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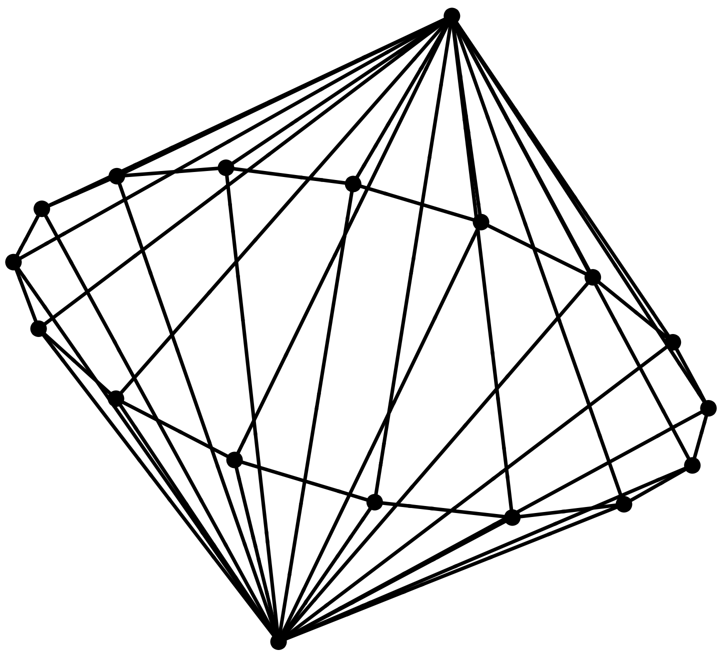
M500 258



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

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Advice to authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to the Editor, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation.

Solution 254.4 – Gaussian binomial coefficients

For positive integer n, define

$$[n]_q \;=\; 1+q+\cdots+q^{n-1}, \qquad [n]_q! \;=\; [1]_q\,[2]_q\,\ldots\,[n]_q$$

and as with the usual binomial coefficient $\displaystyle\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$, define the *Gaussian binomial coefficient* by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \;=\; \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.$$

Show that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q with integer coefficients.

Stuart Walmsley

Introduction The problem concerns a system of polynomials, defined so that it closely parallels the integers, factorials and binomial coefficients.

The binomial coefficients are familiar constructs arising successively from the positive integers $n = 1, 2, 3, \ldots$ and the factorials $n! = 1\cdot 2\cdot 3\cdot \cdots \cdot n$ leading to the binomial coefficients themselves:

$$\binom{n}{k} \;=\; \frac{n!}{k!(n-k)!}. \tag{1}$$

The Gaussian binomial coefficients arise from a system in which polynomials replace integers,

$$[n]_q \;=\; 1+q+q^2+\cdots+q^{n-1},$$

with 'factorials'

$$[n]_q! \;=\; [1]_q\cdot[2]_q\cdot\cdots\cdot[n]_q$$

and hence the Gaussian binomial coefficients,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \;=\; \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{2}$$

The binomial coefficients, which are rational numbers from the definition, may be shown to be integers and the objective of the problem is to show that the Gaussian binomial coefficient, formally a quotient of polynomials, can be simplified to a single polynomial.

Two ways of proving the integer results will be considered and it will be shown that they can be adapted to deal with the case of the polynomials. The first method uses a recurrence relation and the second the properties of factors.

The integer recurrence relation From (1)

$$\binom{n}{k} = \binom{n}{n-k}$$

and

$$\binom{n}{0} = \frac{n!}{0!\,n!} = 1 = \binom{n}{n}$$

with the usual convention that $0! = 1$. The lowest binomial coefficients can then be evaluated in the form of the Pascal triangle.

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \qquad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1$$

It is seen that all these values are integers. It is well known (and easily proved) that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{3}$$

Thus all coefficients are sums of coefficients with lower n. Since the lowest values are integers, all values are integers.

The polynomial recurrence relation The same path is followed for the Gaussian system. From (2),

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q \quad \text{and} \quad \begin{bmatrix} n \\ 0 \end{bmatrix}_q = \frac{[n]_q!}{[0]_q!\,[n]_q!} = \begin{bmatrix} n \\ n \end{bmatrix}_q.$$

It will be assumed that $[0]_q! = 1$ as in the integer case and shown later that this is consistent. Then

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 = \begin{bmatrix} n \\ n \end{bmatrix}_q.$$

In addition

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = \frac{[2]_q!}{[1]_q!\,[1]_q!} = 1 + q$$

so that the top of the 'Gaussian' Pascal triangle can be written thus.

$$\begin{matrix} & & \begin{bmatrix}0\\0\end{bmatrix}_q = 1 & & \\ & \begin{bmatrix}1\\0\end{bmatrix}_q = 1 & & \begin{bmatrix}1\\1\end{bmatrix}_q = 1 & \\ \begin{bmatrix}2\\0\end{bmatrix}_q = 1 & & \begin{bmatrix}2\\1\end{bmatrix}_q = 1+q & & \begin{bmatrix}2\\2\end{bmatrix}_q = 1 \end{matrix}$$

All the terms are polynomials.

