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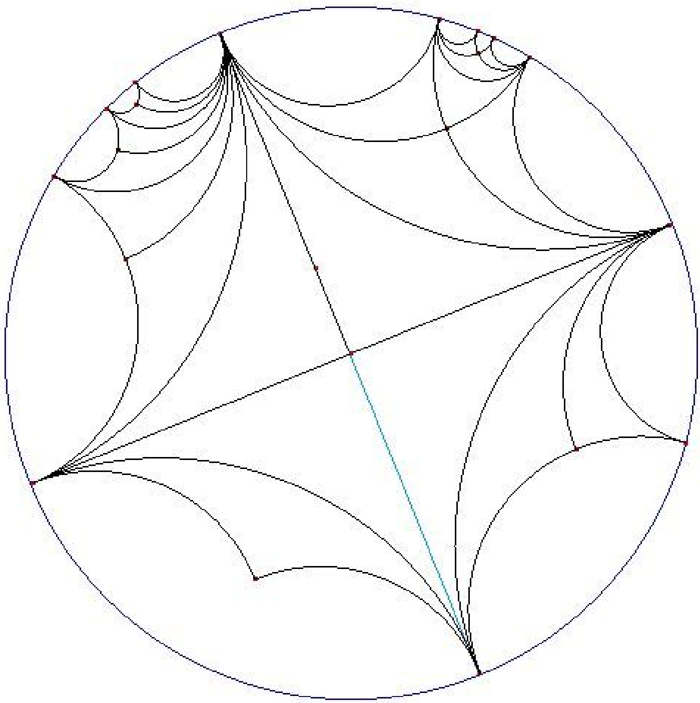
ISSN 1350-8539

# M500 171



## The M500 Society and Officers

**The M500 Society** is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

**The magazine** M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

**The** M500 **Special Issue**, published once a year, gives students' reflections on their previous courses.

**MOUTHS** is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

**The September Weekend** is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

**The Winter Weekend** is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

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**Advice to authors**. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

# A history of time

## David Singmaster

This *History* is based on extracts from my various chronologies, particularly my *Medieval Chronology—From the Greeks to the Renaissance*, which includes most of this material as relating to medieval developments. It is distinct from my chronology of the calendar. The division is basically that the calendar is concerned with the length of the year and the month and the arrangement of days into weeks, months and years, while time is concerned with the length of the day, subdividing it into convenient bits and measuring these bits. There is some overlap.

*Notes*:

1 year = 365.242216 or 365.242197 or 365.24219870 days or
365 days 5 hr 48 min 46 sec = 365.2421990741 days.

1 lunar month = 29.530598 days = 29 days 12 hr 44 min 3.67 sec or
29 days 12 hr 44 min = 29.5305555... days

1 year = 12.368267 lunar months.

The day starts at various times.

At dawn: Babylonian, Indian.

At sunset: Jewish, Greek, Moslem.

At 11 pm: China.

At midnight: Modern general usage, except astronomers use noon.

One source says Italy was using noon in 1937. The calculation of the Jewish sunset depends on three trigonometrical equations!

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***c.* −3000.** Neugebauer says the twelve hours of the day and night were an outcome of the Egyptian decimal system. They had a decade of ten days and then divided the sky into ten-day intervals, leading to 36 ‘decans’. Eighteen of these occur at night, but only twelve are dark enough to be able to observe stars at the horizon, giving twelve hours of night. The day had ten hours, as marked on sundials, but two extra hours of dawn and dusk were added to get twelve hours of day. It is not until Hellenistic times that equal hours for day and night are introduced.

***c.* −2000?** The Babylonians divide the day into six parts and the night into six parts. They are thought to have had sundials from this time.

***c.* −1500.** Oldest extant sundial, from Egypt, now in the British Museum.

**−1550?** Earliest waterclock (clepsydra) invented by the astronomer Amenemhet and described on the walls of his tomb near Luxor, Egypt. He states that a winter night is 14 hours long and a summer night is 12 hours.

***c.* −1300.** A cenotaph of Seti I shows a sundial and describes its use.

**−13C.** Sundial with the name of Pharaoh Herneptah, found in Palestine.

***c.* −1000?** The Jews divide both the day into 3 parts and the night into 3 parts (known as watches). The day watches are called: Morning, Heat of the Day, Cool of the Day.

**−7C.** Biblical book of *Isaiah* is first written mention of a sundial.

**−580.** Anaximander of Miletus teaches how to make sundials. Herodotus says the Greeks learned about the sundial from the Babylonians.

***c.* −425.** Aristophanes: *Ecclesiazusae* mentions a gnomon, showing the idea of the sundial was well known. He also mentions waterclocks.

***c.* −340.** Aristotle mentions the invention of gears, waterclocks and the twelve parts of the night.

***c.* −300.** Waterclocks in use in India.

***c.* −270.** Berossos, a Babylonian astronomer settled in Cos, invents the hemi-cyclic sundial, basically a bowl with a gnomon in the middle.

***c.* −260.** Ktesibos invents various hydraulic and pneumatic devices, including the waterclock. He also invents metal springs and uses gear wheels.

***c.* −230?** Sundials introduced to Rome. The playwright Plautus (d. −184) writes of them as an innovation.

***c.* −180.** Hypsicles divides day into 360 parts in his *On Rising Times.* (I have seen an assertion that the Babylonians divided the zodiac into 360 degrees, *c.* −450.)

**−159.** First waterclocks in Rome.

***c.* −180 / *c.* −125.** Hipparchus. He introduces hours of equal length in the day and the night, but these are only used in scientific work.

***c.* −113.** Oldest extant waterclock, in China.

***c.* −80.** Antikythera mechanism, a Greek calendrical computer, with many gear wheels.

**−1C. or +1C.** Andronicus of Cyrrhestes builds the Tower of the Winds in Athens, a large waterclock.

***c.* 1C.** In New Testament times, both the day and the night are divided into four watches. The day watches were called third hour, sixth hour, ninth hour and twelfth hour, corresponding to the hours at the end of the watch. Cf. −1000.

**2C. or 3C.** There is a portable Roman sundial of this date well preserved at Oxford.

**724.** Yi Xing (= Zhang Sui = I-Hsing = Yi-hsing), astronomer, is said to have built or used a waterclock.

**8C.** Luitprand, a monk at Chartres, is said to have invented the hourglass.

**802.** Haroun al Rashid sends a waterclock to Charlemagne (or in 807).

**807.** Charlemagne orders a 12-hour hourglass.

**850.** Pacificus, archdeacon of Verona, is said to have invented the escapement and the use of weights to drive a clock.

