important it would be, so he neither named it nor went back again to study it further (Sandifer, 2008). The Euler line is an extremely beautiful object which relates three special points in a triangle together in a completely unexpected way. As such I think it is an interesting and challenging topic to be taught in the classroom for good effect.

Word count: 2051

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