The strategy is now to examine the polynomial equivalent of each side of (3) and to see if a modification can be found which leads to equality. We have

$$\begin{bmatrix}n\\k\end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \frac{[n-1]_q!}{[k]_q!\,[n-k]_q!}[n]_q, \tag{4}$$

$$\begin{aligned} \begin{bmatrix}n-1\\k-1\end{bmatrix}_q + \begin{bmatrix}n-1\\k\end{bmatrix}_q &= \frac{[n-1]_q!}{[k-1]_q!\,[n-k-1]_q!}\left(\frac{1}{[n-k]_q} + \frac{1}{[k]_q}\right) \\ &= \frac{[n-1]_q!}{[k-1]_q!\,[n-k-1]_q!} \cdot \frac{[k]_q + [n-k]_q}{[k]_q\,[n-k]_q} \\ &= \frac{[n-1]_q!}{[k]_q!\,[n-k]_q!}\big([k]_q + [n-k]_q\big). \end{aligned}$$

Thus

$$\begin{bmatrix}n\\k\end{bmatrix}_q = \{\text{const}\}[n]_q, \qquad \begin{bmatrix}n-1\\k-1\end{bmatrix}_q + \begin{bmatrix}n-1\\k\end{bmatrix}_q = \{\text{const}\}([k]_q + [n-k]_q).$$

The factor {const} being the same in the two cases. Then

$$[n]_q = 1+q+q^2+\cdots+q^{n-1} = 1+q+q^2+\cdots+q^{k-1}+q^k+q^{k+1}+\cdots+q^{n-1},$$

$$[k]_q + [n-k]_q = 1+q+q^2+\cdots+q^{k-1}+1+q+q^2+\cdots+q^{n-k-1}.$$

If it is assumed that $k \le n-k$, then the number of terms in each is the same and $[n]_q = [k]_q + q^k[n-k]_q$. Correspondingly

$$\begin{bmatrix}n\\k\end{bmatrix}_q = \begin{bmatrix}n\\n-k\end{bmatrix}_q = \begin{bmatrix}n-1\\k-1\end{bmatrix}_q + q^k\begin{bmatrix}n-1\\k\end{bmatrix}_q, \qquad k \le n-k,$$

giving a recurrence relation.

This is consistent for $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ and $\begin{bmatrix} n \\ n \end{bmatrix}_q$ if both are assigned the value 1. Thus all the Gaussian binomial coefficients are sums of coefficients with lower n, some multiplied by a power of q. Since the lowest values are polynomials, all values are polynomials.

Integer factorization Consider the (ordinary) binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}.$$

Without loss of generality consider $k \le n-k$. Then

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{1\cdot 2\cdot \dots \cdot k}; \tag{5}$$

that is, the quotient of the product of k successive integers divided by the product of the the first k integers. For each j in the sequence 1 to k, there is a multiple of j in any sequence of k successive integers. Hence each such j can be cancelled out and the rational number is reduced to an integer. Hence all binomial coefficients are integers.

Polynomial factorization The basic component

$$[n]_q = 1 + q + q^2 + \dots + q^{n-1}$$

can be rewritten

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

In this way, the Gaussian binomial coefficient can be rewritten

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q-1)(q^2-1)\dots(q^k-1)}.$$

Thus each integer j, in the binomial coefficient form (5), is replaced by a polynomial $q^j - 1$. The roots of $q^j - 1$ are the j distinct jth roots of 1:

$$\exp(2\pi i k/j), \qquad k = 0, 1, \dots, j-1. \tag{6}$$

All the roots are non-real, except for $+1$ corresponding to $k = 0$ and -1 for $k = j/2$ when j is even.

Factors which are polynomials with integer coefficients can be found for progressively higher values of j, there being one new factor for each new integer. These are the *cyclotomic polynomials*, the roots of which are the terms (6) for which k is co-prime to j.