***c.* 1000.** Gerbert (Pope Sylvester II) makes a pendulum escapement and is said to have invented the striking clock.

**1088?** Su Sung builds an astronomical waterclock at the Observatory at Kaifeng, using a 3.5 m diameter escapement wheel.

**Early 12C.** Al-Jazari's *Compendium of the Theory and Practice of the Mechanical Arts* describes clocks, musical automata, etc.

***c.* 1280.** First mechanical clocks, in England, then in Europe. Previously there had been many waterclocks and the oldest surviving picture of one is from *c.* 1250. A guild of clockmakers is mentioned in Cologne in 1183 and in 1220, they occupied a street, the Urlogingasse. In 1271, Robert the Englishman wrote a treatise on the state of clockmaking and the attempts to make a purely mechanical clock.

***c.* 1300.** Invention of the mechanical escapement for weight driven clocks, in Northern Europe. These clocks had one hand and a foliot balance.

**1309/1313.** A poem of Francesco da Barberino is earliest reference to an hourglass (with sand).

**1323/1382.** Oresme (published 1868). First(?) person to propose a Date Line. See 1519 / 1522.

***c.* 1330.** Richard of Wallingford begins the first great astronomical clock, at St Albans Abbey. Completed *c.* 1356. The first clock to be clearly described.

**1335/1347.** First mechanical clocks: St Gothard Church, Milan, 1335 (or 1336); Modena (?), 1343; Padua, 1344, by Jacopo Dondi; Monza, 1347. (Dover, 1348?) (Paris, 14C?)

**Mid 14C.** Civil adoption of 24 equal hours for the day, beginning in Italy.

**1350?** Clock at Peterborough Cathedral. Surviving parts are thought to date from 1450 or even 1350. Until 1950, its striking train was the oldest working clock mechanism.

**1352/1354.** Strasbourg Cathedral has an astronomical clock dating from this time. The local Museum has the automaton cockerel from the clock. The clock died in the early 16C and a new one was built, using the original cockerel. The clock stopped about 1788 and was substantially rebuilt in 1838–1842, when the cockerel was retired to the museum and replaced by a new version.

**1365.** Giovanni di'Dondi's treatise on clocks, describing his Atrarium (astronomical clock), built in 1348–1364 at Pavia. It was still extant in 1529, and was repaired. It has recently been reproduced.

**1370.** Henry de Vick builds a clock for Emperor Charles V.

**1379.** The city clock at Rouen, La Grosse Horologe, was built in 1379 and is probably the oldest extant clock. It was converted to pendulum control and ran until 1929 when the works were replaced by electric motors. Another source says it was built in 1389 by Jean de Fealins and was still running in 1942.

**1382.** There is a clock at Dijon brought as a war spoil in 1382. It may represent the oldest extant clock but it is not running and little of the original is left.

**1386.** Mention of a clock at Salisbury Cathedral. Its works are in the nave and still run.

***c.* 1390.** Clock at Wells Cathedral—works are preserved at the Science Museum, London, and still run. The marvellous striking jack and jousting horsemen in the north transept of the Cathedral are probably original and operate at the quarter hours. An older book dates this as 1340.

**?** Strälsund has 'probably the oldest original, still running, astronomical clock in Europe'—but this source gave no date.

***c.* 1430.** First extant European spring-driven clock, built for Philip the Good, Duke of Burgundy. Even if the works are not original, there were others by the 1440s. This clock includes the earliest fusee.

**1497/1500.** A drawing of Leonardo da Vinci seems to be a pendulum escapement for a clock.

***c.* 1500.** Peter Henlein (or Hele) of Nuremberg is said to have invented the mainspring, or at least to have used it to make the first pocket watch in 1510.

**1514.** J. Werner suggests measuring the distance between the moon and a star to determine absolute time and hence longitude—'the method of lunar distances'. See 1524.

**1519/1522.** Magellan's expedition first circumnavigates the earth. They find their reckoning off by a day! Cf. 1323/1382.

**1524.** Gemma Frisius gives first diagram showing the measurement of a lunar distance. See 1514.

***c.* 1525.** Beginning of watch manufacture in Germany.

**1525.** Oldest known spring clock, by Jacob Zech of Prague.

**1530.** Gemma Frisius first proposes determining longitude by use of sea-going clocks. It takes two centuries for such clocks to be developed. (Another source says this idea first occurs in the fourth edition of his *De Principiis Astronomiae & Cosmographiae* in 1553.)

**1540.** Hampton Court clock, possibly by Nicholas Kratzer.

**1581 or 1583.** Galileo notes constancy of the pendulum, but he doesn't use it for timing rolling bodies. Mersenne suggests the idea to Huygens *c.* 1640, leading to Huygens creating the pendulum clock.

***c.* 1585.** Joost Bürgi develops a clock escapement which was a forerunner of the pendulum clock.

***c.* 1600.** Shakespeare: *As You Like It* mentions a portable sundial.

**1641.** Galileo and his son Vincenzio attempt to make a pendulum clock but dies before finishing it. Viviani made a drawing from Vincenzio's description in 1658.

**1641.** A report says Richard Harris made a pendulum clock for St Paul's Church, Covent Garden, but there is no definite evidence of this.

**1657.** Huygens designs a practicable pendulum clock. The first example was constructed by Salomon Coster of The Hague in 1657 and he presented it to the States General. (Does it still exist?) Shortly thereafter, he invented the use of cycloidal cheeks to compensate for the dependence of period on amplitude, as well as inventing the balance spring (though Hooke may have done this earlier).

**1658.** Christmas Day: Huygens starts his pendulum clock. (1657?) Within a few years they were being produced throughout Europe. Apparently he was anticipated by Johann Philipp Treffler, a clockmaker in Augsburg.

**1639/1713.** Thomas Tompion, the father of English watchmaking.

**1667.** Viviani adapts (or installs) Galileo's clock in the Palazzo Vecchio, Florence, with a pendulum.

**1673.** Huygens: *Horologium Oscillatorium* publishes his clock designs.

**1674.** Robert Hooke: *Animadversions* ... describes his invention of the clock driven telescope.

**1675.** Huygens develops a watch, using a balance spring—but Hooke had already described this in 1658 and apparently made some watches.

**1675.** Hooke invents the anchor or recoil escapement, making the pendulum very usable. The first examples are produced by W. Clement in London in 1676.

