Part I: Sudoku – the basics

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Definitions and terminology

By a *sudoku puzzle* we mean one of these things.

			1				4	
					2		9	6
1		8		3		2		
							5	
	2	1	4		3	9	6	
	8							
		2		9		6		1
5	1		3					
	9				8			

Puzzle A

Fill in the blanks to make a Latin square on the symbols $\{1, 2, \ldots, 9\}$ with the extra constraint that each of the 3×3 boxes into which the big 9×9 square is divided contains the symbols $\{1, 2, \ldots, 9\}$.

A *Latin square* is an $n \times n$ square array of n distinct symbols where each row and each column contains precisely n distinct symbols.

A *sudoku square* or *sudoku array* is a 9×9 Latin square on the symbols $\{1, 2, \ldots, 9\}$ with the extra sudoku condition—that the 9×9 boxes also contain $\{1, 2, \ldots, 9\}$. You can also think of a sudoku square as a solved sudoku puzzle.

For definiteness, we always use the word *box* for the 3×3 squares and the word *region* as a general term for rows, columns and boxes.

The displayed symbols of a sudoku puzzle are called *starter digits*. (I prefer the phrase to that hideous word 'givens' used by some writers.) We always insist that a sudoku puzzle has a unique solution.

We call a puzzle *symmetric* if the starter digits occupy diametrically opposite positions. Most published puzzles have this property—for no good reason I can think of other than that they look pretty.

The individual squares are called *cells* or *positions* or some such word with a similar meaning. We use a coordinate system to identify cells. The rows are numbered

0–8, top to bottom, and columns 0–8, left to right. The cell at row r column c is referred to by its coordinates, (r, c). Boxes are labelled by their top left cell. Thus box $(6, 6)$ is the one at bottom right.

Within each box, the cells fall naturally into three types; *centres*, *corners*, and *edges*. A box has four corner cells four edge cells and one centre cell.

By a *row block* we mean three adjacent rows that span three boxes. Similarly, a set of three adjacent columns that span three boxes is called a *column block*.

Symmetries

This section is important. It is absolutely vital that you fully understand everything.

A *symmetry operation* is something you can do to a sudoku square (i.e. a completed sudoku puzzle) that does not disturb the essential sudoku properties of the thing. That is, we want to preserve properties such as the partitioning of the square into regions, the distinctness of the nine symbols and the placing of the symbols in the regions. But we don't care, for instance, if you are doing a sudoku puzzle whilst looking at it in a mirror—it's essentially the same puzzle.

Confirm that the following are valid symmetry operations of a sudoku square.

(i) *Rotation by 90 degrees anticlockwise.* By rotation we mean rotating the symbols of the array about the centre of the array—at position $(4, 4)$. After the rotation the lines marking the borders of the squares are unchanged. But all the symbols have moved round by 90 degrees, columns have become rows and rows have become columns. For instance, in Puzzle A the 6 in cell $(1, 8)$ moves to cell $(0, 1)$.

Of course, you can combine more than one. Two 90 degree rotations make a 180 degree rotation. But 90 degrees is the minimum. Rotation by a smaller non-zero angle such as 45 degrees is impossible.

(ii) *Reflection in the middle row.* The 6 in cell $(1, 8)$ of Puzzle A moves to $(7, 8)$.

(iii) *Reflection in the middle column.* The 6 in cell $(1, 8)$ of Puzzle A moves to $(1, 0)$.

(iv) *Reflection in a main diagonal.* Reflection in the main NW–SE diagonal is also called *transposition*; here the rows and columns are interchanged. The 6 at cell $(1, 8)$ of Puzzle A moves to $(8, 1)$—the coordinates are interchanged. Reflection in the NE–SW diagonal moves it to $(0, 7)$.

(v) *Swap any two row blocks.* We can legitimately interchange any two blocks of three adjacent rows into which the array is naturally divided. This works because (a) swapping rows does not affect the overall contents of any row or column, and (b) only entire boxes (i.e. those contained within the row blocks) get swapped and hence their contents are unaffected. Note that we can't in general swap any two individual rows because that might break up the boxes that they intersect.

(vi) *Swap any two column blocks.*

(vii) *Swap any two rows within a row block.* Under the restriction of operating within a row block this is legitimate because the swapping of the rows does not affect the contents of any of the boxes they intersect.

(viii) *Swap any two columns within a column block.*

(ix) *Any permutation of the symbol set.* Our symbol set will always be 1–9. But it is useful to allow things like changing all 1s to 2s and vice versa.

That completes the list of symmetry operations we will need.

Each symmetry operation X, say, has its own unique *inverse*, X^{-1}. In some cases (as with reflections and swappings) $X = X^{-1}$. Order is usually relevant. In general, doing symmetry operation Y followed by X won't give the same result as X followed by Y. Symmetry operations can be combined in any order. But if you carry out a sequence of operations, $ABCDE$, for example, and want to undo them all, you must do their inverses and do them in the reverse order, $E^{-1}D^{-1}C^{-1}B^{-1}A^{-1}$. All this is familiar to anyone who has done some group theory.

Use of symmetry

We now demonstrate the power of the idea of symmetry by stating and inviting you to prove some theorems.

Theorem 1 *In a sudoku puzzle, the starter digits must contain at least eight distinct symbols.*

Proof. Assume the theorem is false. Hence two symbols are missing.

Use a symmetry operation to show that you can assume the missing symbols are 1 and 2.

What is the effect on the solution of interchanging 1 and 2 in the puzzle?

Conclude that the theorem is proved. □

Hence we can also conclude that 8 is a lower bound to the number of starter digits of a sudoku puzzle.

This next result is interesting and will be useful later.

Theorem 2 *In a sudoku square, a given symbol occurs in box edge positions an even number of times.*

Proof.

Use a symmetry operation to show that it is sufficient to prove the theorem for the symbol 1.

Convince yourself that none of the symmetry operations affect the parity of the number of occurrences of 1 in edge positions in the array. This is not entirely trivial and it is what makes the proof work.

Now imagine you have any sudoku square.

Prove that you can use symmetry operations to move the 1 of box $(0, 0)$ (top left) into its central position, cell $(1, 1)$.

Prove that you can use symmetry operations to move the 1 of box $(3,3)$ (middle of the array) into its central position, cell $(4,4)$ without disturbing the 1 that you previously moved to cell $(1,1)$.

Prove that you can use symmetry operations to move the 1 of box $(6,6)$ (bottom right) into its central position, cell $(7,7)$ without disturbing the 1s that you previously moved to cells $(1,1)$ and $(4,4)$.

You have now moved 1s to the centres of the three boxes along the NW–SE diagonal.

How many 1s are there in edge positions of the resulting array?

Convince yourself that the theorem is proved. □

See if you can use similar ideas to prove the following extension.

Theorem 3 *In a sudoku square, if a symbol occurs k times in box centres then it must occur $6-2k$ times in box edges.*

How to solve sudoku puzzles

We can represent a sudoku puzzle by a *state matrix*

$$\mathcal{S} = \begin{bmatrix} S_{0,0} & S_{0,1} & S_{0,2} & S_{0,3} & S_{0,4} & S_{0,5} & S_{0,6} & S_{0,7} & S_{0,8} \\ S_{1,0} & S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} & S_{1,5} & S_{1,6} & S_{1,7} & S_{1,8} \\ S_{2,0} & S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} & S_{2,5} & S_{2,6} & S_{2,7} & S_{2,8} \\ S_{3,0} & S_{3,1} & S_{3,2} & S_{3,3} & S_{3,4} & S_{3,5} & S_{3,6} & S_{3,7} & S_{3,8} \\ S_{4,0} & S_{4,1} & S_{4,2} & S_{4,3} & S_{4,4} & S_{4,5} & S_{4,6} & S_{4,7} & S_{4,8} \\ S_{5,0} & S_{5,1} & S_{5,2} & S_{5,3} & S_{5,4} & S_{5,5} & S_{5,6} & S_{5,7} & S_{5,8} \\ S_{6,0} & S_{6,1} & S_{6,2} & S_{6,3} & S_{6,4} & S_{6,5} & S_{6,6} & S_{6,7} & S_{6,8} \\ S_{7,0} & S_{7,1} & S_{7,2} & S_{7,3} & S_{7,4} & S_{7,5} & S_{7,6} & S_{7,7} & S_{7,8} \\ S_{8,0} & S_{8,1} & S_{8,2} & S_{8,3} & S_{8,4} & S_{8,5} & S_{8,6} & S_{8,7} & S_{8,8} \end{bmatrix},$$

divided up into regions as in a sudoku square, and where each element $S_{r,c}$ is a set of numbers taken from $\{1,2,3,4,5,6,7,8,9\}$. We need a couple more definitions.

A state matrix $\mathcal{S}$ is *valid* if for each region and each number n there is at most one position in the region which has the single number $\{n\}$ in it. This of course is the sudoku property.

A state matrix $\mathcal{S}$ is *inconsistent* if at least one position is empty.

In its initial state, $\mathcal{S}$ is valid, some positions (the starter digits) have just one number in them and all other positions have $\{1,2,3,4,5,6,7,8,9\}$ in them. In the actual printed version of the puzzle the starter digit positions have their contents printed as big numbers whilst all the other positions will appear as blank squares. It might help to think of $S_{r,c}$ as the set of all those numbers which have not yet been barred from occupying position (r,c). However, $S_{r,c}$ really has no special significance—it is just a set of numbers.

To solve a puzzle we transform $\mathcal{S}$ by making changes under two rules:

(i) the transformations must preserve the validity of $\mathcal{S}$;

(ii) they must leave the positions containing starter digits unchanged.

But otherwise you can do anything you like. If you want to be perverse, you can simply empty the $S_{r,c}$ for all positions not occupied by starter digits—this is a perfectly legitimate transformation of $\mathcal{S}$ even if it doesn't achieve very much. (Remember, you are not allowed to touch cells containing starter digits.) On the other hand, there do exist transformations which genuinely help to solve puzzles. The objective is to achieve a state where each position has exactly one number in it. Ideally, but this is not vital, we want our transformations to have the additional property that the $S_{r,c}$ remain non-empty and do not grow in size.

