Leonhard Euler and (some of) his work

Euler's Greatest Hits – Number theory, Graph theory, Real analysis, Triangles, Complex analysis and much more….

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Euler's Greatest Hits – Number theory

Euler worked without calculators or computers, and so shall we except where * is used.

Pierre de Fermat conjectured in 1650 that all numbers of the form

$F_n = 2^{(2^n)} + 1$ *are prime for non-negative integer n.*

Thus $F_0 = 2^1+1 = 3$ which is prime.

1. Evaluate F_1, F_2, F_3, F_4.
2. Test this conjecture for n = {1, 2, 3}.
3. Describe a procedure for deciding whether F_4 is prime. What difficulties arise?

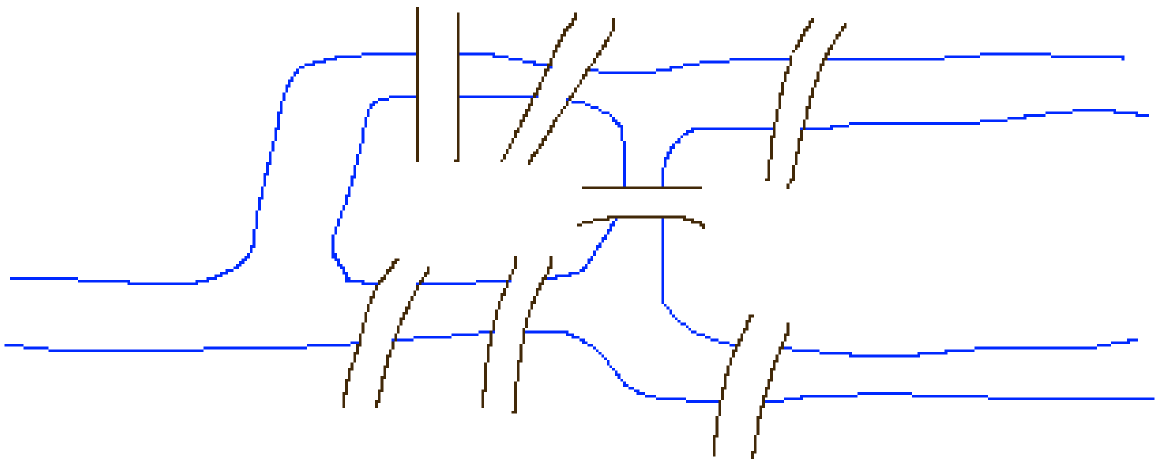
Euler proved in 1732 that F_5, which equals 4 294 967 297, is composite (not prime) by showing that 641 is a factor. In doing so he refuted Fermat's conjecture and did work which led to some huge advances in* ***number theory****.*

4. You may like to attempt Euler's proof, which does not involve random or systematic division. Start with the prime factorisation of 640 and work from there using modulo arithmetic.

Ask for a set of proof cards for you to sort if you get stuck.

Euler's Greatest hits – Graph Theory

The town of Konigsberg on the Baltic Sea was divided into four parts by the river Pregel, and connected by seven bridges. Is it possible to tour Königsberg along a path that crosses every bridge once, and at most once? You can start and finish wherever you want, not necessarily in the same place.



1. Can you do it? If so, shout out loud. Or prove that it is not possible? Try representing the four blocks of land as nodes (vertices), and the bridges as lines (edges) joining the vertices.
2. Explain to your table your results. Then construct a picture (graph) that is possible, and a different one that is not.
3. Now consider the islands or blocks of land. Can we visit each one once, and return to the same place? If yes, how many such paths are there?
   *A path which traverses every edge once and once only is an **Eulerian** path. A path which visits every vertex once and once only is a **Hamiltonian** path.*
   *Euler founded **graph theory** with his work on this problem c1735.*
   Konigsberg is now Kaliningrad in Russia, and two of the bridges were bombed and destroyed in World War II.
4. Remove two bridges of your choice and see if you can now find an Eulerian path.

Euler's Greatest Hits – Analysis

Euler popularised and extended the use of infinite series in analysis

The Basel problem asks for the precise summation of the reciprocals of the squares of the natural numbers, i.e. the precise sum of the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{n \to +\infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right).$$

To follow Euler's argument, the expansion of the sine function is:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

Dividing through by *x*, we have (1):

Now, the roots (intersections with the x-axis) of $\sin(x)/x$ occur precisely at $x = n \cdot \pi$ where $n = \pm 1, \pm 2, \pm 3, \ldots$

If we know that the roots of an equation are a and b, we know that the equation is (x-a)(x-b) = 0, or (1 – x/a)(1-x/b) = 0

So, knowing the roots of the LHS we can write the RHS as an infinite series of factors(2)

If we formally multiply out this product and collect all the x^2 terms, the x^2 coefficient of $\sin(x)/x$ is (3)

But from the original infinite series expansion of $\sin(x)/x$, the coefficient of x^2 is −1/(3!) = −1/6. These two coefficients must be equal, hence (4):

Rearranging the equation gives the exact sum of the reciprocals of the positive square integers(5)

Incidentally, this number, which approximates to 1.644934…., is Riemann (2), massively important in the study of the distribution of prime numbers.*

Euler's Greatest Hits – On triangles (1)

We shall try to prove a very famous result of Euler's concerning the properties of triangles. Much of the preliminary work is classical geometry.