**1675.** Royal Observatory founded at Greenwich by Charles II who commanded Flamsteed 'to find out the so-much-desired Longitude of Places for the perfecting the art of navigation'. (Different sources give slightly different versions of this.) Charles's interest was aroused when a French friend of his mistress, the Duchess of Portsmouth, claimed to have a method for finding longitude. The method did not work, but Jonas Moore, Surveyor General of the Ordnance and an author of mathematical texts, instigated the foundation of an observatory. John Flamsteed was appointed the first Astronomer Royal. The Astronomers Royal lived in Flamsteed House from 1676 to 1948—this was designed by Wren 'for the Observator's habitation and a little for pompe', as Wren himself put it. Tompion later provides two remarkable one-year 'deadbeat escapement' clocks which Flamsteed used to check the constancy of the day in 1676—there are replicas in the Old Observatory and one of the originals was acquired and put on display in 1994. However, below it says Tompion didn't devise the deadbeat escapement until 1725, after Flamsteed's death—?

**1676.** Hooke: *A Description of Helioscopes ...* describes his invention of the balance spring watch.

**1673/1751.** George Graham. Leading clockmaker of his time, becoming FRS. Started working for Tompion, married his niece and became his partner and successor. Graham and Tompion are buried in the same grave in Westminster Abbey. Graham first proposed a compensating pendulum. He made the first version of an orrery, by 1715, for Prince Eugen.

**1707.** Admiral Sir Cloudisley Shovell runs four ships aground on the Gilstone Ledges off the Isles of Scilly, losing about 2000 lives, including his own, due to not knowing their longitude (though other sources say his crew had celebrated a famous victory rather too extensively).

**1714.** Parliament passes *An Act for providing a Publick Reward for such Person or Persons as shall Discover the Longitude at Sea* (The Longitude Act).

**1725.** Graham invents the deadbeat escapement, making pendulum clocks accurate to a few seconds per week. This was the most accurate method for nearly two centuries. The first ones go the Greenwich Observatory.

**1693/1776.** John Harrison. Inventor of the marine chronometer. Awarded the Copley Medal of the Royal Society, 1749.

**1735.** John Harrison builds the first marine chronometer, but it takes until 1759 to get one which really works.

**1741.** Captain G. Anson attempts to make a landfall by sailing along a parallel, but heads the wrong way due to having lost his longitude. Consequently many of his crew perish from scurvy.

**1755/1759.** Harrison builds his fourth chronometer, H4, the first small one. It only lost 5 seconds in a 62-day voyage and is considered the most important watch ever made.

**1761.** Maskelyne tests the method of lunar distances on a voyage to St

Helena to observe the transit of Venus and finds he can always determine longitude to within a degree. This leads to the *The Nautical Almanac* being started in 1767.

**1762.** Harrison claims the Longitude Prize, but it takes eleven years and the intervention of the King before it is all awarded.

**1765.** Thomas Mudge, a student of Graham, invents the lever escapement.

**1767.** *The Nautical Almanac* is started, providing tables of lunar distances for determining longitude.

**1773.** Harrison finally is awarded the Longitude Prize of £20,000.

**1793.** As part of the Revolution and the adoption of the metric system, France adopts a year of twelve 30-day months with five or six holidays at the end—cf. under *Calendars*. The French also tried to decimalize time with 10 hours of 100 minutes and angles with 100 centigrades in a right angle.

**1847.** General Post Office adopts London Time throughout UK.

**1833.** The Time Ball on the roof of the Royal Greenwich Observatory was installed—the first public time-signal, dropping at 1:00 pm GMT.

**1841.** Alexander Bain and Barwise receive first patent for an electric clock.

***c.* 1850.** Airy has the transit circle installed at Greenwich. It defines the Prime Meridian—cf. 1884—and remained in regular use until 1954. There were earlier meridians through earlier instruments. From 1675 to 1720, the Meridian was east of the present one and from 1750 to 1816, it was 19 feet west of the present one.

**1852.** A new clock was installed at Greenwich and telegraphy was added to produce time signals at other places, e.g. on the Electric Telegraph Company in the Strand, London Bridge Station and then via the railways to time balls at other ports.

**1850s.** The clock in the tower of the Houses of Parliament is quite remarkable. The architect, Charles Barry, made certain specifications and asked the leading clockmaker of the day, Benjamin Lewis Vulliamy, to design a clock to fit. Vulliamy produced a design which was never used but would have been a poor time keeper and a feeble striker. Other clockmakers objected to the design not having been open to them and Airy was called in to refine the specifications. Airy specified that the first hour stroke should be accurate to one second. Vulliamy and the Clockmakers' Company protested that this could not be done. But in 1851, an MP, Edmund Beckett Denison, QC, later Sir Edmund Beckett and then Lord Grimthorpe, a passionate amateur of clockmaking who 'knew more about clockmaking than most clockmakers, and more about the mathematics of clocks than even Airy', produced the design which was built by E. J. Dent and his successor

Frederick Dent in 1851–1854. In 1854 it was set running and in 1859 it was installed in the tower.

Denison invented and fitted the 'double three-legged gravity escapement' which separated the time-keeping from the driving of the hands and bells and which was used for all later large clocks until the advent of electric motors. Denison also had to redesign and build the minute hands—Barry's were so heavy that they shook and stopped the clock. Denison made them from flattened copper tubes so they weighed about two hundred pounds each, less than a third of the weight of Barry's. When installed, it kept better time than almost any clock extant, including most astronomical observatory clocks. It gains or loses about a second in ten days and the rate is adjusted by adding or removing a penny to or from the pendulum every two weeks or so. In 1937, it was off by more than a second on only 18 days [Ackermann, p. 690]. It is claimed that the clock was slowed down by five minutes due to a flock of starlings roosting on the minute hand in 1945. The minute hands are 14 ft long, the hour hands are 9 ft long, the pendulum is 13 ft long, giving a 2 second period, and the bob weighs about 400 pounds [Fell, pp. 31–32].

Denison even designed the bells. The first 'Big Ben' of 1856 was miscast and soon cracked. At 14 tons, it was the largest bell ever cast in England. It was melted down and recast by George Mears of Whitechapel in 1858, but small cracks soon appeared. After being silent for some time, it was turned, the cracks were stopped by drilling out the ends, the hammer was lightened and 'Big Ben' was returned to use in 1863 and has continued ever since. (Imperial College has an 1884 example of a clock by Denison.)