	6		1		5		4	
	5			7		1		2
		6		3				4
	8	5	4	6	2	9	7	
4				5		2		
8		7		4			5	
	1		9		3		2	

Puzzle B

Let us begin by considering a specific transformation.

The critical set strategy. If there are k positions in a region that contain k numbers altogether, then remove these k numbers from all other positions in the same region.

The sets of k numbers which have that property are called *critical sets*. Imagine applying the critical set strategy repeatedly until $\mathcal{S}$ is stable. Prove that each region can be partitioned into critical sets and that a partitioning into smaller critical sets is impossible.

Whilst the phrase 'critical set strategy' might be unfamiliar, I am sure that all sudoku addicts use an elementary method based on a couple of special cases, which I shall call *basic transformations*. First let us restrict the sizes of the critical sets to 1. Then we have the simple rule:

Basic transformation 1. If a cell contains $\{n\}$, then remove n from all other cells in the same region.

For instance, if you apply this rule to Puzzle B, above, you can put $\{8\}$ in row 7, column 4. And for the other well-known case we have:

Basic transformation 2. If n is absent from all other cells in a region, then put $\{n\}$ here.

Thus we can put $\{5\}$ in cell $(3, 6)$ because 5 is blocked from everywhere else in row 3 by the $\{4\}$ at cell $(3, 8)$ and by $\{5\}$s present in regions which intersect row 3. Again, this is the critical set strategy in action, this time with sets of size 8: If you examine the contents of the positions in row 3, columns 0–5, 7 and 8, you should find the numbers 1, 2, 3, 4, 6, 7, 8 and 9. Hence we can remove these numbers from $S_{3,6}$.

By the way, I have in the past suggested describing puzzles as 'easy' if they can be completely solved by the two basic transformations. The main reason that you do not need to make notes. Stare at the puzzle until you see where a basic transformation applies; write a number in a square; repeat until solved. So-called 'fiendish' puzzles in newspapers often have this property. Having said that, I have to admit that instances where basic transformation 1 applies are difficult to spot.

The critical set strategy is more powerful than the basic transformations, and seems sufficient to dispose of most published puzzles. Puzzle B, however, is an exception. There are 29 starter digits, the basic transformations yield 20 new numbers, and the general critical set strategy produces 3 more, making a total of 52. You might like to confirm this by actually attempting a solution. See if you end up with the state matrix below, where I have shown the entire contents of $\mathcal{S}$ using small symbols for positions not containing starter digits. As you can see, each region is partitioned into critical sets. In row 5, for example, the partitioning is

$$\{\{4\},\ \{5\},\ \{2\},\ \{3,7\} \cup \{3,9\} \cup \{7,8\} \cup \{1,9\} \cup \{1,6,8\} \cup \{6,8\}\},$$

three sets of size 1 and one set of size 6.

2379	4	1	36	29	8	67	369	5
2379	**6**	239	**1**	29	**5**	8	**4**	79
39	**5**	8	36	**7**	4	**1**	369	**2**
279	27	**6**	78	**3**	19	5	18	**4**
1	**8**	**5**	**4**	**6**	**2**	**9**	**7**	3
4	37	39	78	**5**	19	**2**	168	68
8	9	**7**	2	**4**	6	3	**5**	1
5	**1**	4	**9**	8	**3**	67	**2**	67
6	23	23	5	1	7	4	89	89

Puzzle B state matrix after critical sets and intersecting regions

Once you can see the 'marked up' puzzle completing it is not difficult. If you look at the left-hand column, you will notice that there are no 3s except in the top left-hand box. Therefore the 3 in these regions must go somewhere in the overlap. Hence we can eliminate 3 from all other positions in the box. This is an instance of another general transformation:

Intersecting regions. Let A and B be distinct regions of $\mathcal{S}$. Suppose n occurs in cells common to both A and B but not in any other positions of A. Remove n from all other positions in B.

In our example $S_{1,2} = \{2, 3, 9\}$ can be replaced by $S_{1,2} = \{2, 9\}$. This creates a new critical set in row 1 consisting of $S_{1,2}$ and $S_{1,4}$, containing $\{2, 9\}$, which allows you to remove the 9 from $S_{1,8} = \{7, 9\}$ at the end of row 1. The rest is easy.

However, there exist puzzles for which critical sets combined with intersecting regions will not work. Puzzle A on page 1 is an example. If you apply critical sets and intersecting regions you should get 64 cells done. There is another example below, Puzzle C, which closes under critical sets and intersecting regions at 70 cells. So we define one transformation which will always work, no matter what other methods fail.

Backtracking. Apply the critical sets and intersecting regions procedures until $\mathcal{S}$ is stable.

If $\mathcal{S}$ is inconsistent, do nothing. Otherwise choose any position X containing more than one number. If there are none, the puzzle is solved, so report the solution. Otherwise for each n in position X, perform the following.

Save $\mathcal{S}$. Replace the contents of X by the single number $\{n\}$. Perform the backtracking transformation. Restore $\mathcal{S}$.

			5				3	
	2				3			
9	4			1	7			8
							6	7
		7	2	8	9	3		
2	3							
5			9	6			7	3
			3				8	
	7				5			

Puzzle C

Backtracking is the process of trying things out in a systematic fashion, discarding choices that lead to inconsistent $\mathcal{S}$. It works on any sudoko state matrix, not just genuine puzzles. If allowed to run to completion, it will eventually report the entire solution set. The changes made by backtracking are only temporary; the overall effect on $\mathcal{S}$ is to leave it unchanged.

See if you can complete Puzzle A and Puzzle C by backtracking. If you want to save time, a few pages on I have displayed the state matrices of Puzzles A and C as they would be after applying critical sets and intersecting regions.

There are other strategies that do not involve trial. Here is one involving four intersecting regions. Formulate a general strategy based on this specific case.

> **Four intersecting regions**. If 1 occurs in rows a and b only in the positions where it intersects columns c and d, then remove 1 from all positions in columns c and d that do not intersect rows a and b.

		c		d	
a	no 1s here	$\{1,\ldots\}$	no 1s here	$\{1,\ldots\}$	no 1s here
b	no 1s here	$\{1,\ldots\}$	no 1s here	$\{1,\ldots\}$	no 1s here

Finally, Theorem 3 is of more than academic interest. I have actually come across puzzles where counting centre and edge positions occupied by a particular symbol does help to solve them. So I shall add it to the list.

Now we have six strategies.

(0) the basic transformations,
(1) critical sets,
(2) intersecting regions,
(3) four intersecting regions,
(4) counting centre and edge positions,
(∞) backtracking.

Verifying a sudoku solution

A sudoku puzzle has 27 regions—9 rows, 9 columns, 9 boxes. Wise sudoku fans are well aware of the importance of checking each region as soon as it is completed just to make sure that it really does contain precisely one each of the symbols 1 to 9.

But suppose you are given a completed puzzle, such as the solution to Puzzle A, below. *What is the minimum number of regions you must check to verify that a sudoku square is valid?* And how should you go about doing the checking?

Prove that if you check all nine rows, then you need only check eight columns to confirm that the array is a Latin square. Hence prove that the answer is at most 26.

Now by considering boxes prove that the answer is at most 21.

See if you can prove that the answer is exactly 21—or find a way to do the verification by checking 20 regions or less.

Solutions

2	6	9	**1**	7	5	8	**4**	3
3	5	7	8	4	**2**	1	**9**	**6**
1	4	**8**	69	**3**	69	**2**	7	5
469	3	46	279	8	1	47	**5**	247
7	**2**	**1**	**4**	5	**3**	**9**	**6**	8
49	**8**	5	279	6	79	3	1	247
8	7	**2**	5	**9**	4	**6**	3	**1**
5	**1**	46	**3**	2	67	47	8	9
46	**9**	3	67	1	**8**	5	2	47

Puzzle A after critical sets and intersecting regions

7	8	16	**5**	2	4	9	**3**	16
16	**2**	5	8	9	**3**	7	14	146
9	**4**	3	6	**1**	**7**	5	2	**8**
8	5	9	4	3	1	2	**6**	**7**
14	6	**7**	**2**	**8**	**9**	**3**	5	14
2	**3**	14	7	5	6	8	14	9
5	1	2	**9**	**6**	8	4	**7**	**3**
46	9	46	**3**	7	2	1	**8**	5
3	**7**	8	1	4	**5**	6	9	2

Puzzle C after critical sets and intersecting regions

2	6	9	1	7	5	8	4	3
3	5	7	8	4	2	1	9	6
1	4	8	6	3	9	2	7	5
9	3	6	2	8	1	4	5	7
7	2	1	4	5	3	9	6	8
4	8	5	9	6	7	3	1	2
8	7	2	5	9	4	6	3	1
5	1	4	3	2	6	7	8	9
6	9	3	7	1	8	5	2	4

Solution A

7	8	1	5	2	4	9	3	6
6	2	5	8	9	3	7	4	1
9	4	3	6	1	7	5	2	8
8	5	9	4	3	1	2	6	7
1	6	7	2	8	9	3	5	4
2	3	4	7	5	6	8	1	9
5	1	2	9	6	8	4	7	3
4	9	6	3	7	2	1	8	5
3	7	8	1	4	5	6	9	2

Solution C

Part II: Quasi-magic sudoku puzzles

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		2		3			7	1
4								
			2	6	5			
	4			2	7		1	
9								2
	7		1	9			8	
			3	8	6			
								3
6	8			1		4		

Puzzle A

Magic squares

The presence in a sudoku puzzle of 3×3 squares which must contain precisely the numbers 1–9 suggests that one might be tempted to create more interesting puzzles by insisting that the solution has a magic square in each of the nine boxes.

A *magic square of order* n is a square array of the numbers $1, 2, \ldots, n$ such that every row, column and the two main diagonals sum to the same number, the 'magic sum'. What is the magic sum of a magic square of order 3? 4? … n? Construct a magic square of order 3.

Try to construct a sudoku square with a magic square of order 3 in each box. What goes wrong? This is peculiar to 3; there is no problem with 4—and just to prove it, at the end we present a 16×16 sudoku square with a magic square of order 4 in each box.