Let $f_j = (q^j - 1)$ and let t_j represent the corresponding cyclotomic polynomial. Then

$$\begin{array}{lllllll} f_1 &=& t_1 &=& q-1 & \text{giving} & t_1 = q-1, \\ f_2 &=& t_1 t_2 &=& q^2-1 & \text{giving} & t_2 = q+1, \\ f_3 &=& t_1 t_3 &=& q^3-1 & \text{giving} & t_3 = q^2+q+1; \end{array}$$

in general, for a prime number p, the only factors are 1 and p and so $t_p = q^{p-1} + \cdots + 1$;

$$\begin{array}{lllllll} f_4 &=& t_1 t_2 t_4 &=& q^4-1 & \text{giving} & t_4 = q^2+1, \\ f_6 &=& t_1 t_2 t_3 t_6 &=& q^6-1 & \text{giving} & t_6 = q^2-q+1, \\ f_8 &=& t_1 t_2 t_4 t_8 &=& q^8-1 & \text{giving} & t_8 = q^4+1. \end{array}$$

Then every term in the Gaussian binomial coefficient (4) is given by a product of cyclotomic polynomials, one for each of the factors of the underlying integer. All the factors in the denominator are cancelled by factors in the numerator by an extension of the arguments used for the integer case, reducing the polynomial quotient to a single polynomial, as required.

Tony Forbes

We can continue computing cyclotomic polynomials as above to obtain

$$\begin{aligned} t_9 &= q^6 + q^3 + 1, \\ t_{10} &= q^4 - q^3 + q^2 - q + 1, \\ t_{12} &= q^4 - q^2 + 1, \\ t_{14} &= q^6 - q^5 + q^4 - q^3 + q^2 - q + 1, \\ t_{15} &= q^8 - q^7 + q^5 - q^4 + q^3 - q + 1, \\ t_{16} &= q^8 + 1 \text{ (and in general } t_{2^r} = q^{2^{r-1}} + 1), \\ t_{18} &= q^6 - q^3 + 1, \\ t_{20} &= q^8 - q^6 + q^4 - q^2 + 1, \\ t_{21} &= q^{12} - q^{11} + q^9 - q^8 + q^6 - q^4 + q^3 - q + 1. \end{aligned}$$

At this point (or even before) one might be tempted to conjecture that the coefficients are always ± 1 or 0, and indeed this is true—until you get to

$$\begin{aligned} t_{105} = \; & q^{48} + q^{47} + q^{46} - q^{43} - q^{42} - 2q^{41} - q^{40} - q^{39} + q^{36} + q^{35} + q^{34} \\ & + q^{33} + q^{32} + q^{31} - q^{28} - q^{26} - q^{24} - q^{22} - q^{20} + q^{17} + q^{16} + q^{15} \\ & + q^{14} + q^{13} + q^{12} - q^9 - q^8 - 2q^7 - q^6 - q^5 + q^2 + q + 1. \end{aligned}$$

Solution 244.6 – Flagpole integral

Compute

$$\int_{\alpha}^{\pi/2} \left(\tan\theta - \sqrt{\tan^2\theta - \alpha^2}\right) d\theta,$$

where $0 < \alpha < \pi/2$ is a constant. Ideally an exact solution is desired. We are also interested in a good approximation on the assumption that $2/\alpha^2$ is the Earth's radius in metres.

Steve Moon

To postpone the potential problems as $\theta \to \pi/2$, we first tackle the integral with limits α, β such that $0 < \alpha < \beta < \pi/2$. Thus

$$\int_{\alpha}^{\beta} \left(\tan\theta - \sqrt{\tan^2\theta - \alpha^2}\right) d\theta \;=\; \big[\log\sec\theta\big]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \sqrt{\tan^2\theta - \alpha^2}\, d\theta. \quad (1)$$

Postpone the evaluation of the log sec term for now. In the remaining integral, make the substitution

$$\tan\theta = \alpha x, \qquad d\theta = \frac{\alpha\, dx}{1 + \alpha^2 x^2},$$

so that the limits α and β become

$$a = \frac{\tan\alpha}{\alpha} \quad \text{and} \quad b = \frac{\tan\beta}{\alpha}$$

respectively. Then, after a bit of work,

$$\begin{aligned} -\int_{\alpha}^{\beta} \sqrt{\tan^2\theta - \alpha^2}\, d\theta \;&=\; -\int_{a}^{b} \frac{\sqrt{x^2 - 1}}{(x^2 - 1) + (1 + 1/\alpha^2)}\, dx \\ &=\; -\int_{a}^{b} \frac{dx}{\sqrt{x^2 - 1}} + \int_{a}^{b} \frac{1 + 1/\alpha^2}{(x^2 - 1) + (1 + 1/\alpha^2)} \cdot \frac{dx}{\sqrt{x^2 - 1}}. \quad (2) \end{aligned}$$