**1869.** Charles F. Dowd proposes time zones in the US and Sandford Fleming (*c.* 1878) independently campaigns in Canada. Both the US and Canada correlate with the Greenwich Meridian.

**1880.** Greenwich Mean Time adopted as standard throughout Britain.

**1883.** Noon, 18 November 1883: Standard Time adopted in the US on the initiative of the American Railway Association. Another source gives 13 November. The signals were (are?) generated at the US Naval Observatory in Washington.

**1883–84.** Rome Conference proposes and Washington Conference adopts Greenwich as the international Prime Meridian and hence Greenwich Mean Time as standard—the vote was 22 to 1 with 2 abstentions. It was estimated that 90% of sea charts in use were already based on Greenwich.

**1912.** William Willett first proposes summer (daylight saving) time, first introduced in 1916, after Willett's death. (But Benjamin Franklin is said to have suggested the idea.)

**1918.** US adopts time zones.

**1920.** L. G. Hawkins devises the electric clock controlled teamaker.

**1921.** William Hamilton Shortt, at the Edinburgh Observatory, invents the 'free pendulum'. The resulting Synchronome–Shortt clock is accurate to about a second per year.

**1921.** First telephone time service inaugurated in Paris.

**1924.** 5 February: The 'six-pips' BBC time signals begin.

**1925.** Summer Time Act makes summer time permanent.

**1927.** Radio time signals for correcting chronometers begin.

**1929.** Marrison, a Canadian, develops quartz crystal clock.

**1933.** 14 February: Paris Observatory starts first Speaking Clock service.

**1936.** 24 July: The Talking Clock starts at Holborn Telephone Exchange. The voice was that of Miss Ethel Cain.

**1941.** 4 April: Double Summer Time adopted in UK.

**1948–1958.** The Royal Greenwich Observatory removed to Herstmonceux Castle, Sussex, in 1948–1958, due to smoke and light pollution obscuring the viewing, as well as air pollution attacking the instruments, particularly tarnishing the silver of the mirrors.

The Royal Greenwich Observatory no longer exists as a particular site—it is headquartered at Cambridge and its main observatory is the Northern Hemisphere Observatory on La Palma in the Canary Islands.

**1963.** Time signals become internationally coordinated.

**20C.** Atomic clocks replace astronomical observations as the basis of time-keeping and the earth is found to be running a bit slow leading to the occasional introduction of 'leap-seconds'.

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# Problem 171.1 – Cylinder

A coin is a short, fat cylinder. Throw it up in the air and let it fall on to a flat surface such as a table-top. It will almost certainly land on one of its faces, 'heads' or 'tails'.

Do the same with a long, thin cylinder and it is far more likely to land on its curved surface rather than on one of its flat ends.

*At what radius-to-height ratio will the probabilities be equal?*

There is also an equilibrium position where the cylinder is standing on one of its rims and balanced at an angle such that the centre of gravity is directly above the point of contact with the table-top. However, as it is unstable in this configuration we can safely ignore the possibility of it landing there.

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# Review of *StarOffice* 5.1

## Simon Geard

*StarOffice* is an integrated office package for the desktop PC. I thought I'd say a few kind words about it here particularly from the mathematical point of view.

First though a word about availability. Sun Microsystems have now bought the company (StarDivision) and are giving *StarOffice* away free of charge to anyone. It can be downloaded from their web site http://www.stardivision.com. At 70Mb this can be fairly time consuming, but the cost of being connected to the telephone overnight at 1p per minute is not high compared to buying MS Office! *StarOffice* is currently at revision 5.1 and supports Linux, Windows, OS/2 and Solaris. The 5.1 release is also featured on *PCPlus*'s CD-ROM for both Windows (November 1999) and Linux (December 1999).

Most people who produce mathematical texts do not use a word-processor like *MS Word*. This is because it inflexible and incredibly tedious to do lots of pointing and clicking with the mouse. Instead they prefer to use *LaTex*. This is an incredibly powerful system that is quite up to the task of producing mathematical text books with all their cross references, equation numbers and the like. *LaTex* is a mark-up type of language (as is HTML) and lacks a good WYSIWYG interface (although they do exist, *lyx* for example). The final output is produced after a two-pass process (necessary to do all that cross-referencing) and produces a wonderfully professional looking document.

*StarOffice* is a cross between the two. It does not have all the cross-referencing capabilities of *LaTex* and so you wouldn't write a book with it, but it does provide its own command line input so that you don't have to do all that tedious mouse-clicking. The commands are consistent, logical and well thought through. To access them use *insert>object>formula...*. For example, the final solution of the problem 168.2,

$$s \;=\; \sqrt{\frac{41+\sqrt{1071}}{2}} \;\approx\; 6.07,$$

would be coded as

**s = sqrt{{41 + sqrt{1071}} over 2} approx 6.07**

whilst the equation of the circle $(x-s)^2+(y-s)^2=16$ is

**(x - s ) sup 2 + (y - s ) sup 2 = 16** .

You can also do more complex things. For example the definition of *Hankel's Contour Integral*:

$$\frac{1}{\Gamma(x)} = \frac{\iota}{2\pi} \oint (-t)^{-z} \exp(-t)\, dt$$

is coded as

**1 over {%GAMMA(x)} = {%iota over {2%pi}}lint(-t)sup-z exp(-t) dt**

Matrices are also supported; for example,

$$\begin{pmatrix} 5 \\ 6 \end{pmatrix} \equiv \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 17 \\ 39 \end{pmatrix}$$

is coded as

**left( matrix{5 ## 6} right) equiv {left( matrix{1#2##3#4} right) } sup -1 left( matrix{17 ## 39} right)**

I hope this has given you a flavour of what's possible with the *StarOffice* maths package. I'll do my best to answer specific questions by email but can't promise an instant response. I have put this letter on my Web site so that readers can look at the real thing:

http://www.whiteowl.force9.co.uk/m500

I have also put there Word and HTML versions—but they don't look as good!

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**ADF**—Simon has updated his M500 Web site to include a browser implementation of the Exploring Chaos algorithm described by Barry Lewis in M500 **169**.

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'In a world without walls and fences, who needs windows and gates?'

—Geoff Franklin.

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'Warring bankers in talks with Governor.'

—Headline in *The Times*; spotted by JRH.

