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ISSN 1350-8539

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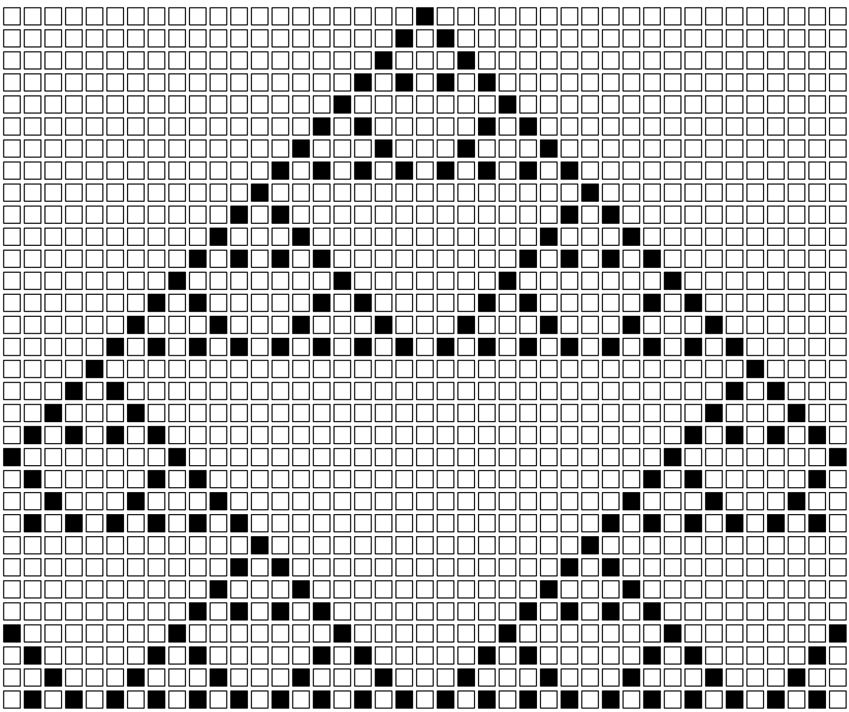

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A New Kind of Science

Stephen Wolfram,
Wolfram Media, Inc., 2002

Sebastian Hayes

Stephen Wolfram is probably known to many readers of M500 as the millionaire designer/inventor of the computer software package MATHEMATICA. However, as a pioneer of a 'new kind of science' he has been markedly less successful. I only heard about it six years after its first publication because of an interview with the great man (who lives as a recluse in Concord, Massachusetts), reported in the *New Scientist*.

Wolfram's main claim to fame is his discovery that 'very simple rules can give rise to considerable complexity'. At first sight, this doesn't sound earth-shattering but, then, neither did the Second Law of Thermodynamics, which in its original formulation by Clausius was simply the innocuous sounding observation that 'Heat does not spontaneously move from a colder to a warmer body'.

Wolfram has made a twenty-five year long study of cellular automata and, partly because of his own software, has been able to investigate their hidden properties more extensively than anyone else. What is a cellular automaton? It presupposes

1. a grid which can be extended indefinitely;

2. an initial 'seed', usually a single cell (square of grid) coloured black with empty cells to left and right all along the first row;

3. a rule which specifies the colour of every cell in each new row according to the colour of the cells in the row above.

In the simplest type of cellular automata there are only two permitted colours, black and white, and a rule attributes the colour of a new cell according to the colour of the three cells directly above it.

The rule can be presented visually, for example

1	2	3	4	5	6	7	8
■■■	■■□	■□■	■□□	□■■	□■□	□□■	□□□
■	■	■	■	■	■	■	□

If we start with a single black cell and the rest of the row empty, i.e. . . . □□□□■□□□□. . . , we must apply rules 7, 6, 4 and 8 obtaining a 'Mayan temple' pattern with each row having two more black squares than the row above it.

. . . □□□□■□□□□. . .
. . . □□□■■■□□□. . .
. . . □□■■■■■□□. . .
. . . □■■■■■■■□. . .
. . . ■■■■■■■■■. . .

Now, since there are two possibilities, ■ and □, for each of the trios numbered 1 to 8 in the top line of the rule, there are altogether $2^8 = 256$ different rules, giving various patterns of black and white cells.

Wolfram labels the first one 0 since, apart from the initial seed ■ which is present in every first row, the rule

■■■	■■□	■□■	■□□	□■■	□■□	□□■	□□□
□	□	□	□	□	□	□	□

produces a completely blank expanse after the initial row. The last rule, number 255, does the opposite and makes everything below the first row black.

■■■	■■□	■□■	■□□	□■■	□■□	□□■	□□□
■	■	■	■	■	■	■	■

In between there are various intermediary patterns.

The patterns (which I am unable to give in detail on my computer) fall into three main categories: **1. patterns of fixed size; 2. growing repetitive patterns; 3. complicated, non-repetitive patterns.**

Of the more complex patterns, quite a few are 'nested': the Sierpinski Triangle, for example, is generated by Rule 90. Wolfram observes that this well-known pattern 'is exactly Pascal's triangle of binomial coefficients reduced modulo 2, [where] black cells correspond to odd binomial coefficients' (Wolfram, p. 870)—something I did not know.

That such simple procedures can generate fractal shapes is in itself surprising but even more surprising is that one or two out of the 256 generated patterns that end up becoming completely random though containing occasional localized pockets of more ordered behaviour. Many other examples illustrate the same phenomenon.

So far, so good. Wolfram seems to have won the first round on points by showing that 'however certain one might be that simple programs could never do more than produce simple behaviour, the pictures of the past few pages [of this book] should forever disabuse one of that notion' (Wolfram, p. 39). Moreover, there seems, in the majority of cases, to be no way of predicting the type of behaviour that a particular rule will produce.