Now

$$\int_{a}^{b} \frac{dx}{\sqrt{x^2 - 1}} \;=\; \left[\cosh^{-1} x\right]_{a}^{b} \;=\; \left[\log\left(x + \sqrt{x^2 - 1}\right)\right]_{a}^{b}.$$

Undo the substitution $\tan\theta = \alpha x$:

$$-\int_{a}^{b} \frac{dx}{\sqrt{x^2 - 1}} \;=\; -\left[\log\left(\frac{\tan\theta}{\alpha} + \sqrt{\frac{\tan^2\theta}{\alpha^2} - 1}\right)\right]_{\alpha}^{\beta}. \quad (3)$$

Combining the log sec term from (1) and the first integral from (2) evaluated as (3) gives

$$\big[\log\sec\theta\big]_\alpha^\beta - \left[\log\left(\frac{\tan\theta}{\alpha}+\sqrt{\frac{\tan^2\theta}{\alpha^2}-1}\right)\right]_\alpha^\beta$$

$$= \left[\log\frac{\alpha}{\sin\theta+\sqrt{\sin^2\theta-\alpha^2\cos^2\theta}}\right]_\alpha^\beta$$

$$= \log\frac{\sin\theta+\sqrt{\sin^2\theta-\alpha^2\cos^2\theta}}{2} \qquad (4)$$

on letting $\beta\to\pi/2$.

We are left with the second integral in (2),

$$J = \int_a^b\left[\frac{1+1/\alpha^2}{(x^2-1)+(1+1/\alpha^2)}\right]\frac{dx}{\sqrt{x^2-1}}.$$

Some 'trial and error' is needed here. The form of the square-bracketed component indicates we might expect an arctan or arctanh in the integral result. Consider the functions

$$f(x)=\frac{x}{\sqrt{x^2-1}} \quad\text{and}\quad g(x)=A\tanh^{-1}(Bf(x))$$

with A and B constants. Then

$$f'(x)=\frac{-1}{(x^2-1)\sqrt{x^2-1}} \quad\text{and}\quad g'(x)=\frac{AB}{\sqrt{x^2-1}}\cdot\frac{1}{B^2x^2+1-x^2}.$$

Now set $A=B=\sqrt{1+\alpha^2}$ so that

$$g'(x) = \frac{1+1/\alpha^2}{\sqrt{x^2-1}}\cdot\frac{1}{x^2+1/\alpha^2},$$

which is the integrand we seek. Hence

$$J = \left[\sqrt{1+\alpha^2}\ \tanh^{-1}\frac{x\sqrt{1+\alpha^2}}{\sqrt{x^2-1}}\right]_a^b.$$

Again, unwind the substitution $\tan\theta=\alpha x$:

$$J = \left[\sqrt{1+\alpha^2}\ \tanh^{-1}\frac{\sqrt{1+\alpha^2}\ \sin\theta}{\sqrt{\sin^2\theta-\alpha^2\cos^2\theta}}\right]_\alpha^\beta.$$

Now we can take the limit as $\beta \to \pi/2$:

$$J \;=\; \sqrt{1+\alpha^2}\left(\tanh^{-1}\sqrt{1+\alpha^2} - \tanh^{-1}\frac{\sqrt{1+\alpha^2}\;\sin\alpha}{\sqrt{\sin^2\alpha - \alpha^2\cos^2\alpha}}\right). \qquad (5)$$

Using the log form of the arctanh function, $\tanh^{-1}x = \dfrac{1}{2}\log\dfrac{1+x}{1-x}$, equality (5) becomes

$$J \;=\; \frac{A}{2}\left(\log\frac{1+A}{1-A} - \log\frac{\sqrt{\sin^2\alpha-\alpha^2\cos^2\alpha}+A\sin\alpha}{\sqrt{\sin^2\alpha-\alpha^2\cos^2\alpha}-A\sin\alpha}\right), \qquad (6)$$

where $A = \sqrt{1+\alpha^2}$, as before. Finally, combining (4) and (6),

$$\int_\alpha^{\pi/2}\left(\tan\theta - \sqrt{\tan^2\theta-\alpha^2}\right)d\theta \;=\; \log\frac{\sin\alpha+\sqrt{\sin^2\alpha-\alpha^2\cos^2\alpha}}{2}$$
$$+\;\frac{A}{2}\log\frac{(1+A)\left(\sqrt{\sin^2\alpha-\alpha^2\cos^2\alpha}-A\sin\alpha\right)}{(1-A)\left(\sqrt{\sin^2\alpha-\alpha^2\cos^2\alpha}+A\sin\alpha\right)}. \qquad (7)$$