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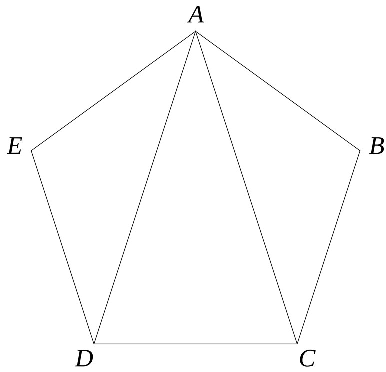
What do you call a person who searches for large primes only for the prize money?

A mersennary.

# Solution 169.2 – Chords



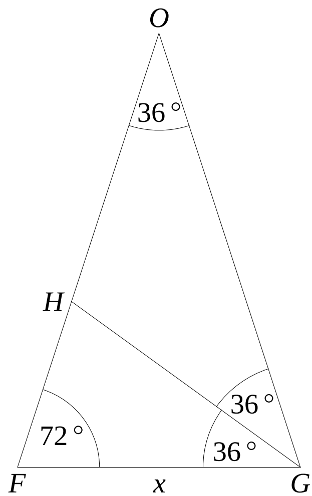

If we have a regular pentagon inscribed in a circle with unit radius, show that the product of the chords from any vertex to each of the others is equal to 5. That is,

$$(AB)(AC)(AD)(AE) = 5.$$

## Sebastian Hayes

First we need the

*Lemma. The ratio of the radius to the side of an inscribed regular decagon is the Golden Section.*




A rough proof is as follows. $FG$ is the side, $OF$ is the radius. Construct $HG = FG$. Then $\angle FOG = 36°$, $\angle OFG = \angle OGF = 72°$ and, since $\triangle FGH$ is isosceles, $\angle FHG = 72°$. Therefore triangles $FGH$ and $GOF$ are similar, $\angle FGH = \angle OGH = 36°$ and $OH = HG$. Hence

$$\frac{FG}{OF} = \frac{FH}{FG},$$

and

$$x^2 = (FG)^2 = OF \times FH = 1 - x.$$

Therefore

$$x = \frac{\sqrt{5}-1}{2},$$

as stated.

In the pentagon, the lengths of the chords are

$$AB = AE = 2\sin 36° = 4\sin 18° \cos 18°$$

and

$$AC = AD = 2\sin 72° = 2\cos 18°.$$

But $2\sin 18^\circ = (\sqrt{5}-1)/2$ because it is the length of the side of a regular decagon inscribed in a unit circle. Hence

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4},$$

$$\cos^2 18^\circ = 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{5+\sqrt{5}}{8},$$

so

$$(AB)(AE)(AC)(AD) = (8\sin 18^\circ \cos^2 18^\circ)^2$$

and

$$8\sin 18^\circ \cos^2 18^\circ = \frac{1}{4}(\sqrt{5}-1)(5+\sqrt{5}) = \sqrt{5}.$$

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## ADF

Let $\zeta = e^{2\pi i/5}$. In the complex plane we can represent the vertices of the pentagon by 1, $\zeta$, $\zeta^2$, $\zeta^3$ and $\zeta^4$. Hence the product of the four chords is given by

$$|(\zeta-1)(\zeta^2-1)(\zeta^3-1)(\zeta^4-1)|.$$

There is of course the simple-minded way of dealing with this expression: just multiply it out by hand. We have

$$\begin{aligned}(\zeta-1)(\zeta^2-1)(\zeta^3-1)(\zeta^4-1) &= (\zeta^3-\zeta^2-\zeta+1)(\zeta^3-1)(\zeta^4-1)\\ &= (\zeta^6-\zeta^5-\zeta^4+\zeta^2+\zeta-1)(\zeta^4-1)\\ &= \zeta^{10}-\zeta^9-\zeta^8+2\zeta^5-\zeta^2-\zeta+1\\ &= 1-\zeta^4-\zeta^3+2-\zeta^2-\zeta+1\\ &= 4-\sum_{j=1}^{4}\zeta^j.\end{aligned}$$

The geometric series sums to $\dfrac{\zeta^5-\zeta}{\zeta-1} = \dfrac{1-\zeta}{\zeta-1} = -1$ and the result follows.

However, **Sebastian Hayes** showed me a superior method which he found in the *Hungarian Problem Book* (translated by Elvira Rapaport and published by Random House, 1963).

Observe that $(\zeta^k)^5 = 1$ for any integer $k$; hence 1 $\zeta$, $\zeta^2$, $\zeta^3$ and $\zeta^4$ are zeros of the polynomial $z^5-1$, which factorizes as

$$z^5-1 = (z-1)(z^4+z^3+z^2+z+1).$$

The numbers 1, $\zeta$, $\zeta^2$, $\zeta^3$ and $\zeta^4$ are all different, and the factor $z-1$ corresponds to $z=1$. Hence $\zeta$, $\zeta^2$, $\zeta^3$ and $\zeta^4$ must be the zeros of the fourth degree polynomial

$$f(z) \;=\; z^4+z^3+z^2+z+1.$$

Therefore $f(z)$ splits into four linear factors

$$f(z) \;=\; (z-\zeta)(z-\zeta^2)(z-\zeta^3)(z-\zeta^4)$$

and if we put $z=1$, we obtain the result we want,

$$(1-\zeta)(1-\zeta^2)(1-\zeta^3)(1-\zeta^4) \;=\; f(1) \;=\; 5.$$

The argument is generalizable—it works just as well with 5 replaced by $N$. Thus we can prove that

$$(1-\xi)(1-\xi^2)(1-\xi^3)\ldots(1-\xi^N) \;=\; N,$$

where $\xi = e^{2\pi i/N}$. In a regular $N$-gon of unit radius, the product of the $N-1$ chords from a given vertex to each of the others is $N$.

---

## Barry Lewis

We need to look at two cases; when the polygon has an even number of sides, and when it has an odd number.