OK, it's impossible. Nevertheless, there is a desperate urge to give the boxes of a standard sudoku square some kind of magic property. We have to bend the rules somehow. Amongst several approaches that suggest themselves, one option is to keep the row-column-diagonal sum property but relax the condition that the sums must be a precise figure.

We require the row, column and diagonal sums of the 3×3 *boxes to be any number in the range* $15 \pm \Delta$, where Δ is a fixed parameter—which must be made known to the solver. The sums do not have to be identical, nor is it necessary for all numbers in the permitted range to occur. We call a square with this property *quasi-magic*, and we use the same qualifier for a sudoku puzzle where all the boxes are quasi-magic squares: a *quasi-magic sudoku puzzle*. If you obtained the solution to Puzzle A at the beginning, you should verify that it is actually quasi-magic with $\Delta = 2$. Having that information at the start would have made it a lot easier to solve!

Symmetry

Confirm that for a quasi-magic sudoku square the following symmetry operations are permissible.

(i) *Rotation by 90 degrees anticlockwise.*

(ii) *Reflection in the middle row.*

(iii) *Reflection in the middle column.*

(iv) *Reflection in a main diagonal.*

(v) *Swap any two row blocks.*

(vi) *Swap any two column blocks.*

(vii) *Swap the upper and lower rows of a row block.*

(viii) *Swap the left-hand and right-hand columns of a column block.*

(ix) *Permutation of the symbol set by the mapping $x \mapsto 10 - x$.* That is, you subtract every number from 10.

The case $\Delta = 1$

As we have seen, we must have $\Delta \geq 1$ for a possible quasi-magic sudoku square. With a little more work we can show that $\Delta = 1$ is not possible.

Let $\Delta = 1$. Now the row, column and diagonal sums are restricted to $\{14, 15, 16\}$.

Prove that that the box centres are restricted to $\{4, 5, 6\}$.

Prove that the box centres of the whole array form a Latin square on the symbols $\{4, 5, 6\}$.

Prove that boxes with a 4 or a 6 in the centre must have 1 and 9 on opposite edges.

Prove that if a box has a 5 in the centre and a 9 in a corner, then there must be a 1 adjacent to the 9.

Deduce that only the following patterns are permitted as valid boxes (plus variations induced by the symmetry operations—rotations and reflections).

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ 9 & 4 & 1 \\ \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \cdot & \cdot \\ 9 & 6 & 1 \\ \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \cdot & \cdot \\ 9 & 5 & \cdot \\ \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} 9 & \cdot & \cdot \\ 1 & 5 & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

Show that there is no way of arranging the 1s and 9s on the entire array.

Therefore quasi-magic sudoku squares with $\Delta < 2$ do not exist.

Properties of quasi-magic sudoku squares with $\Delta = 2$

Henceforth we set $\Delta = 2$, so that the rows, columns and diagonals of the boxes must sum to any one of $\{13, 14, 15, 16, 17\}$. Our next task is to see if quasi-magic sudoku squares have any interesting properties which might be helpful to puzzle solvers. We start with some easy ones.

Prove that 1 and 2 cannot occur together in a box row, column or diagonal. Hence by symmetry nor can 8 and 9.

Prove that a box centre must be one of $\{3, 4, 5, 6, 7\}$.

Prove that quasi-magic squares with centre 3 or 7 must have the following patterns

67	67	2		34	34	8
1	3	9		9	7	1
8	45	45		2	56	56

This is as far as one can get by treating a quasi-magic square in isolation. But when nine of them are arranged as boxes in a quasi-magic sudoku square it is possible to prove much more.

In fact, we have the interesting and surprising fact that 3 and 7 can occur at most once each as box centres in a quasi-magic sudoku square, and if they do both occur, they must occupy box centres in the same row or in the same column.

Another surprising and interesting fact is that, although there exist quasi-magic squares where the 5 is on an edge, in a quasi-magic sudoku square all the 5s occur either in box centres or in box corners.

The only proofs I know are long and complicated. I hope that by presenting them here someone will possibly see something I have missed and produce a shorter and more elegant proof.

Anyway, as a consequence of the no-5-on-a-box-edge rule the number of possible patterns with box centre 3 or 7 is halved to just these four (plus symmetries).

6	7	2		7	6	2		4	3	8		3	4	8
1	3	9		1	3	9		9	7	1		9	7	1
8	4	5		8	4	5		2	6	5		2	6	5

And since they map to each other in pairs under the action of the symmetry operation $x \mapsto 10 - x$ there are really only two.

Distribution of 5 in a quasi-magic sudoku square

First we prove the following.

Theorem 4 *In a quasi-magic sudoku square with $\Delta = 2$, each row block and each column block has the number 5 in one of the box centre positions.*

Proof. Suppose the theorem is false.

Use symmetry to show that we can assume that the box centres in row 1 are either $(3, 7, 4)$, or $(3, 4, 6)$.

Consider the case $(3, 7, 4)$ first.

Use symmetry and what we already know about boxes with 3 or 7 in the centre to show that one of the following four patterns must occur in rows 0–2.

67	67	2	8	1	56	.	.	.
1	3	9	34	7	56	.	4	.
8	45	45	34	9	2	.	.	.

67	67	2	34	9	2	.	.	.
1	3	9	34	7	56	.	4	.
8	45	45	8	1	56	.	.	.

8	1	7	56	56	2	.	.	.
5	3	6	1	7	9	.	4	.
4	9	2	8	34	34	.	.	.

8	1	7	34	9	2	.	.	.
5	3	6	34	7	56	.	4	.
4	9	2	8	1	56	.	.	.

Show that each one blatantly leads to a contradiction.

That leaves just the case $(3, 4, 6)$ to worry about.

Use symmetry and what we already know about boxes with a 3 in the centre to show that there are only two possible patterns for rows 0–2:

67	67	2	.	.	.	.	.	.
1	3	9	.	4	.	.	6	.
8	45	45	.	.	.	.	.	.

and

8	1	6	.	.	.	.	.	.
5	3	7	.	4	.	.	6	.
4	9	2	.	.	.	.	.	.

Show that in the first case (left-hand diagram) there is nowhere for a 2 to go in box $(0, 3)$ without generating a contradiction. Try putting 2 in cell $(1, 3)$ and see what happens. Then try putting 2 in cell $(2, 3)$ and see what happens. Then try putting 2 in cell $(2, 4)$ and see what happens. Use symmetry to prove that you needn't bother to try putting 2 in cells $(1, 5)$ and $(2, 5)$.

That leaves only the second case (right-hand diagram) to consider. Unfortunately this is the troublesome one.

Suppose that cell $(4, 4)$ is 3 or 7.

Show that one of the following four patterns must occur in the top six rows

A

8	1	6	.	.	.	.	.	.
5	3	7	.	4	.	.	6	.
4	9	2	.	.	.	.	.	.
139	.	.	67	67	2	.	.	.
.	.	.	1	3	9	.	.	.
.	.	.	8	5	4	.	.	.

B

8	1	6	2379	.	.	.	.	.
5	3	7	.	4	.	.	6	.
4	9	2	.	.	.	.	.	.
.	.	.	8	1	67	.	.	.
.	.	.	45	3	67	.	.	.
.	.	.	45	9	2	.	.	.

C

8	1	6	.	.	.	.	.	.
5	3	7	.	4	.	.	6	.
4	9	2	.	.	.	.	.	.
12679	.	.	8	3	4	.	.	.
.	.	.	1	7	9	.	.	.
.	.	.	56	56	2	.	.	.

D

8	1	6	.	.	.	.	.	.
5	3	7	129	4	.	.	6	.
4	9	2	.	.	.	.	.	.
.	.	.	8	1	56	.	.	.
.	.	.	34	7	56	.	.	.
.	.	.	34	9	2	.	.	.

[Optional] Show that in each case one can obtain a contradiction. Here is one way to do it.

Pattern A. Cell $(3,0)$ is either 1, 3 or 9, and each leads to a contradiction.

Pattern B. Cell $(0,3)$ is either 2, 3, 7 or 9; 2 and 3 lead to contradictions; 7 implies box $(6,3)$ middle column $= \{5,6,7\}$, which exceeds 17; and 9 implies box $(6,3)$ left column $= \{2,3,7\}$, which is less than 13.

Pattern C. Cell $(3,0)$ is one of $\{1,2,6,7,9\}$ all of which lead to contradictions.

Pattern D. Cell $(1,3)$ is one of $\{1,2,9\}$ all of which lead to contradictions.

We have shown that cell $(4,4)$ is not 3 or 7. Use symmetry to show that the same is true of $(4,7)$, $(7,4)$ and $(7,7)$.

Now use symmetry to show that we can assume that $(4,4)=5$, $(7,4)=6$, $(4,7)=4$ and $(7,7)=5$.

8	1	6	.	.	.	.	.	.
5	3	7	.	4	.	.	6	.
4	9	2	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	67	.	.	5	.	.	4	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	47	.	.	6	.	.	5	.
.	.	.	.	.	.	.	.	.

Suppose $(4,1)=7$.

Show that $(3,0)$ is one of $\{1,2,3,6,9\}$ all of which lead to contradictions.

Hence we can assume $(4,1)$=6.

Show that $(3,0)$ is one of $\{1,2,3,7,9\}$ all of which lead to contradictions.

This completes the proof. □

We use this last result to prove that interesting property of the symbol 5.

Theorem 5 *In a quasi-magic sudoku square with $\Delta = 2$, the number 5 never occurs in a box edge position.*

Proof. Suppose the theorem is false. We assume that there is a 5 in a box edge position.

Use symmetry to show that we can assume that cell $(1,0)=5$.

How many box centres in row 1 have 5 in them?