Playing around with various more restricted, or more extensive, types of cellular automata Wolfram reaches a very significant conclusion:

> Looking at many examples [of mobile automata], a certain theme emerges: complex behaviour almost never occurs except when large numbers of cells are active at the same time, Indeed, there is, it seems, a significant correlation between overall activity and the likelihood of complex behaviour. (Wolfram, p. 76)

Why is this significant? Because, precisely, the behaviour of living creatures involves the co-operation of astronomical numbers of individual cells, and we are at every turn confronted by 'the extreme simplicity of the principle [of the DNA], on the other the endless complexity of the outcome' as one biologist puts it.

Despite—in some ways even because of—the discovery of the DNA and the subsequent genome project, the phenomenon we call 'life' remains as mysterious as ever. It is to the highest degree paradoxical that cellular automata, since they are basically just patterns on a computer screen produced by rules that human beings such as Wolfram specify, have found their commonest and most successful applications in the modelling of living systems. For, whatever else they may be, cellular automata are not the product of random mutation plus natural selection which, as all good Darwinists know, are the two forces responsible for most of what we see around us that moves.

As deliberately designed 'pure' products of human intelligence, cellular automata would seem to have more in common with certain abstract mathematical systems than with physics or biology. And indeed a good deal of the adverse, not to say violent, reaction to Wolfram's claims comes from the mathematical, rather than the scientific, community. Why is this? Basically, because, although mathematics itself is not in crisis—pure mathematics is more flourishing than ever—the relevance of higher mathematics to what goes on in the real world has become a highly troublesome issue that orthodox mathematicians would rather not confront, as they suspect that the answer will not be favourable to them.

Broadly speaking, I fear that the dreadful truth is that 'Nature does not do mathematics' any more than New Labour 'does God', to quote Alisteir Campbell's unforgettable remark. Euclidian geometry has a certain relevance to situations where close packing in regular arrays is significant, and thus to crystallography, but you will look in vain for the standard geometric shapes, the circle, triangle and even the dead straight line in the natural world around you. The extraordinarily varied, irregular and complicated forms of plants and animals could not be a more striking contrast to the simple ideal shapes Euclid studies with such absorption and that Plato immortalized as his 'Ideas'. Mandelbrot's fractals do occasionally look rather more like natural objects, but no plant is self-similar and the sea-horses of the Mandelbrot set have nothing to do with real sea-horses.

'Modern' mathematics—I mean from Descartes and Newton onwards—deals essentially in algebraic formulae which in the vast majority of applications require the completely unrealistic assumption of 'continuity'. The equation of a curve $y = f(x)$ is absolute: it delimits the curve everywhere and for all time (barring certain so-called singularities). As a French physicist I once knew put it, 'It is "our fault" if we cannot see all the features of the curve at a glance, they are essentially all there in the formula.' This is quite different from definition by recursion where a mathematical entity is built up step by step from an initial 'seed' and, although most mathematical functions can be defined recursively, mathematicians have a marked preference for the analytic way of doing things.

As for continuity, we now know that exchanges of energy, which account for practically all physical and chemical behaviour, are not continuous, but must obey quantum laws. Nonetheless, Calculus methods are still employed in, for example, population studies and molecular thermodynamics where we know for a fact that the independent variable can never be smaller than a single molecule or a single human being. It is doubtless because of the long shadow mathematics has thrown over physics, that, even today, it is automatically assumed that Space and Time are 'continuous' even though this is by no means self-evident and has always struck me as being a dead weight that physicists insist on carrying around with them (essentially because they have all been trained in the same mathematical school).

As it happens, cellular automata score on both these points. They are built up step by step, row by row, and the rule is essentially a method for getting from one state of a system to the next, not a formula which is 'true for all time'. This goes some way to explaining the success of applications of cellular automata to living systems, also to systems involving a very large number of individual elements, e.g. fluid mechanics. The key question is: which way of proceeding is Nature's way? In the days when all scientists believed in a supremely intelligent Creator God, as Newton and Boyle and Leibniz did, the analytic approach clearly had the advantage, hence the very idea of Nature 'obeying laws'. As for systems with large numbers of entities, there was, prior to the invention of computers, no alternative to Calculus but now it is much less necessary to solve differential equations and concoct analytic formulae since it is often possible simply to slog it out numerically.

There is, I think, a consensus now that organisms, including ourselves, are not masterminded by a transcendent intelligent Being. But nor do they in general 'know what they are doing' mathematically and scientifically speaking. Humble unicellular organisms perform miracles of chemical engineering that even our current technology cannot even remotely rival. No factory is as efficient and complex as a cell. A cheetah stalking its prey is ignorant of the equations of motion and children easily learn to ride a bicycle without knowing anything about angular momentum or gyroscopic stability. For all this, either 'instinct' (whatever that is) or mere trial and error suffices, and, according to one of the two most successful scientific theories of all time, trial and error suffice to explain most of what we see around us in the organic world.

The great thing about cellular automata is that they are not just simple; they are, some of them at any rate, absolutely simplistic. A child of three could carry out one of Wolfram's 256 rules and build up a pattern with coloured blocks, though she would very rapidly get bored with the activity. And yet some of these simple automata exhibit very great complexity. Although some animals seem to possess a rudimentary sense of number, I cannot conceive of mammals and plants knowing anything at all about calculus. However, I can just about conceive of an organism, or even a *genus*, directing itself to carry out over and over again the basic rules governing the growth of a cellular automaton, and letting the programme run to see what comes up, while natural selection can be depended on to weed out the absolutely unworkable results.