Tony Forbes

To get an approximate formula for small α we compute the first few terms in the Taylor expansion about $\alpha = 0$ of the right-hand side of (7). Rather than do it by hand I shall instead use MATHEMATICA:

$$\int_\alpha^{\pi/2}\left(\tan\theta - \sqrt{\tan^2\theta-\alpha^2}\right)d\theta \;=\; \frac{-2\log\alpha+\log 4-1}{4}\,\alpha^2$$
$$+\;\frac{12\log\alpha-6\log 4-5}{96}\,\alpha^4 + \frac{2\sqrt{6}}{27}\,\alpha^5 + O(\alpha^6). \qquad (8)$$

If $\alpha = \sqrt{2/R}$ and $R = 6378137$ is the Earth's radius in metres, the α^2 term of (8) can be used to obtain a good approximation for the average length of the shadow cast by a 1 metre flagpole at a location where the sun is directly overhead at midday, assuming it is observed between sunrise and sunset on a cloudless day (see Vincent Lynch's analysis in M500 **244**, pp 16–17):

$$\frac{2R}{\pi}\left(\frac{\alpha^2}{2} + \frac{-2\log\alpha+\log 4-1}{4}\,\alpha^2\right) \;\approx\; 5.52635,$$

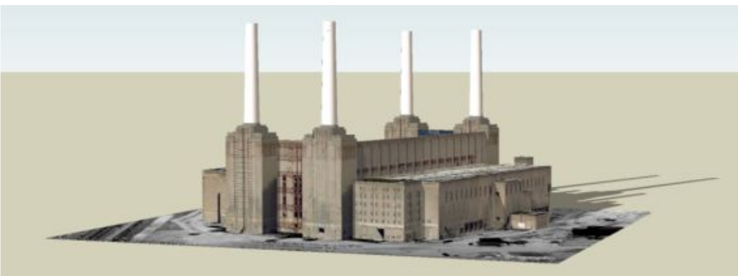
which is correct to five decimal places, possibly more. On reflection, I think the Taylor expansion is too interesting to have delivered as if by magic. So for enlightenment we would be very interested if anyone would like to show how to get at least the first term on the right of (8).

Problem 258.1 – Battersea Power Station

David Singmaster

For those who don't know London, Battersea Power Station is a London landmark beside Chelsea Bridge and the Thames. This was (is?) the world's (or Europe's?) biggest brick building, using 61M bricks. Giles Gilbert Scott was asked to improve the architecture in 1931. Battersea A was started in 1929 and the first part, of 138MW, started work in 1933, with an additional 105MW in Sep 1935. Construction of Battersea B started in 1937. The first part, of 100MW, was in service in 1941. After the war, another 60MW was added and a final 100MW was added in 1953. The final result is a massive rectangular building with four massive chimneys, at the four corners of the rectangular building, which are visible from much of London. The chimneys are 337ft high. Battersea A closed in 1975. Battersea B closed on 31 Oct 1983. It was planned to be converted into a theme park by 1990, but the idea fell victim to a recession and the building remains half open to the elements. A friend recently described it as looking like a dead table.

The chimneys are basically at the corners of a rectangle, whose long sides run approximately north–south. The dimensions are approximately 50m $\times$ 160m. The problem arises because one sees the chimneys on the skyline as one drives into London from the west and one notices that the relative positions of the chimneys shift as one drives along the north side of the Thames. It appears to me that there will be some point where the chimneys will appear regularly spaced along the skyline. Is this true? If so, where does one have to be to see this effect? For consistency, let us label the four chimneys as A, B, C, D, going clockwise from A at the SW corner which we take as the origin of a coordinate system. So A is at $(0, 0)$, B is at $(0, 160)$, C at $(50, 160)$, D at $(50, 0)$. John Sharp has found a site which does architectural views and has sent the view below, but there are no dimensions given. I want to be able to go to a correct viewpoint and take a photo.