Use Theorem 4 to derive a contradiction. □

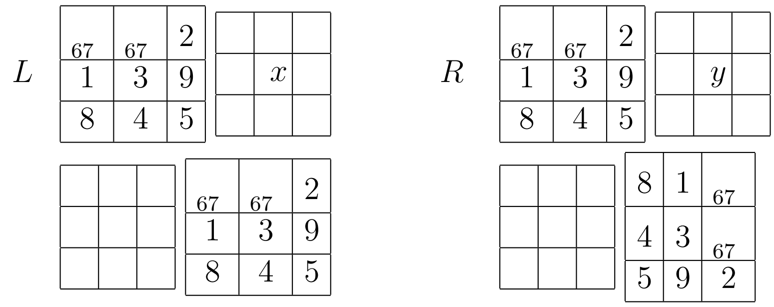
Box centres 3 and 7 in a quasi-magic sudoku square

Theorem 6 *In a quasi-magic sudoku square with $\Delta = 2$, the numbers 3 and 7 can each occur at most once each as a box centre.*

Proof. [Optional] It suffices to deal with 3; the result for 7 follows by symmetry.

Suppose the theorem is false. Use symmetry to show that we can assume box centre 3s occur in cells (1,1) and (4,4).

Remembering the possible patterns with box centres 3 or 7, and using Theorem 5, show that the part of the array consisting of the top left 36 cells contains one of the following patterns, denoted by L and R (left and right, respectively).



We consider seven cases: array L, $x = 5, 6$ or 7, or array R, $y = 4, 5, 6$ or 7.

Taking each case in turn, prove that no solution exists.

Here is one way, but some of the details are a bit complicated.

Array L, $x = 5$. First, all of $(0,4) = 1$, $(0,5) = 1$ and $(2,5) = 1$ lead to contradictions. Hence $(2,4) = 1$ and $(1,3) = 2$. But then $(1,5) = 8$ for otherwise the middle row of box $(0,6)$ exceeds 17. Then $(0,4) = 9$, $(0,3) = 4$, $(0,5) = 3$ and $(2,5) = 6$. But now the right column of box $(6,3)$ sums to only 12.

Array L, $x = 6$. First, $(3,3) = 6$ and $(3,4) = 7$. Since $(0,3) = 9$, $(0,4) = 9$ and $(2,3) = 9$ lead to contradictions, we assume $(2,4) = 9$. Then $(0,4) = 1$, $(2,5) = 3$, $(2,3) = 2$, $(0,5) = 8$, $(1,3) = 7$, $(1,5) = 4$, $(0,3) = 5$, $(3,3) = 6$, $(3,4) = 7$, $(1,7) = 5$ and $(7,4) = 5$. Next, $(3,1) = 1$, $(3,2) = 1$ and $(5,2) = 1$ lead to a contradictions. So $(5,1) = 1$. Hence $(4,0) = 2$, $(2,7) = 2$, and $(3,6) = 1$ or 5, both leading to contradictions.

Array L, $x = 7$. The proof goes as for array L, $x = 6$ except that the 6 and 7 are swapped in cell pairs $\{(1,3),(1,4)\}$ and $\{(3,3),(3,4)\}$. This interchange does not affect the argument.

Array R, $y = 4$. This leads immediately to a contradiction since there is nowhere to put a 5 in box $(0,3)$.

Array R, $y = 5$. There is nowhere to put a 4 in column 4 since $(6,4) = 4$, $(7,4) = 4$, $(8,4) = 4$ and $(0,4) = 4$ all lead to contradictions.

Array R, $y = 6$. This leads almost immediately to a contradiction.

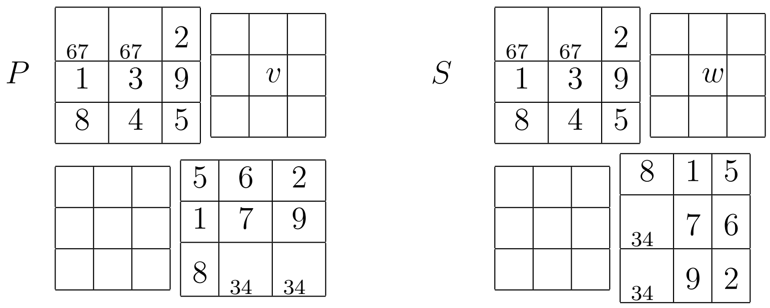
Array R, $y = 7$. So does this. $\square$

Box centres 3 and 7 together in a quasi-magic sudoku square

Theorem 7 *In a quasi-magic sudoku square with $\Delta = 2$, the numbers 3 and 7 can occur together as box centres only in the same row or column.*

Proof. [Optional] Suppose the theorem is false.

Use symmetry to show that we can assume cell $(1, 1) = 3$ and $(4, 4) = 7$. Remembering the possible patterns with box centres 3 or 7, and using Theorem 5, show that the part of the array consisting of the top left 36 cells contains one of the following patterns, denoted by P and S (port and starboard, respectively).



We consider five cases: array P, $v = 4$ or 5, or array S, $w = 4, 5$ or 6.

Taking each case in turn, prove that no solution exists.

Here is one way.

Array P, $v = 4$. All of $(1, 3) = 2$, $(2, 3) = 2$ and $(2, 4) = 2$ lead to contradictions.

Array P, $v = 5$. First, $(2, 3) = 2$ leads to a contradiction. If $(1, 3) = 2$, then $(1, 5) = 8$ (otherwise the middle row of box $(0, 6)$ exceeds 17); but now the left column of box $(6, 3)$ exceeds 17. Hence $(2, 4) = 2$. Then $(0, 5) = 1$, $(1, 3) = 4$, $(0, 3) = 3$, $(0, 4) = 9$, $(2, 3) = 7$, $(2, 5) = 6$ and $(1, 5) = 8$. But by Theorem 6, $(1, 7)$ can't be 7, so it must be 6 and this leads to a contradiction.

Array S, $w = 4$. Both $(2, 3) = 2$ and $(2, 4) = 2$ lead to contradictions. Hence $(1, 3) = 2$. But now the middle row of box $(0, 6)$ exceeds 17.

Array S, $w = 5$. All of $(1, 3) = 2$, $(2, 3) = 2$ and $(2, 4) = 2$ lead to contradictions.

Array S, $w = 6$. Likewise. □

More properties of quasi-magic sudoku squares

There are further restrictions on the distribution of the box centre digits.

We already know from Theorem 4 that every row block and column block must have a 5 in one of its box centres.

Prove that in a row block the only possible box centres are

$$\{3, 5, 6\},\ \{3, 5, 7\},\ \{4, 5, 6\},\ \{4, 5, 7\},\ \{3, 4, 5\} \text{ and } \{5, 6, 7\}.$$

We shall now show that the last two do not occur. By symmetry it suffices to deal with $\{3, 4, 5\}$.

Using symmetry and Theorem 4, show that it suffices to consider the following row block pattern.

67	67	2	.	.	.	.	.	.
1	3	9	.	4	.	.	5	.
8	4	5	.	.	.	.	.	.

Show that there is nowhere for the 2 to go in the middle row.

Hence the only allowed box centres in a row block or column block are

$$\{3,5,6\},\ \{3,5,7\},\ \{4,5,6\},\ \{4,5,7\},$$

and they all do occur.

Prove that if a box centre is 4 or 6, then $\{4,5,6\}$ either lies on a diagonal or occupies three positions in the form of a knight's move.

Also it is a fact that a 5 in a box centre forces $\{4,5,6\}$ on a diagonal of the box. However, the only proof I have is to ask the computer to solve four quasi-magic sudoku puzzles with just these starter digits (in the top left box):

(a)

.	4	.
.	5	.
.	.	.

(b)

4	6	.
.	5	.
.	.	.

(c)

4	.	6
.	5	.
.	.	.

(d)

4	.	.
.	5	6
.	.	.

Computer says no in each case.

Prove that when $\{4,5,6\}$ occurs on a diagonal, $\{1,2,3\}$ and $\{7,8,9\}$ each occupies one of two broken diagonals into which the six remaining cells of the box are partitioned. In this case, after allowing for symmetries there is essentially only one admissible pattern.

$$\begin{array}{ccc} a & d & g \\ h & b & f \\ e & i & c \end{array} \qquad \begin{aligned} \{a,b,c\} &= \{4,5,6\} \\ \{d,e,f\} &= \{1,2,3\} \\ \{g,h,i\} &= \{7,8,9\} \end{aligned}$$

Prove that if the array has no boxes with 3 or 7 in the centre, then the nine box centres form a Latin square on $\{4,5,6\}$. Hence prove that each box must have $\{4,5,6\}$ along a diagonal.

Summary

(i) The number 5 cannot occur in a box edge position (Theorem 5).

(ii) The number 3 cannot occur more than once as box centres (Theorem 6). Nor can 7 by symmetry. And when 3 or 7 do occur, the boxes with them as centres must conform to one of the four patterns.

6	7	2		7	6	2		4	3	8		3	4	8
1	3	9		1	3	9		9	7	1		9	7	1
8	4	5		8	4	5		2	6	5		2	6	5

(iii) A 3 and 7 can both occur as box centres only in the same row or column. (Theorem 7).

(iv) The only allowed box centres in a row or column are

$$\{3,5,6\},\ \{3,5,7\},\ \{4,5,6\} \text{ and } \{4,5,7\}.$$

(v) Only the following patterns occur in boxes with centre 4, 5 or 6. But note however that we do not yet have a hand-written proof for the one with a 5 in the centre.

4	·	·		5	·	·		5	9	2		5	·	·		5	1	8
·	5	·		·	4	·		3	4	6		·	6	·		7	6	4
·	·	6		·	·	6		8	1	7		·	·	4		2	9	3

With all this information to hand you should now be able to solve with relative ease Puzzles B, C and D (as well as A) before advancing on to the more difficult E, F and G. Having only four starter digits Puzzle G looks truly fiendish!