Wolfram has a point. Just possibly, something akin to the procedures that drive cellular automata could take the inorganic into the organic, i.e. produce life. At this very moment, a Swiss professor, Henry Markham, is heading a multi-million IBM backed project to develop a true 'artificial mind', not just a chess-playing computer programme but something that has consciousness and a sense of personal identity. 'Markham believes,' a correspondent writes, 'that consciousness is probably something that "emerges" given a sufficient degree of organized complexity' (*Daily Mail*, January 4 2010). This is precisely Wolfram's contention; indeed Wolfram goes rather further in that he seems to suggest that something we might reasonably call 'life' is automatically going to emerge from certain types of system whether we like it or not. (Some scientists such as Dr Blackmore have seriously suggested that new types of 'life' are being spawned already by the Internet.)

Because of the conclusions Wolfram derives from his observation of, or, better, experimentation with, cellular automata, the contemporary analogy between the human brain and a computer, which has become something of a tired cliché, takes on new life. For Wolfram believes that there are plenty of systems based on precisely formulated simple rules which nonetheless, perhaps after a considerable lapse of time, exhibit highly complex 'interesting' behaviour which is entirely unpredictable (and he gives examples of this). He suspects that the brain is one such system and proposes this as the solution to the age-old problem of Free Will versus Necessity.

> Even though all the components of our brains presumably follow definite laws, I strongly suspect that their overall behaviour corresponds to an irreducible computation whose outcome can never in effect be found by reasonable laws.
>
> ... As a whole our brains still manage to behave with a certain apparent freedom.
>
> Traditional science has made it very difficult to understand how this can possibly happen .... [But] in fact there can be vastly more to the behaviour of a system than one could ever foresee just by looking at its underlying rules. And fundamentally this is a consequence of the phenomenon of computational irreducibility.
>
> For if a system is computationally irreducible this means that in effect there is a tangible separation between the underlying rules for the system and its overall behaviour. ... And it is in this separation, I believe, that the basic origin of the apparent freedom we see in all sorts of systems lies. (Wolfram, 750–751)

This claim constitutes a decisive break with the basic presuppositions of scientific thinking during the last five hundred years, inasmuch as Wolfram denies that the universe, or us, are predictable even in theory—though Gödel, Quantum Indeterminacy and Chaos Theory have prepared the way for this grand conclusion. Oddly, Wolfram faces two ways at once: he denies on the one hand that there is anything 'special' about human beings, while at the same time, by identifying them as 'computationally irreducible systems' he provides them with 'free will' and capacity for development in unpredictable ways.

Wolfram—and for that matter Dawkins—will have to work a bit harder if they want to convince me that man-made patterns on a computer screen are really analogous to organisms that are subject to the constraints of actual, as opposed to virtual, space, but he has certainly convinced me that *some* features of his automata throw light on the dark mysteries of growth and form.

When moving on to cosmology—Wolfram is nothing if not ambitious—he abandons the idea of a grid progressively filling up with coloured cells. He models Space/Time as ‘a giant network of nodes ... with a fixed number of connections’ and departs noticeably from the near universal assumption of Space/Time continuity by giving an estimate of the size of a basic Space/Time ‘causal link’, namely ‘an elementary distance of 10^{-35} metres and an elementary time interval of around 10^{-43} seconds’ (Wolfram, p. 520).

A New Kind of Science is a very long book (1,200 large pages) but it is simply and fluently written and, in the main text, contains *absolutely no mathematical formulae*—though there is plenty of advanced mathematics and computer speak in the **Notes** (300 pages long) for those who might otherwise be tempted to immediately dismiss this very ambitious work that claims to prepare the ground for the coming scientific paradigm.

A version of this article has appeared on the website of Sebastian Hayes at www.sebastianhayes.co.uk.

If anyone is familiar with cellular automata and would like to collaborate in a project to apply them in a new direction, I would be grateful if he or she would enter into account with me via sebastianhayes@tiscali.co.uk — **SH**.

Problem 235.1 – Roots

Let n be an integer, $n \geq 2$. Let S_j denote the sum of the jth powers of the reciprocals of the roots of the equation

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \;=\; 0.$$

Prove that

$$S_2 \;=\; S_3 \;=\; \ldots \;=\; S_n \;=\; 0 \quad \text{and} \quad S_{n+1} \;=\; \frac{1}{n!}.$$

For example, with $n = 2$ the roots are $-1 \pm i$, their reciprocals are $-\dfrac{1}{2} \pm \dfrac{i}{2}$, the squares of the reciprocals are $\pm\dfrac{i}{2}$ and the cubes $\dfrac{1}{4} \pm \dfrac{i}{4}$.

Differential equations and trigonometric functions

Dennis Morris

Consider the basic Euclidean trigonometric functions $\{\sin\phi, \cos\phi\}$. Form the sum of these and differentiate:

$$y = \sin x + \cos x, \quad \frac{dy}{dx} = \cos x - \sin x, \quad \frac{d^2y}{dx^2} = -\sin x - \cos x,$$

leading to the differential equation

$$\frac{d^2y}{dx^2} = -y,$$

which, obviously, has the solution we started with. This differential equation and solution are thus associated with 2-dimensional Euclidean space.

Similarly, if we consider the basic trigonometric functions of space–time, we get the differential equation and solution

$$\begin{aligned} s &= \cosh t + \sinh t, \\ \frac{ds}{dt} &= \sinh x + \cosh x, \\ \frac{ds}{dt} &= s. \end{aligned}$$

In the case of the 3-dimensional natural space, $C_3L^1H^2_{[j=1,k=1]}$, we have the basic trigonometric functions $\{\nu A(b,c),\ \nu B(b,c),\ \nu C(b,c)\}$. Summing these and differentiating gives

$$\begin{aligned} y &= \nu A(b,c) + \nu B(b,c) + \nu C(b,c), \\ \frac{\partial y}{\partial b} &= \nu C(b,c) + \nu A(b,c) + \nu B(b,c), \\ \frac{\partial y}{\partial b} &= y. \end{aligned}$$

Similarly,

$$\frac{\partial y}{\partial c} = y \quad \Rightarrow \quad \frac{\partial y}{\partial c} = \frac{\partial y}{\partial b}.$$

Thus, these differential equations and solution are associated with the C_3 group and with the $C_3L^1H^2$ natural space.