1. Draw any triangle. Join the vertices of the triangle to the midpoints of the opposite sides. Do this several times. What two things do you notice?

   *The lines you have drawn are the **medians** of the triangle.*

2. Can you prove or disprove (i) that the medians of a triangle are concurrent (intersect at one point)? (ii) that they intersect in the same ratio?

There are several approaches to these problems.

3. Consider the regions into which the medians divide the triangle. Make a conjecture and prove it, perhaps by elementary formulas about triangles.

*The point where the medians intersect is called the **centroid** or **geocentre**.*

4. Why do you think it is called the geocentre? Give an argument to convince someone else.

***Leonhard Euler** 15 April 1707 – 18 September 1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function. He is also renowned for his work in mechanics, fluid dynamics, optics, and astronomy.*

Euler spent most of his adult life in St. Petersburg, Russia, and in Berlin, Prussia. He is considered to be the pre-eminent mathematician of the 18th century, and one of the greatest mathematicians ever. He is also one of the most prolific mathematicians ever; his collected works fill 60–80 volumes. A statement attributed to Pierre-Simon Laplace expresses Euler's influence on mathematics: "Read Euler, read Euler, he is the master of us all."

Euler’s Greatest Hits – Complex analysis

A Taylor series expansion for any real or complex function can be found by using the power series:

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots.$$

A Taylor series about 0, when a=0, is called a Maclaurin series.

1. Work out Maclaurin series for sin x, e^x, and cos x
2. By substituting ix for x, write down a series for e^{ix}
3. Now, by multiplying and adding, form a series for cos x + i sin x. What do you notice?

This is a wonderful relationship. Euler also introduced our modern approach to trigonometric functions and their notation.

4. What happens if x = π ?

This is the very famous and beautiful Euler Identity, which brings together the five most important numbers in mathematics – the additive identity; the multiplicative identity; the base of natural logarithms (named by and for Euler); the ratio of the circumference of a circle to its diameter; and the square root of minus one.

- *Euler was clearly a genius from an early age. In 1723 he gained his masters’ degree, with a dissertation comparing the natural philosophy systems of Newton and Descartes.*
- *On his father's wishes, Euler furthered his education by enrolling in the theological faculty, but devoted all his spare time to studying mathematics.*
- *By 1740 he had lost the sight of one eye.*
- *He fathered thirteen children, of whom five survived infancy.*

Mathematical notation created or popularized by Euler

e	the base of the natural logarithm, a constant equal to 2.71828...
i	the "imaginary unit", equal to the square root of -1
$f(x)$	the function f as applied to the variable or argument x
Σ	sigma, the sum or total of a set of numbers
a, b, c x, y, z	a, b, c are constants, such as the sides of a triangle; x, y, z are variables or unknowns in an equation
sin, cos, tan, *cot, sec, csc*	trigonometric functions for sine, cosine, tangent, cotangent, secant, cosecant
π	pi, the ratio of a circle's circumference to its diameter

In 1766, Euler accepted an invitation from Catherine the Great to return to the St. Petersburg Academy, and spent the rest of his life in Russia.

However, his second stay in the country was marred by tragedy, including a fire in 1771 which cost him his home (and almost his life); the loss of sight in his second eye the same year; and the death in 1773 of his wife of 40 years, Katharina. He later married Katharina's half-sister, Salome Abigail, and this marriage would last until his death from a brain haemorrhage in 1783.

Euler's Greatest Hits – On Triangles (2)

*An **altitude** of a triangle is a line drawn from a vertex perpendicular to the opposite side.*

1. Investigate the altitudes of a triangle by drawing different acute and obtuse angled triangles and their altitudes. Make a conjecture and prove it if you can.

*The altitudes of a triangle meet at the **orthocentre**.*

2. One and only one circle can be drawn through the vertices of any triangle. How might you find the centre of such a circle?

This is the 'exploding firework' problem. Three children light a firework then run away in straight lines in different directions. When the firework explodes they are at points A, B, C. How do you locate the firework, mathematically?