A magic sudoku square

A = 10
B = 11
C = 12
D = 13
E = 14
F = 15
G = 16

A	F	1	8	3	6	9	G	D	C	7	2	5	4	E	B
3	6	C	D	A	F	4	5	8	1	E	B	G	9	7	2
G	9	7	2	8	1	E	B	A	F	4	5	3	6	C	D
5	4	E	B	D	C	7	2	3	6	9	G	A	F	1	8
4	A	F	5	G	3	6	9	2	D	C	7	B	1	8	E
D	7	2	C	5	A	F	4	B	8	1	E	6	G	9	3
6	G	9	3	B	8	1	E	5	A	F	4	D	7	2	C
B	1	8	E	2	D	C	7	G	3	6	9	4	A	F	5
1	8	A	F	9	G	3	6	7	2	D	C	E	B	5	4
C	D	3	6	4	5	A	F	E	B	8	1	7	2	G	9
7	2	G	9	E	B	8	1	4	5	A	F	C	D	3	6
E	B	5	4	7	2	D	C	9	G	3	6	1	8	A	F
8	5	B	A	F	E	2	3	1	4	G	D	9	C	6	7
9	C	6	7	1	4	G	D	F	E	2	3	8	5	B	A
2	3	D	G	C	9	5	8	6	7	B	A	F	E	4	1
F	E	4	1	6	7	B	A	C	9	5	8	2	3	D	G

Quasi-magic sudoku puzzles

In addition to the usual sudoku rules, the rows, columns and diagonals within in each box must sum to any one of 13, 14, 15, 16, 17.

					4			
9								
			5					
3		4		6	2			
			8	3		7		4
					7			
								7
			3					

Puzzle B

	4	5	6					
						6		
		2						
			3					
				4				
					9			
						7		
		7						
					7	8	3	

Puzzle C

	6							
	2	7						
		4	6					
				3				
					5	8		
						4	9	
							5	

Puzzle D

						9		
				8				
		9	2	5	7	3		
				1				
		2						

Puzzle E

				8				
					2			
8				7				9
			8					
				4				

Puzzle F

					5			
								7
		6						
				3				

Puzzle G

Solutions

5	9	2	4	3	8	6	7	1
4	3	6	9	7	1	2	5	8
8	1	7	2	6	5	9	3	4
3	4	8	6	2	7	5	1	9
9	6	1	8	5	3	7	4	2
2	7	5	1	9	4	3	8	6
7	2	4	3	8	6	1	9	5
1	5	9	7	4	2	8	6	3
6	8	3	5	1	9	4	2	7

Solution A

5	3	8	2	9	4	1	7	6
9	4	1	6	7	3	8	5	2
2	7	6	5	1	8	4	3	9
3	8	4	7	6	2	5	9	1
7	5	2	1	4	9	3	6	8
6	1	9	8	3	5	7	2	4
1	9	5	4	2	7	6	8	3
8	6	3	9	5	1	2	4	7
4	2	7	3	8	6	9	1	5

Solution B

8	4	5	6	7	1	2	9	3
1	3	9	2	5	8	6	7	4
6	7	2	9	3	4	5	1	8
5	9	1	3	8	6	4	2	7
3	6	8	7	4	2	9	5	1
7	2	4	5	1	9	3	8	6
4	8	3	1	9	5	7	6	2
2	5	7	8	6	3	1	4	9
9	1	6	4	2	7	8	3	5

Solution C

1	9	5	4	2	7	6	8	3
8	6	3	9	5	1	2	4	7
4	2	7	3	8	6	9	1	5
9	3	4	6	1	8	5	7	2
2	5	8	7	3	4	1	6	9
6	7	1	2	9	5	8	3	4
3	8	6	5	7	2	4	9	1
7	4	2	1	6	9	3	5	8
5	1	9	8	4	3	7	2	6

Solution D

2	7	6	4	9	1	5	8	3
9	5	1	3	6	8	2	4	7
4	3	8	7	2	5	9	1	6
7	2	5	6	8	3	4	9	1
1	4	9	2	5	7	3	6	8
6	8	3	9	1	4	7	2	5
5	9	2	1	7	6	8	3	4
3	6	7	8	4	2	1	5	9
8	1	4	5	3	9	6	7	2

Solution E

5	3	7	1	8	4	9	2	6
9	6	2	7	5	3	1	4	8
1	8	4	6	2	9	5	7	3
3	7	5	4	9	2	6	8	1
8	4	1	3	7	6	2	5	9
6	2	9	8	1	5	7	3	4
4	9	3	2	6	7	8	1	5
2	5	8	9	3	1	4	6	7
7	1	6	5	4	8	3	9	2

Solution F

9	1	5	4	2	7	6	8	3
2	4	7	8	6	3	1	5	9
6	8	3	1	9	5	7	2	4
7	2	4	6	1	9	5	3	8
3	6	8	7	5	2	9	4	1
5	9	1	3	8	4	2	6	7
1	7	6	5	4	8	3	9	2
8	5	2	9	3	1	4	7	6
4	3	9	2	7	6	8	1	5

Solution G

Part III: 6-region sudoku puzzles

Tony Forbes

5				2	1		6	
					3	8	4	
							1	7
	9		8	4		1		
	5						3	
		1		6	9		8	
2	1							
	6	9	5					
	4		1	7				3

Puzzle A

Four coordinates and six regions

Instead of labelling rows and columns 0–8, let us imagine that we are using base 3 numbers, 00, 01, 02, 10, 11, 12, 20, 21, 22. Or, looking at it another way, we use a two-coordinate system. For rows, the first digit, or coordinate, is the row block number and the second coordinate is the row number within the row block. Similarly for columns.

Now a cell has *four* coordinates, (a, b, c, d): a = row block number; b = row number within row block a; c = column block number; d = column number within column block c. To recover the original coordinates you just multiply by 3 and add; $(a, b, c, d) \to (3a + b, 3c + d)$.

A region is the set of cells obtained by fixing a certain pair of coordinates.

Identify the set of cells where a and b are fixed and c and d each run from 0 to 2.

Identify the set of cells where c and d are fixed and a and b vary.

Identify the set of cells where a and c are fixed and b and d vary.

So we have identified rows, columns and boxes by fixing two out of the four cell coordinates. Using $*$ to denote a coordinate that varies from 0 to 2, we have rows : $(a, b, *, *)$, columns : $(*, *, c, d)$, boxes : $(a, *, c, *)$.

But there are three further ways of fixing two coordinates.

Describe the set of cells obtained by fixing b and c: $(*, b, c, *)$.

Describe the set of cells obtained by fixing a and d: $(a, *, *, d)$.

Describe the set of cells obtained by fixing b and d: $(*, b, *, d)$.

We shall now introduce these three new types of sets of cells as extra regions, making six regions altogether. We call the new region types *split rows*, *split columns* and *split boxes*.

A split row consists of three rows of three, spaced three apart in one of the three column blocks (like the cells marked 'r' and 'x' in the array, below left). A split column consists of three columns of three, spaced three apart in one of the three row blocks (like the cells marked 'c' and 'x'). A split box is a 3 x 3 square array of cells, spaced three apart in both directions (like the cells marked 'b').

		0			1			2			c
a	b	0	1	2	0	1	2	0	1	2	d
	0	.	.	c	.	.	c	.	.	c	
0	1	r	r	x	.	.	c	.	.	c	
	2	b	.	c	b	.	c	b	.	c	
	0	.	.	.	.	.	.	.	.	.	
1	1	r	r	r	.	.	.	.	.	.	
	2	b	.	.	b	.	.	b	.	.	
	0	.	.	.	.	.	.	.	.	.	
2	1	r	r	r	.	.	.	.	.	.	
	2	b	.	.	b	.	.	b	.	.	

region type	coordinates
row	$(a, b, *, *)$
column	$(*, *, c, d)$
box	$(a, *, c, *)$
split row	$(*, b, c, *)$
split column	$(a, *, *, d)$
split box	$(*, b, *, d)$

We define a *6-region sudoku square* as a 9×9 Latin square where every row, column, box, split row, split column and split box contains the symbols $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. A *6-region sudoku puzzle* is a sudoku puzzle with the additional constraint that the split rows, split columns and split boxes must also contain the symbols 1–9.

[Acknowledgement. This part is based on an idea of Robert Connelly of Cornell University and communicated to me by Peter Cameron of QMUL.]

Symmetry

Valid symmetry operations for a 6-region sudoku square:

(i) Rotation by 90 degrees anticlockwise.

(ii) Reflection in the middle row. Reflection in the middle column. Reflection in a main diagonal.

(iii) Swap any two row blocks. Swap any two column blocks.

(iv) Any permutation of the symbols.

(v) Any permutation of the four coordinates.

Show that swapping two rows is never a symmetry operation.

The last one, (v), is especially interesting because it can change the shapes of the regions. We are already familiar with one special case: transposition is equivalent to interchanging the row and column coordinates, $(a, b, c, d) \to (c, d, a, b)$. But now we have four coordinates and hence 24 permutations; so there is plenty of scope for experimentation. Consider the permutation of the coordinates that sends the symbol at position (a, b, c, d) to position (b, c, a, d). Examine what effect it has on various regions. For example, you should get successively row, box, split row, and back to row.

1	2	3	4	5	6	7	8	9
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.

→

1	2	3	.	.	.	.	.	.
4	5	6	.	.	.	.	.	.
7	8	9	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.

→

1	2	3	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
4	5	6	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
7	8	9	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.

abcd
bcad

0123
1203

1203

Prove that any permutation of the coordinates in a 6-region sudoku square preserves the property that all 54 regions contain the symbols 1–9.

Prove that symmetric 6-region sudoku puzzles remain symmetric under any permutation of the four coordinates. This might be a little tricky, so here's one way of dong it.

Let π be any permutation of the four coordinates. We know that the operation $(a, b, c, d) \mapsto \pi((a, b, c, d))$ preserves the 6-region sudoku property.

Now define another operation on the four coordinates,

$$\mu : (a, b, c, d) \mapsto (2 - a, 2 - b, 2 - c, 2 - d).$$

Show that applying μ to a cell moves it to its diametrically opposite position (rotation by 180°). Hence after applying μ to a symmetric 6-region puzzle, the result is also a symmetric 6-region puzzle.

Prove that π and μ commute; that is $\mu(\pi((a, b, c, d))) = \pi(\mu((a, b, c, d)))$ for any cell coordinates (a, b, c, d). For example, if π maps (a, b, c, d) to (b, c, d, a), we have:

$$\begin{array}{ccc} (a,b,c,d) & \xrightarrow{\quad\pi\quad} & (b,c,d,a) \\ \downarrow \mu & & \downarrow \mu \\ (2-a,2-b,2-c,2-d) & \xrightarrow{\quad\pi\quad} & (2-b,2-c,2-d,2-a) \end{array}$$

How to solve 6-region sudoku puzzles

Draw an empty 9×9 array. Put a 1 in cell $(0, 1, 0, 1)$ (i.e. centre of top left box). Then put an 'x' in all positions of the array where a 1 cannot go under the 6-region sudoku rules.