Solution 255.2 – Bomb

(i) A bomb is released from position $(0,0,h)$ by an aircraft travelling at velocity $\mathbf{v}$ relative to the ground. The wind has velocity $\mathbf{w}$. Air resistance may be ignored. Assuming the ground is flat, where will it land and how long will it take to get there?

(ii) As (i) but air resistance is not ignored. The bomb, travelling at velocity $\mathbf{u}$ relative to the air, experiences an acceleration of $-\mathbf{u}|\mathbf{u}|k$ for some small constant k.

For example, when $\mathbf{v} = (v,0,0)$ and $\mathbf{w} = (0,0,0)$ the landing point and drop time are (I (TF) think)

$$\left(\frac{1}{k}\log\left(1+\sqrt{k/g}\,v\,\cosh^{-1}e^{kh}\right),0,0\right) \quad \text{and} \quad \frac{\cosh^{-1}e^{kh}}{\sqrt{gk}},$$

which degenerate to $(v\sqrt{2h/g},0,0)$ and $\sqrt{2h/g}$ when $k=0$.

Tommy Moorhouse

Rather than frame the solution in militaristic terms I would like instead to consider the question of a parcel of emergency aid dropped from an aeroplane. The reader should be in no doubt that the equations are the same but the outcome is more humane.

First of all we consider the simplest case of a parcel released at a height h with zero velocity. The parcel will accelerate under the influence of gravity, with the air resistance growing as the speed increases. If we label the vertical direction z (measured from the ground up) and the z-velocity as $u = \dot{z}$ we have

$$\dot{u} \;=\; -g + ku^2.$$

Note the sign of the second term: since u is going to point downwards the air resistance will act upwards (in the positive z-direction). We make the substitution

$$u \;=\; \frac{-1}{k}\frac{\dot{\psi}}{\psi}$$

and find that $\ddot{\psi} = kg\psi$. The solution is

$$\psi(t) \;=\; A\sinh(t\sqrt{gk}) + B\cosh(t\sqrt{gk})$$

with arbitrary constants A and B. Substituting back we get

$$u \;=\; \frac{-1}{k}\sqrt{gk}\frac{A\cosh(t\sqrt{gk}) + B\sinh(t\sqrt{gk})}{A\sinh(t\sqrt{gk}) + B\cosh(t\sqrt{gk})}.$$

At $t = 0$ we have $u = 0$ so $A = 0$ and

$$u = \frac{-1}{k}\sqrt{gk}\tanh(t\sqrt{gk}).$$

Note that this leads to a 'terminal velocity' of $-\sqrt{g/k}$. To find z we use $\dot{z} = u$ to integrate and find

$$z = \frac{-1}{k}\log(\cosh(t\sqrt{gk})) + z_0.$$

Since $z(0) = h$ we have $z_0 = h$. When $z = 0$ we have $-\log(\cosh(t\sqrt{gk}))/k = -h$ so $t = \cosh^{-1}(e^{hk})/\sqrt{gk}$.

The general case is only slightly more involved, and the result above for the transit time remains true. First we choose the z-direction to be vertically upwards, and the x-direction to be in the horizontal direction of the initial velocity V. Write $u = \dot{z}, w = \dot{x}$. Our equations of motion are then the set

$$\begin{aligned}\dot{u} &= -g + ku^2,\\ \dot{w} &= -kw^2.\end{aligned}$$

We only have to worry about the second equation, having solved the first. Rearranging $\dot{w} = -kw^2$, we have

$$-\frac{\dot{w}}{w^2} = k.$$

The left-hand side is just the time derivative of $1/w$ so we have, using the initial condition $w(0) = V$,

$$w(t) = \frac{V}{kVt + 1}.$$

(It is interesting to see how a substitution similar for the one we used to find u above works to give the same result.) Integrating $\dot{x} = w$ we find

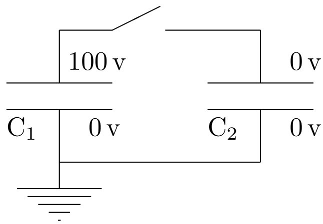
$$x(t) = \frac{1}{k}\log(kVt + 1) + C$$

and since $x(0) = 0$ we have $C = 0$. At the time the parcel hits the ground

$$x = \frac{1}{k}\log\left(\frac{V\sqrt{k}}{\sqrt{g}}\cosh^{-1}(e^{hk}) + 1\right).$$

Note. This type of nonlinear second order ordinary differential equation was studied by Painleve and others, and the linearization is associated with Miura.