***Odd:*** $N=2n+1$. The required diagonals, by the cosine rule, are

$$d_1^2 \;=\; 1^2+1^2-2\times1\times1\cos\frac{2\pi}{2n+1} \;=\; 2\left(1-\cos\frac{2\pi}{2n+1}\right),$$
$$\ldots,$$
$$d_n^2 \;=\; 1^2+1^2-2\times1\times1\cos\frac{2\pi n}{2n+1} \;=\; 2\left(1-\cos\frac{2\pi n}{2n+1}\right).$$

So, by symmetry, the required chord product is

$$d_1^2\ldots d_n^2 \;=\; 2^n\left(1-\cos\frac{2\pi}{2n+1}\right)\ldots\left(1-\cos\frac{2n\pi}{2n+1}\right). \qquad (1)$$

Now consider the equation $x^{2n+1}-1=0$. This has the complex roots

$$x \;=\; \cos\frac{2k\pi}{2n+1}+i\sin\frac{2k\pi}{2n+1}, \quad k=\pm1,\pm2,\ldots,\pm n,$$

and the real root $x = 1$ ($k = 0$). So we have

$$x^{2n+1} - 1 = (x-1)\dots \left(x - \cos\frac{2k\pi}{2n+1} - i\sin\frac{2k\pi}{2n+1}\right)\left(x - \cos\frac{2k\pi}{2n+1} + i\sin\frac{2k\pi}{2n+1}\right)\dots.$$

If we divide through by $x - 1$, and expand the RHS in conjugate pairs, we get

$$x^{2n} + x^{2n-1} + \dots + x + 1 = \dots\left(x^2 - 2x\cos\frac{2k\pi}{2n+1} + 1\right)\dots,$$

and when $x = 1$ this gives

$$\begin{aligned} 2n+1 &= \left(1 - 2\cos\frac{2\pi}{2n+1} + 1\right)\left(1 - 2\cos\frac{4\pi}{2n+1} + 1\right)\dots \\ &= 2^n\left(1 - \cos\frac{2\pi}{2n+1}\right)\left(1 - \cos\frac{4\pi}{2n+1}\right)\dots. \end{aligned}$$

So the required product (1) is

$$d_1^2 d_2^2 \dots d_n^2 = 2n + 1 = N.$$

***Even:*** $N = 2n$. We have

$$d_1^2 = 2\left(1 - \cos\frac{\pi}{n}\right), \quad \dots, \quad d_{n-1}^2 = 2\left(\frac{1 - \cos(n-1)\pi}{n}\right), \quad d_n = 2,$$

the last result because this diagonal is a diameter of the unit circumscribing circle. Again, by symmetry, the required product is

$$d_1^2 \dots d_{n-1}^2 d_n = 2^{n-1}\left(1 - \cos\frac{\pi}{2n+1}\right)\left(1 - \cos\frac{2\pi}{2n+1}\right)\dots 2. \qquad (2)$$

Note that the final diagonal is not one of a symmetric pair and so is not squared. By the same process as for the odd number of sides, but this time considering the equation $x^{2n} - 1 = 0$, dividing through by $x^2 - 1$, expanding in conjugate pairs, and taking $x = 1$ we get

$$n = 2^{n-1}\left(1 - \cos\frac{\pi}{2n+1}\right)\left(1 - \cos\frac{2\pi}{2n+1}\right)\dots\left(1 - \cos\frac{(n-1)\pi}{2n+1}\right).$$

So the required product (2) is:

$$d_1^2 d_2^2 \dots d_{n-1}^2 d_n = 2n = N.$$

# Solution 168.3 – Fraction

Find decimal digits $a$, $b$, $\ldots$, $g$ such that $\dfrac{4251935345}{abc1935defg} = \dfrac{425345}{abcdefg}$.

## Barbara Lee

The solution

$$\frac{4251935345}{91219355185} = \frac{425345}{9125185}$$

can also be expressed as:

$$\frac{5 \times 7 \times 97 \times 1252411}{5 \times 7 \times 2081 \times 1252411} = \frac{5 \times 97 \times 877}{5 \times 87 \times 2081},$$

so it is just a matter of getting your prime factors in their correct places.

Here are two more small ones to add to our collection.

$$\begin{aligned} 1784\ /\ 5798 &= 184\ /\ 598 \qquad \text{(delete 7)} \\ 7136\ /\ 17171 &= 736\ /\ 1771 \qquad \text{(delete 1).} \end{aligned}$$

You can go on for ever if there are infinitely many primes.

---

# Solution 169.1 – Three people

> There are three people. One tells lies. One tells truths. One alternates. They get paid for each word. So find the cheapest and most elegant set of questions to identify them all correctly.

## Barbara Lee

Let the three people be $T$, $L$, $A$. Ask each: $Q$—Are you truthful on alternate occasions only? Two possible sets of answers ($T$, $L$, $A$):

1. No, Yes, Yes
2. No, Yes, No

If you got 1 for $Q$, you identify $T$ and you know that $A$ truthed. Follow by asking $L$ and $A$: Did you answer $Q$ truthfully? $L$ says Yes, $A$ says No, and they are both lying this time, so you identify them.

If you got 2 for $Q$, you identify $L$ and you know that $A$ lied. Follow by asking $T$ and $A$: Did you answer $Q$ truthfully? $T$ says Yes, $A$ says No, and they are both truthing, so you identify them. **Five words** to pay for.

---

## Ralph Hancock

Ask any one of them:

1. Is the Pope Jewish?
2. Is the Pope Catholic?
3. Is your friend on the left more truthful than your friend on the right?

The liar answers 1 and 2 as yes and no, so that you know it's him and that he will answer 3 wrongly, so you can identify them all. The truthful person answers 1 and 2 as no and yes, so you know it's him and that he will answer 3 correctly, so again you have them all.

The alternator answers both 1 and 2 in the same way, so you know both that it's him and whether his answer to 3 will be true, so again you've got all three.

**Total cost** @ 1000000 zlotys per word: 3000000 zlotys.

---

**JRH**: Where do these liars and alternators live? I'd ask them questions like: Can I take your wallet? By the way, assuming that the three people are as they are in any form of communication, not just speech, you can identify them at no cost by adapting either Ralph's or Barbara's solution. E.g. (Ralph), instruct one to:

1. Smile if the Pope is Jewish
2. Smile if the Pope is Catholic
3. Point to your more truthful friend

If you think they don't know the Pope, you do something else, like stand them in a box (but the Pope is funnier):

1. Smile if there are 17 people in the box
2. Etc.

Ralph's is shorter than Barbara's, but it assumes the three all know each other's ways, which Barbara's doesn't. Also, you can make Ralph's set of questions smaller, therefore more elegant as the question required, by discarding 2 and asking 1 twice, or vice versa.