Perhaps you obtained a diagram like the one on the left, below.

Now look at the right-hand diagram and explain why the cells marked 'z' must also be 1-free.

Formulate and prove a general rule based on this special case.

Prove that the two main diagonals of a 6-region sudoku square each contain the symbols 1–9.

1-free cells

x	x	x	.	x	.	.	x	.
x	1	x	x	x	x	x	x	x
x	x	x	.	x	.	.	x	.
.	x	.	.	.	.	.	.	.
x	x	x	.	x	.	.	x	.
.	x	.	.	.	.	.	.	.
.	x	.	.	.	.	.	.	.
x	x	x	.	x	.	.	x	.
.	x	.	.	.	.	.	.	.

More 1-free cells

x	x	x	.	x	.	.	x	.
x	1	x	x	x	x	x	x	x
x	x	x	.	x	.	.	x	.
.	x	.	z	.	z	z	.	z
x	x	x	.	x	.	.	x	.
.	x	.	z	.	z	z	.	z
.	x	.	z	.	z	z	.	z
x	x	x	.	x	.	.	x	.
.	x	.	z	.	z	z	.	z

Prove that a 6-region sudoku puzzle must have at least 8 starter digits.

Unlike normal sudoku, the minimum is actually attained. Suppose you are given a symmetric 6-region sudoku puzzle with 8 starter digits (such as Puzzles B, C, D, below). Because 8 is even, the central position, $(1, 1, 1, 1)$, will not contain a starter digit. What must the centre digit be?

Formulate a general principle which halves the work involved in solving a symmetric 6-region sudoku puzzle.

Magidoku puzzles

We have already seen one way of giving some magic-like property to the boxes of a standard sudoku square. As an alternative to quasi-magic squares (where the rows, columns and diagonals of the boxes sum to approximately 15), we can insist that just the rows and columns of the boxes sum to 15 (exactly). The diagonal sums are unrestricted. The boxes are then *semi-magic squares* and one could call the resulting array a semi-magic sudoku square. However, John Bray of QMUL, who first took up this idea, has coined the name *magidoku*.

What are the symmetries of a magidoku square?

Determine all possible semi-magic squares.

Armed with this information you can attempt magidoku puzzles E, F and G.

The constraint imposed by the limited number of available semi-magic squares is quite severe. Hence magidoku puzzles are usually sparse. Indeed it is possible to have less than eight starter digits, such as Puzzle G. Because permutations of the symbol set are generally not permitted, the proof of the lower bound of 8 that works for standard sudoku does not apply to magidoku puzzles.

Constant row sum sudoku puzzles

For yet another variation on the sudoku-with-magic-squares theme, let the boxes have the property that their rows add up to 15. The column and diagonal sums are unrestricted. We call these *constant row sum sudoku puzzles*.

What are the symmetries of a constant row sum sudoku square?

Show that the boxes must conform to one of two patterns (plus symmetries).

α

1	5	9
2	6	7
3	4	8

β

1	6	8
2	4	9
3	5	7

Prove that the boxes in a row block have either all pattern α or all pattern β.

You may now attempt puzzles H, I, J, K and L. But before you do, observe that puzzles can be classified by their three row block box patterns. There are four types: $\{\alpha, \alpha, \alpha\}$ (all row blocks have box pattern α), $\{\alpha, \alpha, \beta\}$, $\{\alpha, \beta, \beta\}$ and $\{\beta, \beta, \beta\}$. See if you can identify the box patterns of Puzzles H–L before you start to solve them.

A curiosity. Puzzles where the box patterns are either all α or all β require significantly more starter digits than puzzles with a mixture of α and β. Compare H and K with I, J and L. I would be most interested if anyone can provide an explanation.

6-region sudoku puzzles

							7	
					6			4
2								5
1			8					
	9							

Puzzle B

Every row, column, box, split row, split column and split box contains the symbols 1–9.

						3		
	6							
	9				1			
			4				2	
							7	
		5						

Puzzle C

	5						4	
						3		
8								2
		9						
	7						6	

Puzzle D

Magidoku puzzles

Usual sudoku rules with the extra constraint that the row and columns of each box must sum to 15.

		6	2					
								6
		1						
						5		
5								
					3	6		

Puzzle E

							4	
					7			
		9						
			3		8			
						6		
			6					
	8							

Puzzle F

								4
6								
					1			
								6
		5						
						6		
			3					

Puzzle G

Constant row sum sudoku puzzles

Usual sudoku rules with the extra condition that the rows of each box must sum to 15.

			3				5	1
			7			4		
			1			2		
3							1	
	2						3	
	9							7
		7			9			
		5			4			
8	3				6			

Puzzle H

3					6	2		
2					5			8
1								
				4				
								5
8			6					9
		5	3					6

Puzzle I

2			5					
		5					9	
5				6				
		3		9		2		
				8				9
	6					3		
					3			7

Puzzle J

5					6		8	4
			8			9		
	3	4	1			6		
	8				7			
	2						3	
			3				7	
		7			8	5	9	
		8			9			
1	5		6					3

Puzzle K

		7	9					
		4			7			
	4				8	7		
6								9
		5	4				6	
			7			9		
					6	5		

Puzzle L

Solutions

5	8	4	7	2	1	3	6	9
1	7	2	6	9	3	8	4	5
9	3	6	4	5	8	2	1	7
6	9	3	8	4	5	1	7	2
4	5	8	2	1	7	9	3	6
7	2	1	3	6	9	5	8	4
2	1	7	9	3	6	4	5	8
3	6	9	5	8	4	7	2	1
8	4	5	1	7	2	6	9	3

Solution A

4	3	1	7	8	5	2	6	9
8	5	7	6	9	2	1	4	3
9	2	6	3	1	4	5	7	8
5	7	8	9	2	6	3	1	4
2	6	9	4	3	1	7	8	5
1	4	3	8	5	7	6	9	2
6	9	2	1	4	3	8	5	7
3	1	4	5	7	8	9	2	6
7	8	5	2	6	9	4	3	1

Solution B

5	8	3	9	4	7	6	1	2
4	7	9	1	2	6	3	5	8
2	6	1	8	3	5	7	9	4
7	9	4	2	6	1	8	3	5
6	1	2	5	8	3	9	4	7
3	5	8	4	7	9	1	2	6
1	2	6	3	5	8	4	7	9
8	3	5	7	9	4	2	6	1
9	4	7	6	1	2	5	8	3

Solution C

6	1	8	4	3	9	5	2	7
3	9	4	2	7	5	1	8	6
7	5	2	8	6	1	9	4	3
2	4	5	7	8	6	3	9	1
8	6	7	9	1	3	4	5	2
1	3	9	5	2	4	6	7	8
9	7	3	1	5	2	8	6	4
5	2	1	6	4	8	7	3	9
4	8	6	3	9	7	2	1	5

Solution D

3	5	7	6	8	1	4	9	2
8	1	6	2	4	9	3	5	7
4	9	2	7	3	5	8	1	6
6	8	1	3	5	7	9	2	4
7	3	5	4	9	2	1	6	8
2	4	9	8	1	6	5	7	3
5	7	3	1	6	8	2	4	9
9	2	4	5	7	3	6	8	1
1	6	8	9	2	4	7	3	5

Solution E

5	9	1	4	8	3	7	2	6
7	2	6	9	1	5	3	4	8
3	4	8	2	6	7	5	9	1
1	5	9	7	2	6	8	3	4
6	7	2	3	4	8	1	5	9
8	3	4	5	9	1	6	7	2
9	1	5	6	7	2	4	8	3
4	8	3	1	5	9	2	6	7
2	6	7	8	3	4	9	1	5

Solution F

1	5	9	6	2	7	3	8	4
6	7	2	8	4	3	5	1	9
8	3	4	1	9	5	7	6	2
7	2	6	9	5	1	4	3	8
3	4	8	2	7	6	9	5	1
5	9	1	4	3	8	2	7	6
9	1	5	7	6	2	8	4	3
4	8	3	5	1	9	6	2	7
2	6	7	3	8	4	1	9	5

Solution G

6	7	2	3	4	8	9	5	1
1	5	9	7	6	2	4	8	3
4	8	3	1	9	5	2	7	6
3	4	8	6	2	7	5	1	9
7	2	6	9	5	1	8	3	4
5	9	1	4	8	3	6	2	7
2	6	7	5	1	9	3	4	8
9	1	5	8	3	4	7	6	2
8	3	4	2	7	6	1	9	5

Solution H

6	8	1	4	9	2	5	3	7
3	5	7	8	1	6	2	9	4
2	4	9	7	3	5	6	1	8
1	6	8	5	7	3	9	4	2
5	7	3	2	4	9	8	6	1
4	9	2	1	6	8	3	7	5
8	3	4	6	2	7	1	5	9
9	1	5	3	8	4	7	2	6
7	2	6	9	5	1	4	8	3

Solution I

2	9	4	5	3	7	1	6	8
1	8	6	4	2	9	7	3	5
3	7	5	6	1	8	4	9	2
5	1	9	7	6	2	8	4	3
8	4	3	1	9	5	2	7	6
6	2	7	3	8	4	5	1	9
7	6	2	9	5	1	3	8	4
4	3	8	2	7	6	9	5	1
9	5	1	8	4	3	6	2	7

Solution J

5	9	1	7	2	6	3	8	4
6	7	2	8	4	3	9	1	5
8	3	4	1	9	5	6	2	7
4	8	3	2	6	7	1	5	9
7	2	6	9	5	1	4	3	8
9	1	5	3	8	4	2	7	6
2	6	7	4	3	8	5	9	1
3	4	8	5	1	9	7	6	2
1	5	9	6	7	2	8	4	3

Solution K

2	6	7	9	5	1	3	8	4
8	3	4	6	2	7	1	9	5
5	9	1	3	8	4	2	7	6
9	4	2	1	6	8	7	5	3
6	1	8	5	7	3	4	2	9
3	7	5	4	9	2	8	6	1
7	5	3	2	4	9	6	1	8
1	8	6	7	3	5	9	4	2
4	2	9	8	1	6	5	3	7

Solution L

Part IV: Latin squares

Tony Forbes

		7			1		3	4
				8	6	1	2	
	8			2			5	3
		5		7		9		
2	9			3			6	
	3	4	8	6				
1	2		4			7		

Mutually orthogonal Latin squares

Behold, two Latin squares of order 3 (left, below).