In the case of the 3-dimensional natural space, $C_3L^1E^2_{[j=1,k=-1]}$, we have the basic trigonometric functions $\{\nu A^*(b,c),\ \nu B^*(b,c),\ \nu C^*(b,c)\}$, where we have added an asterisk to avoid confusion with the trigonometric functions of the above algebra. The differential relations of this algebra are

$$\begin{aligned}
\frac{\partial \nu A^*}{\partial b} &= \nu C^*, & \frac{\partial \nu A^*}{\partial c} &= \nu B^*, & \frac{\partial \nu B^*}{\partial b} &= \nu A^*, \\
\frac{\partial \nu B^*}{\partial c} &= -\nu C^*, & \frac{\partial \nu C^*}{\partial b} &= -\nu B^*, & \frac{\partial \nu C^*}{\partial c} &= \nu A^*.
\end{aligned}$$

Summing these trigonometric functions and differentiating gives

$$\begin{aligned}
y &= \nu A^*(b,c) + \nu B^*(b,c) + \nu C^*(b,c), \\
\frac{\partial y}{\partial b} &= \nu C^*(b,c) + \nu A^*(b,c) - \nu B^*(b,c), \\
\frac{\partial^2 y}{\partial b^2} &= -\nu B^*(b,c) + \nu C^*(b,c) - \nu A^*(b,c), \\
\frac{\partial^3 y}{\partial b^3} &= -\nu A^*(b,c) - \nu B^*(b,c) - \nu C^*(b,c), \\
\frac{\partial^3 y}{\partial b^3} &= -y.
\end{aligned}$$

Similarly

$$\frac{\partial^3 y}{\partial c^3} = -y$$

and

$$\begin{aligned}
\frac{\partial^2 y}{\partial b\,\partial c} &= \nu A^*(b,c) + \nu B^*(b,c) + \nu C^*(b,c), \\
\frac{\partial^2 y}{\partial b\,\partial c} &= y.
\end{aligned}$$

These three differential equations,

$$\frac{\partial^3 y}{\partial b^3} = -y, \quad \frac{\partial^3 y}{\partial c^3} = -y, \quad \frac{\partial^2 y}{\partial b\,\partial c} = y,$$

are thus also associated with the group C_3 but, in this case, with the $C_3L^1E^2$ natural space. From these three, we get equations like

$$\frac{\partial^3 y}{\partial c^3} = -\frac{\partial^2 y}{\partial b\,\partial c}.$$

At this point, I confess that I know very little about differential equations, but the equation immediately above looks hard to solve if one does not know the answer. If anyone has seen this equation, or similar, before, I would be grateful to know of it.

In general, since every finite group has its own particular set of trigonometric functions, it has, associated with it, a particular set of differential equations, and so we may speak of the dihedral differential equations or the quaternion differential equations.

Since the quaternion rotation matrix is the Lie group SU(2), which is associated with (particle physics) isospin, there is a particular set of differential equations associated with isospin. These isospin equations are not quite as simple as the ones associated with the C_3 group.

The basic quaternion (isospin) trigonometric functions are

$$\begin{aligned} Q_A &= \cos\sqrt{b^2+c^2+d^2}, & Q_B &= \frac{b\sin\sqrt{b^2+c^2+d^2}}{\sqrt{b^2+c^2+d^2}}, \\ Q_C &= \frac{c\sin\sqrt{b^2+c^2+d^2}}{\sqrt{b^2+c^2+d^2}}, & Q_D &= \frac{d\sin\sqrt{b^2+c^2+d^2}}{\sqrt{b^2+c^2+d^2}}, \end{aligned}$$

so that $Q_B/b = Q_C/c = Q_D/d$. They have the differential relations

$$\begin{aligned} \frac{\partial Q_A}{\partial b} &= -Q_B, \quad \frac{\partial Q_A}{\partial c} = -Q_C, \quad \frac{\partial Q_A}{\partial d} = -Q_D, \\ \frac{\partial Q_B}{\partial b} &= \frac{1}{b^2+c^2+d^2}\left(b^2 Q_A + \left(c^2+d^2\right)\frac{Q_B}{b}\right), \\ \frac{\partial Q_B}{\partial c} &= \frac{bc}{b^2+c^2+d^2}\left(Q_A - \frac{Q_B}{b}\right), \\ \frac{\partial Q_B}{\partial d} &= \frac{bd}{b^2+c^2+d^2}\left(Q_A - \frac{Q_B}{b}\right) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 Q_B}{\partial c\,\partial d} &= \frac{-bcd}{(b^2+c^2+d^2)^2}\left(3Q_A + \left(b^2+c^2+d^2-3\right)\frac{Q_B}{b}\right), \\ \frac{\partial^2 Q_B}{\partial b^2} &= \frac{b}{(b^2+c^2+d^2)^2}\left(3(c^2+d^2)Q_A - \left(b^4+(b^2+3)(c^2+d^2)\right)\frac{Q_B}{b}\right). \end{aligned}$$

Thus the differential equation

$$\frac{\partial^3 y}{\partial b\,\partial c\,\partial d} = \frac{cd}{(b^2+c^2+d^2)^2}\left(3by + \left(3-b^2-c^2-d^2\right)\frac{\partial y}{\partial b}\right)$$

has solution $y = Q_A$. And we can construct many other differential equations similarly. Such are the isospin differential equations. (I wonder if they are of any use.)