---

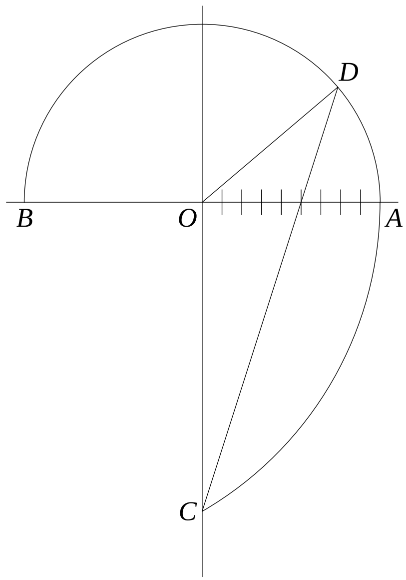
**ADF**: I agree that elegance of the questions is important and what better way of measuring it than mega-zlotys per word? Thus Ralph works out at 25 Mz (including the responses), and Barbara 19, assuming that you would substitute an intelligible set of words, such as 'my previous question', for '*Q*'. Jeremy's device lets him get away with 18 Mz but in some sense we are not entirely happy with his contribution. If we insist that all words count, can anyone do better?

# An early protractor

## Dirk Bouwens

This is a method of constructing angles that was told me by a builder, who had it from his father and grandfather. It appears to be even older than that implies. I have measured some angles it makes and found they are correct to within measurement error. Are they right in theory, and if so, why?

Let a horizontal and a vertical line cross at $O$. On the right arm of the cross mark nine equi-spaced points. Label the ninth point $A$. Using centre $O$ and radius $A$ draw a semicircle to hit the horizontal line at $B$. With centre $B$ draw an arc from $A$ to continue the semicircle to the vertical line at $C$. That is the construction.




If you draw a line from $C$ through the $n$th of the nine points reading from outside in and continue it to strike the semicircle at say $D$, then the angle $DOA$ measures $10n$ degrees.

For more accurate angles the number of points can be increased, but it must always be a multiple of nine.

William G. Stewart: Which branch of mathematics means 'triangle measurement'?

Sixth former: Pythagoras.

—*Fifteen to One for Schools*, Ch. 4. Spotted by **JRH**.

Seeing a sign on a pub announcing 'a pint, some food and a friendly word for only £4.99' Sir Clement Freud dropped in. After being handed the pint and a meat pie he asked "Where is the friendly word?" The barman leaned over and said "I wouldn't eat the pie if I were you, sir."

—*The Times Diary*; spotted by **EK**.

# March March march

## JRH

A good candidate for the 'Complex complex complex' series (M500 **169** 22) is the March March march. The March March march is a long, flat, pointless walk across the Fens from the town of March to Cambridge, a distance of about thirty miles. It takes place in March, often but not always on the last Saturday. It was invented by Jonathan Partington in 1979, and has no purpose other than to be called the March March march. Some March March marchers have become prominent mathematicians, as you might expect.

The nature, history, routes, customs, participants, recipes, poems and so on of the March March march are reported at

http://www.bath.ac.uk/~masgks/march.html#March99.

---

# Complex complex complex

We asked for silly sentences containing a repeated-often word.

## Ralph Hancock

If you were to make a heap of the debris from a carpet trimming machine, and the heavy labour gave you a haemorrhoid, this would be a *pile pile pile*. And if you had very mild opinions about a rather unfashionable pop group, you might be a *Wet Wet Wet wet*. One of the inventions of Dr Strabismus of Utrecht was a screw for screwing screws into other screws. When I was in Italy in the 1970s, the Italian government were paying interest on the loan they had taken out to pay the interest on the loan they had taken out to pay the interest on the loan they had taken out to finance public spending. Maybe they have taken out a further loan by now. But that's only real life, and therefore not important.

---

**JRH** writes: How about if they threw you in the river for expressing the mild opinions about the pop group?

---

**Ralph** again: If the person who had moderate opinions on the pop group was a South African, and he climbed out of the river into his car and drove home, you might say that *Mr de Wet, wet Wet Wet Wet wet, wet the upholstery.*

---

**JRH** again: How about if Wet Wet Wet themselves happened along at the moment he got out of the river, and stood round him as he shook himself like a dog?

# M500

## Dave Cartwright

[Dave Cartwright, singer/guitarist/raconteur, came to the recent Revision Week-end at Aston to entertain. He is a friend of JRH, but has avoided mathematics at every opportunity since they were at different schools together more than 35 years ago. He read this poem during his performance.]

M500 brings to mind
A life I hoped I'd left behind.

It doesn't mean a lot to me
(A letter, one; numbers, three),

And yet the combination seems
To wake some cold, unpleasant scenes

Of sleepless nights, in days gone by,
When tossing, turning, moaning, I

Tried hard to make an ounce of sense
Of algebra.
With impudence

I'd fill my pages, thick and fast.
'The neatest rubbish in the class',

My tutors, one by one would bark
With heaving sigh and cross-red mark.

And in the staff room how they grinned—
Miscalculations underpinned

By true solutions (courtesy
Of whomsoever sat by me)!

Cosines, tangents, vectors, $\pi$s;
Simpson's Rule and Cartwright's Lies;

Mathematics always meant
A test of what I could invent,

And knowing what I didn't then,
I left it all to fellow men,

Who, pen in mouth and head bowed down,
Worked it out, without a sound.

I'm older, wiser, now, you see,
But in my mind, indelibly,

Calculus forever blooms
Like hieroglyphics from the tombs.

So, M500, here's to you;
Let $a$ be 1, but let me be, too.

---

# Exploring chaos

## Ken Greatrix

Barry Lewis's article in M500 **169** described Sierpiński's Gasket, which I first heard about (at least in computerized form) from the Royal Institution Christmas Lectures, 1997, presented by Professor Ian Stewart.

The puzzle that I find with this iteration is that even if you start in the 'white' area, you end up putting the dots in the 'black' area. Can anyone say why this transition should occur?

Also in Stewart's lectures was the Langton Spider. This is a random walk generated from the following rules: Starting on a square-tiled surface which is all black, take a step forward to the next tile. If the new tile is black, turn 90° left, if white, turn 90° right. Change the colour of the tile just vacated. Repeat the stepping, turning and changing iterations indefinitely. I leave it to the reader to compile a suitable program to follow this process and discover the strange occurrences after about 10,000 steps. Perhaps someone could write an article about it.

Getting back to Barry's article; Sierpiński's Carpet is analogous to the gasket. Start with a square of carpet and divide it into nine smaller squares, then remove the centre square. Repeat with each new square formed. The Menger Sponge is a three-dimensional analogue of the carpet.

Further information on the above and other suggestions for computer images can be obtained from *Chaos* by James Gleick, published by Sphere Books.

Final note: Having re-read the section on fractal images, I find that the design of the Eiffel Tower is fractal. This would seem to be the original idea behind my Problem 169.5, but it is about seven years since I read the book, so I am not consciously aware of having used it.