Solution 256.5 – Lost energy



The diagram represents the initial state of a circuit containing two capacitors of C farads each, with 100 volts across C_1. When the switch is closed C_1 loses charge to C_2 until they equalize at 50 volts across each capacitor.

Initially the total energy in the system is the $100^2C/2 = 5000C$ joules stored in C_1. But when the circuit has settled down after the switch is closed, the energy is split between the two capacitors at $50^2C/2$ joules each, making a total of $2500C$ joules. What has happened to the other $2500C$ joules?

Mike Lewis

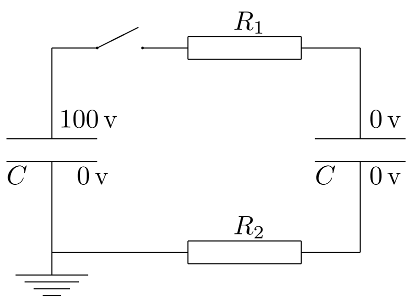
Voltage is defined as the electrical potential energy per unit charge, measured in joules per coulomb (= volts). The energy stored in C_1 is thus potential energy. The change in energy represents the work expended when half the charge moves from C_1 to C_2 and is analogous to the loss in potential energy that results from a mass being moved from a height to a lower height.

If the two charged capacitors C_1 and C_2 are carefully disconnected and reconnected in series instead of parallel then the voltage across the combination is 100 volts and the capacity is $C/2$. The energy is unchanged at $100^2(C/2)/2 = 2500C$ joules. In rewiring the capacitors no charge moved thus no potential energy was lost or gained.

Tony Forbes

I think the answer lies in the wiring. With no resistance, equalization occurs by ∞ amps flowing for 0 seconds. So the energy lost in the wires is given by $\int_0^0 \infty^2 \cdot 0\,dt$, a finite quantity which is obviously equal to $2500C$.

To be more realistic, let's give the upper and lower wires resistances R_1 and R_2 respectively so that we can model the situation with a slightly more complicated diagram, right.



This is more or less precisely the circuit that was analysed in M500 **246** by Mike Lewis. So I can just quote his final result. At time t after the switch is closed the current is

$$i(t) \;=\; \frac{100\, e^{-t/T}}{R_1 + R_2}, \qquad \text{where} \qquad T \;=\; \frac{(R_1 + R_2)C}{2}.$$

We can now calculate the energy lost:

$$\int_0^\infty i(t)^2 (R_1 + R_2) dt \;=\; \int_0^\infty \frac{100^2 e^{-2t/T}}{R_1 + R_2} dt \;=\; 2500C,$$

which is independent of R_1 and R_2, and neatly explains how the $2500C$ joules goes missing—at least for conductors with positive resistance.

Tommy Moorhouse

The energy stored in the final configuration is half that stored in the starting configuration. We will show that the energy difference is accounted for by electromagnetic radiation. To do this we imagine that a resistor is inserted in series with the capacitors on the high voltage side. It does not matter what the value of the resistance is, and we can see this simply from the fact that the charge in the system is constant: the presence of the resistor does not influence the initial and final configurations.

We work from the expressions for the voltages, charges and currents as the system evolves after the switch is closed. Call the initial charge on the first capacitor Q, and call the time-varying charge on this capacitor $Q_1(t)$, with $Q_1(0) = Q$. Similarly the charge on the second capacitor is $Q_2(t)$ and $Q_2(0) = 0$.

The voltage across the resistor varies with time, but if the current at time t is $I(t)$ we have $V(t) = I(t)R$.

Now $I(t) = -\dot{Q}_1(t) = \dot{Q}_2(t)$. If the voltage across capacitor 1 is $V_1(t)$ and that across capacitor 2 is $V_2(t)$ we have

$$V(t) \;=\; V_1(t) - V_2(t) \;=\; \frac{1}{C}(Q_1(t) - Q_2(t))$$

so that

$$I(t) \;=\; \frac{1}{RC}(Q_1(t) - Q_2(t)).$$

We see that as well as the conservation law $Q_1(t) + Q_2(t) = Q$ we have

$$\begin{aligned} \dot{Q}_1(t) - \dot{Q}_2(t) &= -2I = -\frac{2}{RC}(Q_1(t) - Q_2(t)), \\ Q_1(t) - Q_2(t) &= Qe^{-2t/RC}. \end{aligned}$$

Thus $Q_1(t) = Q(1 + \exp(-2t/RC))/2$ and

$$I(t) = \frac{Q}{RC}e^{-2t/RC}.$$

The instantaneous power converted and radiated by the resistor is $V(t)I(t) = RI(t)^2$ and we need to integrate this over the time after the switch is closed to get the total energy radiated:

$$E = R\frac{Q^2}{(RC)^2}\int_0^\infty e^{-4t/RC}dt = \frac{Q^2}{4C}.$$

This amounts to half of the energy originally stored. If the resistance is low the discharge time is faster and the energy is radiated over a shorter time. In a naive circuit without resistance the discharge is practically instantaneous. In this case the energy would be radiated by the electrons accelerating (synchrotron radiation).