0	1	2
1	2	0
2	0	1

0	1	2
2	0	1
1	2	0

$\rightarrow$

00	11	22
12	20	01
21	02	10

Combining the symbols of both squares together we get the single square on the right. Check that every ordered pair of symbols is present. Thus we have here a pair of *orthogonal* Latin squares.

Even more impressive is the next example involving Latin squares of order 5.

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

0	1	2	3	4
2	3	4	0	1
4	0	1	2	3
1	2	3	4	0
3	4	0	1	2

0	1	2	3	4
3	4	0	1	2
1	2	3	4	0
4	0	1	2	3
2	3	4	0	1

0	1	2	3	4
4	0	1	2	3
3	4	0	1	2
2	3	4	0	1
1	2	3	4	0

Now we have *four* Latin squares and furthermore they are mutually orthogonal in pairs. You could verify that manually. There are six pairs of squares and each pair generates 25 symbol pairs. A lot of things to check. But in fact we don't have to do anything because later on we will prove that it must be so. If you look closely, you will see that the squares have been created in a very orderly manner. See if you can guess the pattern before reading on.

Here is a formal definition. For any square array X, Latin or otherwise, denote by $X(r, c)$ the symbol at row r, column c. A set of $k \geq 2$ Latin squares $L_1, L_2, \ldots, L_k$

of order n on the same symbol set is a set of *mutually orthogonal Latin squares* if for any pair L_i, L_j, the set

$$\{(L_i(r,c), L_j(r,c)) : r = 0, 1, \ldots, n-1,\ c = 0, 1, \ldots, n-1\}$$

is the set of all n^2 ordered pairs of symbols.

There is a well known theorem which says that you can have at most $n - 1$ mutually orthogonal Latin squares of order n and that the maximum is attained whenever n is a prime power. Since 3 and 5 are primes (and therefore prime powers) they should have their complete quota, and this is indeed the case, as we have seen. The examples given were actually generated by the formula

$$L_j(r,c) = j\,r + c \bmod p, \quad j = 1, 2, \ldots, p-1, \quad r, c = 0, 1, \ldots, p-1, \qquad (*)$$

which works for any prime p.

Imagine you have an $n \times n$ array L_j generated by formula (*) for some number j. Given a row number r, show that if $L_j(r,x) = L_j(r,y)$ then $x = y$. Hence prove that all the rows of L_j have distinct symbols.

Do something similar to show that all the columns of L_j have distinct symbols.

Deduce that (*) generates Latin squares.

Now imagine you have two $n \times n$ Latin squares L_i and L_j generated by formula (*) for some numbers i and j. Given two pairs of row and column numbers, (r_1, c_1) and (r_2, c_2), show that if $L_i(r_1, c_1) = L_i(r_2, c_2)$ and $L_j(r_1, c_1) = L_j(r_2, c_2)$, then $r_1 = c_1$ and $r_2 = c_2$.

Deduce that L_i and L_j are mutually orthogonal.

Thus we have proved that (*) really does generate $p - 1$ mutually orthogonal Latin squares.

We can also prove that you can never have more than $n - 1$ mutually orthogonal Latin squares of order n.

First prove or at least convince yourself that two orthogonal Latin squares remain orthogonal if you relabel the symbols of one of them.

Suppose you have some mutually orthogonal Latin squares of order n. By relabelling if necessary, we can assume that the first rows are all $0, 1, 2, \ldots, n-1$. Where can 0 go in the second row of the first square? Having decided that, consider where 0 can go in the second row of the second square. Having placed the 0 in the second rows of the first two squares, what positions are available in the third square. Conclude that you run out of positions after placing $n - 1$ zeros in second rows. Hence there can be no more than $n - 1$ mutually orthogonal Latin squares.

0	1	2	3	...
	0			

0	1	2	3	...
		0		

0	1	2	3	...
			0	

...

Mutually orthogonal sudoku squares

We shall now ask ourselves whether we can do something similar for sudoku squares. Do sets of mutually orthogonal sudoku squares exist? What about 6-region sudoku squares?

Sudoku squares are Latin squares with extra conditions. So we expect no more than eight mutually orthogonal sudoku squares. However, we can refine this estimate down a bit.

Use a similar technique to what you did for Latin squares to prove that there can be at most six mutually orthogonal sudoku squares.

Similarly, prove that there can be at most four mutually orthogonal 6-region sudoku squares.

Let us see how many we can actually create. We start by generating mutually orthogonal Latin squares of order 9. Unfortunately there is a problem. Formula (*) won't work because 9 is not prime. But 9 is a prime power; therefore somehow we should be able to obtain eight Latin squares, the maximum number possible.

However, formula (*) does work provided we operate in GF(9), the Galois field of order 9. The elements of GF(9) are the numbers $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ but to learn how to add and multiply them we shall have to go on a little diversion into the theory of finite fields.

Addition in GF(9). Let us add 5 and 8. First convert them to base 3: $5 \to 12$ (expressing the fact that $5 = 1 \times 3 + 2$), and $8 \to 22$. Now add them together as numbers in base 3 but *without carrying*. Thus $12 + 22 = 01$. Finally, convert back to decimal: $01 = 0 \times 3 + 1 = 1$. Hence $5 + 8 = 1$.

Produce a complete addition table for GF(9).

Multiplication in GF(9). This is a lot worse than addition. Let us multiply 5 by 8. Convert to base 3: $5 \to 12$, $8 \to 22$. Next, do a long multiplication in base 3 to get four cross-products:

200	$= 10 \times 20$	from the 1 of 12 and the first 2 of 22
20	$= 10 \times 2$	from the 1 of 12 and the second 2 of 22
10	$= 2 \times 20$	from the 2 of 12 and the first 2 of 22
1	$= 2 \times 2$	from the 2 of 12 and the second 2 of 22

Add in GF(9) to get $200 + 020 + 010 + 001 = 201$. Now we're in trouble because we have a three-digit number 201 and we want the answer to have two digits. What we do in general is *remove the 'hundreds' digit and subtract it modulo 3 from the units position of the rest of the number*. (This works because $x^2 + 1$ is irreducible modulo 3. What we are really doing is putting $x = 10$ and equating to zero. Thus $100 + 1 = 0$; or $100 = -1$.) In our example $201 \to 01 - 2 = 02$. Finally, convert to decimal: $02 \to 2$. Hence $5 \times 8 = 2$.

Now draw up a multiplication table for GF(9). It helps to know that even in GF(9) multiplication is commutative, multiplying something by 0 or 1 has the usual effect, and $a + a = 2a$. But beware, $a + a + a = 0$, not $3a$.

With all this in place use the formula (*) to generate 8 mutually orthogonal

Latin squares, not forgetting that you must work in GF(9). (Or convince yourself that (*) generates the ones I have provided.)

How many are sudoku squares?

How many are 6-region sudoku squares?

If you found 6 and 4, respectively, you have done well; that is best possible.

From Latin squares to affine planes

Start with a Latin square of order 3, $L = \begin{array}{|ccc|} \hline 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \\ \hline \end{array}$. Assemble a set of triples of integers taken from $\{0, 1, \ldots, 8\}$ as follows.

Three triples: $\{x, 3 + x, 6 + x\}$, one for each of $x = 0, 1, 2$.

Nine more triples: $\{3a, 3b+1, 3L(a, b)+2\}$, one for each combination a, b, where $a = 0, 1, 2$ and $b = 0, 1, 2$. Recall that $L(a, b)$ is the number at row a, column b of Latin square L.

You should have 12 triples altogether.

Check they have the property that each unordered pair of integers 0–8 occurs in precisely one triple.

This is an example of a Steiner system, indeed a Steiner triple system. There are nine numbers, or *points*, namely $\{0, 1, \ldots, 8\}$, and twelve triples or *blocks*. The main defining property is that *each pair occurs exactly once*. It's an example of a Steiner system $S(2, 3, 9)$.

More generally, a *Steiner system* $S(t, k, v)$ consists of a set of v points and a set of k-element sets of points (called blocks) subject to the condition that every t-element set of points occurs in exactly one block.

They form an important class of block designs, and they actually have some uses in the design of experiments. A Steiner system $S(2, k, v)$ is a balanced incomplete block design; 'balanced' because every pair is represented the same number of times (once), and 'incomplete' because not every k-tuple is present The idea is that if you want to compare 9 washing machines and you have twelve people to to the testing, you can give each person three washing machines as dictated by the blocks of the $S(2, 3, 9)$. Because of the pairs property, each pair of machines will get compared by someone.

The construction you have just done shows one way of creating an $S(2, 3, 9)$ system out of a Latin square.

We can do the same thing to create an $S(2, 4, 16)$.

We will first generate three mutually orthogonal Latin squares of order 4 using the standard formula

$$L_j(r, c) = j\,r + c, \quad j = 1, 2, 3, \quad r, c = 0, 1, 2, 3, \qquad (**)$$

working in the Galois field GF(4). The elements of GF(4) are $\{0, 1, 2, 3\}$.

Addition in GF(4) is like addition in GF(9). Convert the two numbers to base 2 (binary); add without carrying; convert back to decimal. Thus $2 + 3 \rightarrow 10 + 11 \rightarrow$

$01 \to 1$.

Multiplication in GF(4) is only slightly less hideously complicated than in GF(9). Convert to binary; do a long multiplication to get four cross-products; add the cross products in binary without carrying. If the result has two digits, fine. If the result has three digits, remove the 'hundreds' digit (which will be a 1) and add 11 (in GF(4)) to the two-digit number that remains. (This works because $x^2 - x - 1$ is irreducible modulo 2.) Convert to decimal. Thus $3 \times 3 \to 11 \times 11 \to 100 + 10 + 10 + 1 \to 101 \to 01 + 11 \to 10 \to 2$.