A curious sequence

Tony Forbes

Given a polynomial $p(x)$ and a positive integer r, define $S_n(p(x), r)$ by

$$S_n(p(x)\,;r) \;=\; \frac{1}{n}\sum_{k=0}^{n-1} p(k)\binom{2k}{k}^r.$$

As a subscriber to NMBRTHRY, the internet forum for number theorists, I became aware of an interesting conjecture. Zhi-Wei Sun of Nanjing University asserted that $S_n(21x+8, 3)$ is not only an integer but an integer divisible by $4\binom{2n}{n}$. The conjecture was soon proved by Kasper Andersen of the University of Aarhus by showing that

$$\frac{1}{4n\binom{2n}{n}}\sum_{k=0}^{n-1}(21k+8)\binom{2k}{k}^3 \;=\; \sum_{k=0}^{n-1}\binom{n+k-1}{k}^2.$$

Of course the big question is: What's so special about $21x+8$? After some experimentation using MATHEMATICA I found in addition to $S_n(21x+8, 3)$ a few more examples of possible (i.e. not yet rigorously proved by me) integer-only sequences $S_n(p(x), r)$ with $r \ge 2$, namely

$$S_n(5x^4+6x^3+x^2, 2),\; S_n(15x^5+14x^4-5x^3, 2),\; S_n(15x^2+16x+4, 2),$$
$$S_n(15x^3+17x^2+4x, 2),\; S_n(15x^6+50x^5+31x^4, 2),\; S_n(25x^4-29x^2-8x, 2),$$
$$S_n(42x^6+47x^5+31x^3, 2),\; S_n(45x^6-47x^4+50x^3, 2),\; S_n(50x^4-43x^2+4, 2).$$

The list is not complete. There are further examples with $r = 2$, but sequences with $r > 2$ seem to be rare—I can't find any more.

If we allow the exponent $r = 1$, we get vast numbers of polynomials producing integer sequences: $3x+2$, $3x^2+6x+2$, $3x^2+9x+4$, $3x^2+15x+8$, $3x^2+18x+10$, $3x^2+24x+14$, $3x^2+27x+16$, $3x^2+33x+20$, $3x^2+36x+22$, $3x^2+42x+26$ and many more. It is possible that the sum might yield to analysis, perhaps by someone extremely familiar with the binomial coefficients. So (in addition to finding further examples with $r > 1$) I offer as a challenge the problem of characterizing the polynomials $p(x)$ that produce integer sequences of the form

$$\frac{1}{n}\sum_{k=0}^{n-1} p(k)\binom{2k}{k}.$$

Triskaidekaphilia

Bryan Orman

It is not the purpose of this article to discuss whether the number thirteen is unlucky or even lucky, but to present some unusual arithmetical results concerning this number.

This number appears quite often in mathematical contexts, for example, Euclid's Thirteen Books of *Elements* and The Thirteen Archimedean Semi-Regular Polyhedra. Of course there are many everyday references to the number thirteen; a baker's dozen and even a 'thirteen' which, prior to 1825, was an Irish shilling worth thirteen pence. Some of the more interesting ones are listed at the end of this article for amusement.

The number thirteen is the smallest *emirp* (a prime whose reverse is also prime), which is interesting, but not as interesting as the fact that $12! + 1$ is divisible by 13^2.

By Wilson's Theorem, if p is a prime then $(p-1)! \equiv -1 \pmod{p}$. The Wilson quotient is defined to be $W(p) = ((p-1)!+1)/p$ and a Wilson prime is a prime satisfying $W(p) \equiv 0 \pmod{p}$. The first three Wilson primes are 5, 13 and 563, with no others less than $5 \cdot 10^8$. So 13 is quite special in this context.

Now consider the following results involving the number thirteen:

$$13 = 2^2 + 3^2, \quad 13^2 = 5^2 + 12^2, \quad 13^3 = 9^2 + 46^2, \quad 13^4 = 119^2 + 120^2.$$

The first thing we observe is the fact that all four involve the sum of just two squares. A theorem of Lagrange states that any positive integer can be written as the sum of four squares, some of which may be zero, so it is not true that just two squares would suffice for all positive integers.

We note that if two integers, N and M, are each the sum of two squares, $N = a^2 + b^2$ and $M = c^2 + d^2$ say, then so is their product. The identity

$$(a^2 + b^2)(c^2 + d^2) \;=\; (ac + bd)^2 + (ad - bc)^2 \tag{i}$$

establishes this property. The further identity

$$(a^2 + b^2)(c^2 + d^2) \;=\; (ac - bd)^2 + (ad + bc)^2 \tag{ii}$$

shows that it can be done in two ways. It follows that if N is the sum of two squares then all its positive integer powers will also be the sum of two squares. Evidently only primes need be considered since, by the Fundamental Theorem of Arithmetic, every positive integer $n > 1$ can be expressed as the product of primes. Since a prime of the form $4k+3$ cannot be represented as the sum of two squares, the list of primes of interest is further reduced. Finally, and most importantly, a prime of the form $4k+1$ can be represented as the sum of two squares, and the representation is unique. All these basic results are to be found in David Burton's *Elementary Number Theory*, Allyn and Bacon, 1980.

Hereafter it is assumed that $N = 4k+1$ (prime) and therefore the sum of two squares, $N = a^2 + b^2$. If N is the sum of two consecutive squares then we write $N = a^2 + (a+1)^2$. We now examine each of the four results in turn.

$13 = 2^2 + 3^2$

As 13 is a prime of the form $4k+1$, where $k = 3$, it can be represented as the sum of two squares, but it is also the sum of two consecutive squares. Is this true for all $N = 4k+1$? If we write $4k+1 = a^2 + (a+1)^2$, then $k = \frac{1}{2}a(a+1)$, which is a triangular number. So 13 is not special in this sense. You may like to generate primes that can be written as the sum of two consecutive squares by choosing some values of k.