*The centre of this circle is called the **circumcentre**.*

3. What can you say about the geocentre, orthocentre and circumcentre of an equilateral triangle?

4. Draw any triangle on a large piece of paper. Find the three centres, carefully and accurately measuring where necessary. What do you notice?

5. Make a conjecture and prove it.

6. Have a guess as to what this is called.

If you have time, read this short article. There is a problem at the end.

The distribution of prime numbers

"*Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the mind will never penetrate" (Leonhard Euler)*

Prime numbers, the basic building blocks of number, display a total lack of regularity to the casual, or even learned, observer. To mathematicians, who seek pattern in everything, patterns or structures of primes can be found – somewhat diffuse, usually approximate, and always difficult, but patterns nevertheless. Through the prime number theorem, the use of integrals and floor functions, and Dirichlet's work on primes in arithmetic progressions, to the Riemann hypothesis, the search continues for ultimate answers – ultimate answers which, when one considers the discoveries of Godel in the twentieth century, may not be attainable.

Early mathematical findings on primes were extensive: there could be only one even prime; primes seemed to become rarer as numbers got larger; and Euclid laid out a seminal proof by contradiction that there exist an infinity of primes. Sometimes two primes occurred with a difference of two – e.g. 101, 103. It is a measure of the complexity of research into prime numbers that it is not known whether there is an infinity of such twin primes.

After the Greeks, little progress was made in the study of primes until the seventeenth century (Fermat wrongly thought he had found a formula for prime numbers, as shown earlier in this booklet); the flowering of European mathematics in the eighteenth century (Euler, Lagrange, Gauss) and then Dirichlet in the early nineteenth century. Much searching was done to find polynomials which could generate primes. While some polynomials represent infinitely many primes (e.g. 4x + 1, which we have proved on the course) no simple formula exists to produce **all** the primes.

Legendre and Gauss, at the same time and independently, noticed that as a number (x) increased, the number of primes less than it, denoted by π (x), could be approximated by the ratio between the number and its natural logarithm (x/log x) – and that the ratio in turn between π (x) and (x/log x) appeared to be tending to a limit of 1. Attempts by these famous mathematicians, and others, to prove their conjecture did not succeed – until 130 years later, when, building on work by Chebyshev, Riemann and many others, some forgotten, Hadamard and Poussin (in 1896) independently proved that the limit does exist and is 1. This is the famous **prime number theorem**.[1]

One difficulty with a real analysis approach to π(x) is that it is a step function, that is, it proceeds by jumps (and it has been shown that such jumps can be arbitrarily large – the distance between primes can take any value given sufficiently large x; perhaps astonishingly, a sequence of n consecutive composite numbers can be found for **any** n)[2].

Gauss had done work on the area under the graph of 1/log x, showing that between 2 and large x it approximated to x/log x. Even later, in the mid twentieth century, Selberg and Erdos discovered an elementary proof of the prime number theorem using no complex functions and only elementary calculus.

[1] There are many consequences to this. One is that, because 1/ log x -> 0 as x -> ∞, then the ratio π(x) / x -> 0, that is, primes thin out as numbers get larger.
[2] The first number in such a sequence is (n+1)! + 2.

It is essential to finish with the Riemann hypothesis[3]. Euler had introduced the **zeta function** in about 1740 – the sum of the reciprocals of a certain power of the natural numbers. He showed that it could be expanded as a product involving all the primes. That is, *some expression not involving primes could be equated with one involving all the primes* – fertile ground for research into the distribution of primes.

Riemann, in 1859, extended the function to the complex plane, giving the famous **Riemann zeta function**. For real values of the power the function has zeroes at the negative even integers, but these are regarded as trivial solutions. In the complex numbers with positive real part some other zeroes do occur, and these are the important ones. Riemann found that the real part of any solution seemed always to be ½. This remains unproven to this day. However, if it were proved (and no counterexamples have been found) it would provide yet further structure to the primes. Some have speculated[4] that it could lead, for instance, to ways of factorising very large numbers in a reasonable time. Since present-day internet security relies on the practical impossibility of such an unravelling, much is at stake. [5]

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[3] Technical details are in an appendix.
[4] E.g. the TV personality and mathematician Marcus du Sautoy
[5] The Riemann hypothesis remains an unsolved *Millennium Problem* – with $1 million available to a solver. Mathematicians would do it for nothing, if they could.

References and Bibliography

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http://www.claymath.org/millennium/

Appended note on Riemann/Euler: the LHS is the **Riemann zeta function**; the RHS is the **Euler product** which runs through all the primes. Euler discovered the relationship in 1737. All the billion or more currently found complex solutions for s when the zeta function equals zero have ½ as the real part of the complex number s = a + bi.

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

Question: Have a further look at the note above. How does this brilliant piece of work by Euler immediately establish the infinitude of primes?