Draw up addition and multiplication tables for GF(4). (Or verify the ones I have supplied.)

Use them and formula (**) to create three mutually orthogonal Latin squares, L_1, L_2, L_3, of order 4.

Use the first two of them to construct an $S(2, 4, 16)$ system as follows.

Four blocks: $\{x, 4 + x, 8 + x, 12 + x\}$, one for each of $x = 0, 1, 2, 3$.

Sixteen blocks: $\{4a, 4b+1, 4L_1(a, b)+2, 4L_2(a, b)+3\}$, one for each combination of $a = 0, 1, 2, 3$ and $b = 0, 1, 2, 3$.

Now you have 20 blocks altogether. Check that every pair occurs precisely once.

Confirm that the first four blocks cover all 16 points.

See if you can partition the other 16 blocks into four sets of four such that each set covers the 16 points.

You should succeed. A design that has this property is called *resolvable*, and the sets of blocks which cover all the points are called *resolution classes*.

Verify that the $S(2, 3, 9)$ is resolvable.

Based on the two examples you have just done (for $n = 3$ and $n = 4$), see if you can formulate a general method of constructing an $S(2, n, n^2)$ from $n - 2$ mutually orthogonal Latin squares of order n.

A block design constructed in this way is also called an *affine plane*. It is a kind of finite geometry. The points and lines of the geometry are the points and the blocks of the design. From the way the construction works, the design is always resolvable. The resolution classes can be thought of as sets of parallel lines, each of which covers the entire point set. By analogy with the Euclidean plane, each set of parallel lines represents a 'direction' in the affine plane.

The construction works for all prime powers. So the next one would be the $S(2, 5, 25)$. (If you have time, you could try constructing it from three of the four mutually orthogonal Latin squares of order 5 on page 1.) However 6 fails. There are no mutually orthogonal Latin squares of order 6, and indeed the design $S(2, 6, 36)$ does not exist. The same is true for the next non-prime-power, 10. There is no $S(2, 10, 100)$. Eleven is a prime power, but 12 is not, and there's work to be done here—the existence of an $S(2, 12, 144)$ is an open problem.

Construction of the affine plane $S(2, 3, 9)$ from Latin square $\begin{array}{|ccc|} \hline 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \\ \hline \end{array}$:

x			
0	0	3	6
1	1	4	7
2	2	5	8

$a\ b$			
0 0	0	1	2
0 1	0	4	5
0 2	0	7	8

$a\ b$			
1 0	3	1	5
1 1	3	4	8
1 2	3	7	2

$a\ b$			
2 0	6	1	8
2 1	6	4	2
2 2	6	7	5

GF(4) arithmetic:

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Three mutually orthogonal Latin squares of order 4:

L_1

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

L_2

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

L_3

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

Construction of the affine plane $S(2, 4, 16)$:

x				
0	0	4	8	12
1	1	5	9	13
2	2	6	10	14
3	3	7	11	15

$a\ b$				
0 0	0	1	2	3
0 1	0	5	6	7
0 2	0	9	10	11
0 3	0	13	14	15
1 0	4	1	6	11
1 1	4	5	2	15
1 2	4	9	14	3
1 3	4	13	10	7

$a\ b$				
2 0	8	1	10	15
2 1	8	5	14	11
2 2	8	9	2	7
2 3	8	13	6	3
3 0	12	1	14	7
3 1	12	5	10	3
3 2	12	9	6	15
3 3	12	13	2	11

From affine planes to projective planes

We have seen how to construct affine planes from mutually orthogonal Latin squares. But the real affine plane of Euclidean geometry can be extended to create the real projective plane by adding the points at infinity where parallel lines meet. As you can imagine, we can do precisely the same thing with our finite affine planes, the $S(2, n, n^2)$ systems.

Take the affine plane $S(2, 3, 9)$. There are 9 points and 12 lines. Each line has 3 points.

If you succeeded in resolving the 12 blocks of the $S(2, 3, 9)$ into sets of parallel lines, you should have these resolution classes, or at least something similar.

0	3	6
1	4	7
2	5	8

0	1	2
3	4	8
6	7	5

0	4	5
3	7	2
6	1	8

0	7	8
3	1	5
6	4	2

We shall add four new points, 9, 10, 11 and 12.

Attach $8+i$ to each of the blocks in the ith set of parallel lines, for $i = 1, 2, 3, 4$. So the first set becomes $\{\{0, 3, 6, 9\}, \{1, 4, 7, 9\}, \{2, 5, 8, 9\}\}$.

You now have twelve 4-point blocks. The parallel lines have met at the points at infinity, 9, 10, 11 and 12. Not only that but the points at infinity make a line at infinity, namely $\{9, 10, 11, 12\}$. Just as with the Euclidean plane, we add the line at infinity to the system.

Verify that (i) we have 13 points; (ii) thirteen 4-point blocks; (iii) every pair of points 0–12 occurs in precisely one block.

Hence we have a Steiner $S(2, 4, 13)$ system, the finite projective plane of order 3.

If you verified that the $S(2, 4, 16)$ is resolvable, you probably obtained these sets of parallel lines.

0	4	8	12
1	5	9	13
2	6	10	14
3	7	11	15

0	1	2	3
4	13	10	7
8	5	14	11
12	9	6	15

0	5	6	7
4	9	14	3
8	1	10	15
12	13	2	11

0	9	10	11
4	5	2	15
8	13	6	3
12	1	14	7

0	13	14	15
4	1	6	11
8	9	2	7
12	5	10	3

Use this resolution of the affine plane $S(2, 4, 16)$ to create the finite projective plane of order 4, the Steiner system $S(2, 5, 21)$.

Formulate a general method for constructing a finite projective plane of order n from the affine plane of order n, $S(2, n, n^2)$.

Construct the affine plane $S(2, 2, 4)$ and the projective plane $S(2, 3, 7)$.

GF(9) plus and times tables

$+$	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	6
2	2	0	1	5	3	4	8	6	7
3	3	4	5	6	7	8	0	1	2
4	4	5	3	7	8	6	1	2	0
5	5	3	4	8	6	7	2	0	1
6	6	7	8	0	1	2	3	4	5
7	7	8	6	1	2	0	4	5	3
8	8	6	7	2	0	1	5	3	4

$\times$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	1	6	8	7	3	5	4
3	0	3	6	2	5	8	1	4	7
4	0	4	8	5	6	1	7	2	3
5	0	5	7	8	1	3	4	6	2
6	0	6	3	1	7	4	2	8	5
7	0	7	5	4	2	6	8	3	1
8	0	8	4	7	3	2	5	1	6

using irreducible polynomial $x^2 + 1$

Eight mutually orthogonal Latin squares of order 9

L_1

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	5	3	4	8	6	7
3	4	5	6	7	8	0	1	2
4	5	3	7	8	6	1	2	0
5	3	4	8	6	7	2	0	1
6	7	8	0	1	2	3	4	5
7	8	6	1	2	0	4	5	3
8	6	7	2	0	1	5	3	4

L_2

0	1	2	3	4	5	6	7	8
2	0	1	5	3	4	8	6	7
1	2	0	4	5	3	7	8	6
6	7	8	0	1	2	3	4	5
8	6	7	2	0	1	5	3	4
7	8	6	1	2	0	4	5	3
3	4	5	6	7	8	0	1	2
5	3	4	8	6	7	2	0	1
4	5	3	7	8	6	1	2	0

L_3

0	1	2	3	4	5	6	7	8
3	4	5	6	7	8	0	1	2
6	7	8	0	1	2	3	4	5
2	0	1	5	3	4	8	6	7
5	3	4	8	6	7	2	0	1
8	6	7	2	0	1	5	3	4
1	2	0	4	5	3	7	8	6
4	5	3	7	8	6	1	2	0
7	8	6	1	2	0	4	5	3

L_4

0	1	2	3	4	5	6	7	8
4	5	3	7	8	6	1	2	0
8	6	7	2	0	1	5	3	4
5	3	4	8	6	7	2	0	1
6	7	8	0	1	2	3	4	5
1	2	0	4	5	3	7	8	6
7	8	6	1	2	0	4	5	3
2	0	1	5	3	4	8	6	7
3	4	5	6	7	8	0	1	2

L_5

0	1	2	3	4	5	6	7	8
5	3	4	8	6	7	2	0	1
7	8	6	1	2	0	4	5	3
8	6	7	2	0	1	5	3	4
1	2	0	4	5	3	7	8	6
3	4	5	6	7	8	0	1	2
4	5	3	7	8	6	1	2	0
6	7	8	0	1	2	3	4	5
2	0	1	5	3	4	8	6	7

L_6

0	1	2	3	4	5	6	7	8
6	7	8	0	1	2	3	4	5
3	4	5	6	7	8	0	1	2
1	2	0	4	5	3	7	8	6
7	8	6	1	2	0	4	5	3
4	5	3	7	8	6	1	2	0
2	0	1	5	3	4	8	6	7
8	6	7	2	0	1	5	3	4
5	3	4	8	6	7	2	0	1

L_7

0	1	2	3	4	5	6	7	8
7	8	6	1	2	0	4	5	3
5	3	4	8	6	7	2	0	1
4	5	3	7	8	6	1	2	0
2	0	1	5	3	4	8	6	7
6	7	8	0	1	2	3	4	5
8	6	7	2	0	1	5	3	4
3	4	5	6	7	8	0	1	2
1	2	0	4	5	3	7	8	6

L_8

0	1	2	3	4	5	6	7	8
8	6	7	2	0	1	5	3	4
4	5	3	7	8	6	1	2	0
7	8	6	1	2	0	4	5	3
3	4	5	6	7	8	0	1	2
2	0	1	5	3	4	8	6	7
5	3	4	8	6	7	2	0	1
1	2	0	4	5	3	7	8	6
6	7	8	0	1	2	3	4	5

```
L1:  1 1 0 1 1 0    L2:  1 1 0 1 1 0    L3:  1 1 1 0 0 1    L4:  1 1 1 1 1 1
L5:  1 1 1 1 1 1    L6:  1 1 1 0 0 1    L7:  1 1 1 1 1 1    L8:  1 1 1 1 1 1
```