$13^2 = 5^2 + 12^2$

Since $13 = 2^2 + 3^2$, identity (ii) gives

$$13^2 \;=\; 13 \cdot 13 \;=\; (2^2+3^2)(2^2+3^2) \;=\; (2\cdot 2-3\cdot 3)^2+(2\cdot 3+3\cdot 2)^2 \;=\; 5^2+12^2$$

and we note that identity (i) does not give a second representation. Generally, if $N = a^2 + b^2$ then $N^2 = (a^2 - b^2)^2 + (2ab)^2$, which is just the identity for Pythagorean triples.

Although 13 is the sum of two consecutive squares, 13^2 is not. Is it possible for a prime $N = 4k+1$ to have its square, N^2, represented as the sum of two consecutive squares? If $N^2 = A^2 + (A+1)^2$ then $A = \frac{1}{2}(\sqrt{2N^2-1} - 1)$. Since A is an integer we write $2N^2 - 1 = M^2$ so that $A = \frac{1}{2}(M-1)$, with M necessarily odd. Now $M^2 - 2N^2 = -1$ is a Fermat–Pell equation and its solution, in positive integers $M_n, N_n)$, $n = 1, 2, \ldots$ is given by $M_n + N_n\sqrt{2} = (1+\sqrt{2})^{2n+1}$. The first three solutions are

$(M_1 = 7,\ N_1 = 5)$ giving $5^2 = 3^2 + 4^2$,

$(M_2 = 41,\ N_2 = 29)$ giving $29^2 = 20^2 + 21^2$,

$(M_3 = 239,\ N_3 = 169)$ giving $169^2 = 119^2 + 120^2$.

Continuing, can both N and its square N^2 be represented as the sum of two consecutive squares? If $N^2 = A^2 + (A+1)^2 = \big(a^2 + (a+1)^2\big)^2$ then, by identity (ii), we have $N^2 = (2a+1)^2 + \big(2a(a+1)\big)^2$.

So quite simply $A = 2a + 1$ and $A + 1 = 2a(a + 1)$ and these have the unique, positive solution $a = 1$, with $A = 3$. Thus $N = 5$ is the unique prime with the above property; $5 = 1^2 + 2^2$ and $5^2 = 3^2 + 4^2$.

$13^3 = 9^2 + 46^2$

We know that N^3 can be expressed as the sum of two squares and so a formula for N^3 would be helpful. To this end we write $N^3 = NN^2$ so that $N^3 = (a^2 + b^2)((a^2 - b^2)^2 + (2ab)^2)$ and, applying identities (i) and (ii), two representations for are obtained:

$$N^3 \;=\; (a^3 + ab^2)^2 + (b^3 + a^2b)^2, \quad N^3 \;=\; (a^3 - 3ab^2)^2 + (b^3 - 3a^2b)^2.$$

Furthermore, for $N = 13$ we have $a = 2$ and $b = 3$ so that $13^3 = 26^2 + 39^2 = 9^2 + 46^2$. Since 13^3 is not the sum of two consecutive squares it would be interesting to ask whether both N and N^3 can be represented as the sum of two consecutive squares? To this end, the equation $N^3 = A^2 + (A + 1)^2 = \bigl(a^2 + (a + 1)^2\bigr)^3$ needs to be solved for A in terms of a. Using the second representation of N^3 above with $b = a + 1$ gives

$$A^2 + (A + 1)^2 \;=\; (2a^3 + 6a^2 + 3a)^2 + (2a^3 - 3a - 1)^2.$$

And so, either $A = 2a^3 + 6a^2 + 3a$ and $A + 1 = 2a^3 - 3a - 1$, giving $3a^2 + 3a - 1 = 0$, which has no integer solutions. Or $A + 1 = 2a^3 + 6a^2 + 3a$ and $A = 2a^3 - 3a - 1$, giving $a(a+1) = 0$, with the trivial result $1^3 = 0^2 + 1^2$! So $N^3 = A^2 + (A + 1)^2$ has no solution in the positive integers.

It is left to the reader to show that the first representation for N^3 also leads to no solution in the positive integers.

$13^4 = 119^2 + 120^2$

This looks more promising since 13^4 is expressed as the sum of two consecutive squares. A general formula for N^4 is now necessary for the investigation. Complex factorization is extremely helpful here:

$$N^4 \;=\; A^2 + B^2 \;=\; (a^2 + b^2)^4, \quad (A + iB)(A - iB) \;=\; (a + ib)^4(a - ib)^4.$$

There are just two identifications here for $A + iB$. Setting $A + iB = (a + ib)^4 = a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4$ leads to

$$A \;=\; a^4 - 6a^2b^2 + b^4 \quad \text{and} \quad B \;=\; 4a^3b - 4ab^3. \tag{iii}$$

With $a = 2$ and $b = 3$ these give $13^4 = 119^2 + 120^2$.

Setting $A + iB = (a + ib)^3(a - ib)$ leads to

$$A \;=\; a^4 - b^4 \quad \text{and} \quad B \;=\; 2ab(a^2 + b^2). \tag{iv}$$

With $a = 2$ and $b = 3$ these give $13^4 = 65^2 + 156^2$, which is the same as $65^2 + 156^2 = 13^2(5^2 + 12^2) = 13^2 13^2 = 13^4$!