---

William G Stewart: Mathematics. The square on the hypotenuse is equal to the sum of the squares on the other two sides. How is that tenet popularly known?

Contestant: O.H.M.S. —*Fifteen to One*, Ch 4.

# Problem 171.2 – $7n + 1$

Recall the $3n + 1$ problem, a.k.a. the Syracuse algorithm, introduced into these pages by Jane Kerr [M500 **162** 2] and analysed in depth by Ken Greatrix [M500 **166** 1]. The $7n + 1$ problem is defined in a similar manner. Start with any positive integer. If even, divide by two. If divisible by three, divide by three. (If both, do both.) If not divisible by either two or three, multiply by 7 and add 1. Thus

$$n \;\to\; \begin{cases} n/2 & \text{if } n \text{ is divisible by 2} \\ n/3 & \text{if } n \text{ is divisible by 3} \\ 7n + 1 & \text{otherwise} \end{cases}$$

Continue to apply the formula to get a sequence of integers. It seems that eventually we reach either the cycle $1 \to 8 \to 4 \to 2 \to 1$, or a much longer one that contains the number 19. As with $3n + 1$, the problem is to decide whether or not there exists a starting $n$ that generates a sequence which increases without bound, or perhaps goes into some new cycle. The problem is not trivial. Starting with 31, the sequence takes over 3000 iterations before it settles down in the 19-loop.

---

# Fermat's Last Theorem

## A reply to Bob Margolis's criticism

## Peter L. Griffiths

Most of the criticism on page 25 of Bob Margolis's article [M500 **170** 25–27] can be answered by pointing out that near the beginning of my original article I mention that $t$ is an integer $> 1$, and later, I mention that $n > 2$. I am not sure that I follow Bob's argument on page 26 about the inequality relationship of $c$. Possibly the presentation may have been clearer if I had expressed $e$ not as a constant but as varying with $n$, so that the upper limit for $e$ would be 2.7182818 and the lower limit would be $(1 + 1/2)^2 = 2.25$.

Both the lower as well as the upper limit for $e$ exceed 2, so there is only one inequality apart from the obvious $c > a$, namely $a + a/n > c$, in other words $c$ has an upper limit of $a + a/n$. The infinitesimal possibility of $c$ encountering an integer when $a > n$ is not a problem for me, it is a problem for those who have not included it in their 'proofs' of Fermat's Last Theorem.

As regards Goldbach's Conjecture, it might be helpful if it were expressed as 'every number is the arithmetic mean of two odd primes, counter-examples gratefully received.' Some odd primes may be better at participating in arithmetic means than others.

---

# Twenty-five years ago

## From M500 18

## Vera Keates

Later in my lengthy and aimless wanderings in the Forest of Calculus I met a curious and rather difficult individual, L. Nathaniel Logz by name. He is a great exponent of the need for the integration of all primitive functions, or so he told me. Personally, I didn't understand him at all and the directions he gave me confused me still further. I went backwards and forwards, substituting one path for another, but all to no avail. Sometimes, indeed, I would just stand stationary at a point, unable to decide where to turn. In the centre of the Forest I was surprised to discover an aqueduct, whose construction I could not understand. It was soon after this that I stumbled somehow out of the Forest, only to find myself at the same place where I had entered it. How to avoid it and so progress? I did not know. But by now I was absolutely exhausted and so I lay down on the ground to sleep. It was a disturbed rest, for I was troubled by dreams; at first I thought I was running a race with a tortoise, but even here I could not win. Later a crowd of barbers appeared on the scene, all shaving each other, and then one started to shave off the goatsbeard that bordered the R. Heimer. ...

---

## J.E.C.

*A Foundation student at Bangor,*
*At four in the morning, felt anger,*
*"The tutors", he said,*
*"Have all gone to bed,*
*The OU should prohibit such langour!"*

---

# Winter Week-end

## Norma Rosier

This is an annual residential Weekend to dispel the withdrawal symptoms due to courses finishing in October and not starting again until February. It is an opportunity to get together with friends, old and new, and do some interesting mathematics.

The

**nineteenth M500 Society WINTER WEEK-END**

will be held at

**Nottingham University**

from

**Friday to Sunday, 7 to 9 January 2000**.

Ian Harrison is running it and the main theme this year will be **Time**. Topics that could be included range from Special Relativity to early navigation methods. It promises to be as much fun as ever!

The cost is £128 for M500 members, £133 for non-members. This includes accommodation and all meals from dinner on Friday to lunch on Sunday. The event is very popular and places are limited. Please send a stamped, addressed envelope for booking form to

**Norma Rosier**.

# Advice to authors

## ADF

Stocks are running low again. So it's time for our almost-periodic plea for more publishable material.

As always, we welcome contributions on anything related to mathematics. You can be as informal as you like. We need articles, stories, anecdotes, half-developed ideas, problems, solutions, fillers, in fact anything to do with mathematics, or the OU Maths Faculty, and at any level from trivia to serious research. We are especially interested in stuff that can be understood by first-year students.

Our procedure is simple. You write, we print. Your prose doesn't have to be brilliant; if necessary, we will edit articles for clarity and we will correct any mathematical errors (on the assumption, of course, that we are sufficiently alert to spot them).

Unlike most other mathematics journals there is no refereeing process. If you think you have squared the circle, or decided the twin-prime conjecture, or if you can only manage an elementary proof of FLT, send it to us. If we think it will inspire discussion and debate, we'll print it.

Please send your contribution to one of us: **Tony Forbes**, **Jeremy Humphries**, **Eddie Kent**. See 'page 0' for our addresses. We would appreciate long articles on a PC diskette, or via e-mail, as well as on paper. It doesn't matter what software you used to prepare your article provided that the text is also available as a plain ASCII file. Don't worry about getting the mathematical bits perfect. We have to reset all the equations anyway.

Alternatively, you can send us camera-ready copy (and tell us very clearly that's what it is) if you want your work to appear exactly as you left it. In which case we ask that you follow the general style of the magazine and be prepared for at least one round of proof correction.

If you are not on the MOUTHS list, and if you are willing to correspond with readers, remember that you need to give us permission to publish your postal address with your article. (If you don't, we won't.) Alternatively, you might seriously consider getting your details added to MOUTHS.

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That's it. As this is the last M500 of 1999, I wish you all a Happy Christmas and a Prosperous New Whatever-you-call-it-when-the-thousands-digit-of-the-year-changes.

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