What about consecutive squares? Is it possible for other values of N to have both N and N^4 expressed as the sum of consecutive squares? Using (iii), either $A = a^4 - 6a^2b^2 + b^4$ and $B = A + 1 = 4a^3b - 4ab^3$, or $A = a^4 - 6a^2b^2 + b^4$ and $B = A - 1 = 4a^3b - 4ab^3$. As $b = a + 1$ the first pair of equations leads to $f(a) = 2a^4 - 6a^2 - 4a - 1 = 0$, and this has no solution in the positive integers since $f(1) = -9$, $f(2) = -1$, $f(3) = 95$ and $f(a) > 0$ for $a > 3$. The second pair of equations lead to $a^4 - 3a^2 - 2a = 1$ or $a(a+1)^2(a-2) = 0$. This equation has the unique solution $a = 2$ (positive integer), giving $b = 3$, and eventually $|A| = 119$ and $|B| = 120$. Using (iv) it can be shown that there are no further solutions in positive integers.

So 13 *is a very special integer in that both it, and its fourth power, can be expressed as the sum of two consecutive squares and no other integer has this property.* This is a remarkable uniqueness property. If you have yet to appreciate the special nature of the number 13 then consider the following results concerning squares.

> The only squares among the Fibonacci numbers are 1 and 144 and the only squares among the Pell numbers are 1 and 169.

These have been proved.

Markov numbers are the union of the solutions (x, y, z), called Markov triples, to the Markov equation

$$x^2 + y^2 + z^2 \ = \ 3xyz.$$

This equation entered the mathematical literature in a paper by Markov (A. A. Markoff, Sur les formes quadratiques binaires indefinies, *Math. Ann.* **15** (1879), pp. 381–406 and **17** (1880), pp. 379–400), although interested readers would find J. H. Conway and R. K. Guy, *The Book of Numbers*, Springer, New York 1996, pp. 187–189, more accessible!

The first few Markov triples are $(1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 5)$, $(1, 5, 13)$, $(2, 5, 29)$, ... So the first few Markov numbers are 1, 2, 5, 13, 29, 34, 89, 169, 233, 433, 610, ... We note that both 13 and its square, 169, are Markov numbers.

> *I have conjectured that* 13 *is the only Markov number whose square* 169 *is also a Markov number.*

This is still an open question for number theorists and, if it is true, it would establish 13 as a special one!

A Thirteen Miscellany

In Christianity there were thirteen at the last supper.

The Great Seal of the United States has 13 arrows, 13 stars, 13 olive leaves with 13 olives. These form a triangle over the eagle with the number 13 at each vertex.

In Sweden, Trettondagsafton is thirteen day eve, that is, twelfth night.

In the Spanish speaking world it's Tuesday the thirteenth; 'En Martes, ni te cases ni te embarques', that is, 'On Tuesday, neither get married nor start a journey'.

When the Earth revolves once, the Moon revolves thirteen times.

Anagram: ELEVEN + TWO = TWELVE + ONE.

13,333,333,333,333 is divisible by 13 and its quotient is 1,025,641,025,641 and is a prime.

$p^{12} - q^{12}$ is divisible by 13 only when p and q are not divisible by 13.

Problem 235.2 – Quartic roots

Let α, β, γ and δ be the roots of the quartic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e \;=\; 0.$$

Show that the equation

$$\sqrt[3]{\alpha\beta + \gamma\delta - x} + \sqrt[3]{\alpha\gamma + \beta\delta - x} + \sqrt[3]{\alpha\delta + \beta\gamma - x} \;=\; 0$$

has the solution

$$x \;=\; \frac{2\left(c^3 - 2ad^2 - 2b^2e + 3ace\right)}{a\left(3c^2 - 4bd + ae\right)}.$$

Problem 235.3 – Odd pairs

Show that

$$\sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \frac{1}{(2i+1)^4} \frac{1}{(2j+1)^4} \;=\; \frac{\pi^8}{16 \cdot 8!}.$$

Problem 235.4 – Matrix

Tony Forbes

Construct an $n \times n$ matrix as follows. Partition n into n_1 and n_2, $n_1, n_2 \geq 2$, and divide the matrix into four parts. The top left part is an $n_1 \times n_1$ matrix with a on the diagonal and c everywhere else. The bottom right part is an $n_2 \times n_2$ matrix with b on the diagonal and d everywhere else. The rest of the matrix elements are e. Also we insist that a, b, c, d and e are integers satisfying $a \neq b$, $a \neq c$ and $b \neq d$.

In every example I have created I have observed that the rank of the matrix is either n or $n-1$. So here is the problem: Either prove that the matrix has rank at least $n-1$, or find a counter-example. Also it would be nice to know exactly when rank $n-1$ occurs.

Let us fix $n = 16$, $a = 5$, $b = 6$, $c = 1$ and $d = e = 2$. Then here is what the matrix looks like when $n_1 = 6$ and $n_2 = 10$:

$$\begin{bmatrix}
5 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 5 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 5 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 5 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 5 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 5 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6
\end{bmatrix}.$$

This particular matrix is actually $A \cdot A^{\mathrm{T}}$, where A is the adjacency matrix of a certain generalized 2-design. As a consequence, it has rank 15. Rank 15 also occurs when $n_1 = 12$ (for which I have no explanation) but the rank is 16 for all other values of n_1.

Thanks to **Derek Patterson** of Queen Mary, University of London for the idea behind this problem.

Snow and sausages

Tony Huntington

Dressed like extras for a cheap remake of *Scott of the Antarctic*, a select group braved their way through the January snow to Florence Boot Hall at the University of Nottingham to participate in the annual M500 Winter Recreational Mathematics Weekend. Mel and Angela guided us through a selection of 'investigations' which formed the theme of the weekend. These were simply-worded problems that lend themselves to 'extensions'. There were no 'right' answers, and no limits to how far you chose to take the investigation. My thanks to Mel Starkings and Angela Allsopp for an entertaining and challenging weekend, to Rob Rolfe who organized and presented our Friday Night Quiz, and to Diana Maxwell for arranging everything and for ensuring that all ran smoothly.

On the Friday evening, Mel set us thinking about the mathematics of sausages. Suppose that you have a string of sausages (these are, of course, mathematical sausages and so all are of equal length); then, in general, the string could be arranged in a triangular shape. Given a known number of sausages in the string, then the sausage number is the maximum number of different (non-congruent) triangles that you can construct from a given string. If we let N be the number of sausages, and $f(N)$ be the sausage number, then what is $f(42)$, and $f(2010)$? Is there a general formula?

A little bit of thought reveals that the sausages are not essential to the problem (although they are essential to the cooked breakfast that we enjoyed in the Dining Hall on both mornings, in my opinion). What we are considering is triangles with three integer sides. Let the three sides be a, b and c. Then

$$a + b + c \;=\; N.$$

If we are only considering real triangles (i.e. each triangle encloses a real, finite area), then each side must be at least 1 unit long, so

$$a, b, c, N \;\in\; \mathbb{Z}^+ \quad \text{and} \quad N \geq 3.$$

Now, without loss of generality, let us assume that the lengths of the three sides are such that $a \geq b \geq c$. Then

$$a < b + c, \qquad a < N - a, \qquad a < \frac{N}{2}.$$

And the other two sides will be less than, or equal to, a. Using integer arithmetic, we can define the upper bound on a as

$$A_U \;=\; (N-1) \div 2.$$

The lower bound on a will occur when $a = b = c$ (if that is possible). In integer arithmetic this can be expressed as

$$A_L = (N + 2) \div 3.$$

And so we can say that

$$a \in \{A_L \cap A_U\}.$$

The number of elements in the set of a is

$$N_a = A_U - A_L + 1.$$

Each of the elements in the set of a will appear in at least one triangle, so N_a represents a lower bound on $f(N)$.

Now considering the 'next longest' side, b, its upper bound must be a (otherwise it would be longer than a and so a would not be the longest side), so

$$B_U = a.$$

The lower bound on b will occur when $b = c$ (if that is possible). In integer arithmetic this can be expressed as

$$B_L = (N - a + 1) \div 2.$$

And similarly

$$b \in \{B_U \cap B_L\}.$$

So, for a given N, we can find the largest and smallest values of a. Thus a is integer and will take on each of the integer values between A_L and A_U. For each value of a, we can find the largest and smallest values of b. Indeed, b is integer and will take on each of the integer values between B_L and B_U. And for each combination of a and b, we can find c as $c = N - (a + b)$, and so we can define each possible triangle in turn, and then count them to find the sausage number, $f(N)$. Incidentally, because of the way that the sides of the triangles are found, they must all be mutually non-congruent. The discussion above does not lead to a general formula for $f(N)$, but is the basis of an algorithm for finding $f(N)$ for any given N.

Using this algorithm, the values of $f(N)$ for N between 3 and 13, with intermediate values, are shown in the big table on the next page. And values of $f(N)$ for N between 3 and 32 are on the little table below it.

N	A_U	A_L	B_U a	B_L	b	c	$f(N)$
3	1	1	1	1	1	1	1
4	1	2					0
5	2	2	2	2	2	1	1
6	2	2	2	2	2	1	1
7	3	3	3	2	2 3	2 1	2
8	3	3	3	3	3	2	1
9	4	3	3 4	3 3	3 3 4	3 2 1	3
10	4	4	4	3	3 4	3 2	2
11	5	4	4 5	4 3	4 3 4 2	3 3 2 1	4
12	5	4	4 5	4 4	4 4 5	4 3 2	3
13	6	5	5 6	4 4	4 5 4 5 6	4 3 3 2 1	5

N	$f(N)$	N	$f(N)$	N	$f(N)$	N	$f(N)$	N	$f(N)$
3	1	9	3	15	7	21	12	27	19
4	0	10	2	16	5	22	10	28	16
5	1	11	4	17	8	23	14	29	21
6	1	12	3	18	7	24	12	30	19
7	2	13	5	19	10	25	16	31	24
8	1	14	4	20	8	26	14	32	21

Applying this algorithm to the case where $N = 42$ we can find that

$$f(42) = 37.$$

This is about the limit if calculating sausage numbers by this method by hand. As N increases, so the number of individual calculations needed to find $f(N)$ increases. I wrote a small computer program based on this method to find $f(N)$. This enabled me to answer the question, What is $f(2010)$? And the answer is ... 84,169.

This is the 'sledgehammer' approach to maths and is frowned upon by Luddite purists. So, is there a general formula for $f(N)$?

If all this talk of sausages has your mouth watering you will surely like to know that next January there will be another **M500 Winter Weekend** at Nottingham University. See below for details.

M500 Winter Weekend 2011

A Weekend of Mathematics and Socializing

Join with fellow mathematical enthusiasts for a weekend of mathematical fun. If you are interested in mathematics and want a fantastic weekend, then this is for you, accessible to anyone who has studied mathematics even if you're just starting. The **thirtieth M500 Society Winter Weekend** will be held at

Florence Boot Hall, **Nottingham University**

$\mathbf{7^{th}}$–$\mathbf{9^{th}}$ **January 2011**.

The overall theme will be PROOF. Cost: £190 to M500 members, £195 to non-members. You can obtain a booking form from the M500 site.

http://www.m500.org.uk/winter/booking.pdf

If you have no access to the internet, send a stamped addressed envelope to

Diana Maxwell.

We will have the usual extras. On Friday Tony Huntington is running a pub quiz with Valuable Prizes, and for the sing-song on Saturday night we urge you to bring your favourite musical instrument (and your voice). Hope to see you there.

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04.01

Cover: Stephen Wolfram's Rule 90. See Sebastian Hayes's article, pp 1–7. And here is what Rule 30 looks like.