John Davidson

This is an example of an idealized problem with an inherent snag in its specification. Consider a capacitor C having a terminal voltage $v(t)$ across it. The charging current $i(t)$ supplied to it is given by $i(t) = Cdv/dt$ and to avoid 'infinities' it is usually valid to conclude that the terminal voltage $v(t)$ is *continuous*. However, in the circuit shown for Problem 256.5, an idealization arises because of the lack of circuit resistance (including the switch). It follows that, when the switch is closed, the terminal voltages of the two capacitors are equal and hence both capacitors have discontinuities in their respective terminal voltages; that is, the action of the idealized switch in the idealized circuit is to 'force' discontinuities in what would otherwise be continuous variables. In such a situation the 'short cut' reasoning is via a conservation law, which in this context is that of conservation of charge: since $Q = CV$ it follows that before the switch is closed the total charge in the system is given by

$$Q_{\mathrm{TOT}} = C_1V_1 + C_2V_2 = C \times 100 + C \times 0 \text{ coulombs} = 100C \text{ coulombs}.$$

After the switch is closed the total charge redistributes itself, and since the two capacitors are forced to have the same terminal voltage, say V_{AFTER}, it follows that

$$Q_{\text{TOT}} = C_1 V_{\text{AFTER}} + C_2 V_{\text{AFTER}} = (C_1 + C_2) V_{\text{AFTER}} = 2C V_{\text{AFTER}}.$$

For conservation of charge,

$$Q_{\text{TOT}} = 100C = 2CV_{\text{AFTER}} \;\Rightarrow\; V_{\text{AFTER}} = 50 \text{ volts}.$$

The total energy stored in the system is then

$$\frac{1}{2}(C_1 + C_2)V_{\text{AFTER}}^2 = \frac{1}{2} \times 2C \times 50^2 \text{ J} = 2500C \text{ J}.$$

But now suppose that, due to the wires and/or the contacts of the switch, the original circuit is allowed to possess a resistance R. If the circuit current is $i(t)$ when the switch is closed, in the direction so as to discharge capacitor C_1 and charge capacitor C_2, then [by a familiar calculation; see above or Mike Lewis's article in M500 **246** — TF], with $C_1 = C_2 = C$ and $T = RC/2$, the energy dissipated is

$$E = \int_0^\infty i(t)^2 R\,dt = \frac{100^2}{R} \int_0^\infty e^{-2t/T} dt = 2500C \text{ J}.$$

This quantity is evidently independent of the actual value of R. Two points are perhaps worth mentioning: firstly, any physically-realizable circuit must necessarily occupy non-zero space and would therefore also contain circuit inductance—the resulting oscillatory current would then radiate energy with the circuit behaving as an antenna. Secondly, the redistribution of energy due to the idealized switch being closed *instantaneously* violates general relativity.

An analogous mechanical problem arises when it is supposed that two incompressible masses M_1 and M_2 collide and, in so doing, somehow latch together instantaneously, thereby possessing a common velocity. The usual requirement of continuity of velocity is avoided via conservation of momentum, leading to the conclusion that, for two equal masses M colliding where the first mass has a velocity of 100 ms^{-1} and the second mass is stationary, the common velocity after latching is 50 ms^{-1}. The corresponding 'lost' kinetic energy is then readily shown to be $2500M$ J, but if it is now supposed that the two masses are connected by a dashpot (rather than by an instantaneous latching mechanism), it can be shown that the energy dissipated in the dashpot is independent of the dashpot coefficient and is equal to $2500M$ J.

Two failed attempts to dispose of Fermat's Last Theorem

Bruce Roth

Nearly twenty years ago I went to a lecture given by Dr Simon Singh on Fermat's Last Theorem as part of the launch of his, then, new book. It was very much in the news at the time due to the *Horizon* programme in the wake of Andrew Wiles's famous proof. Two things really stand out in my memory. The first was when Dr Singh stated that the sum of two cubes could never be a cube, only to be heckled by someone two seats to my left, "What about

$$(-1)^3 + 1^3 \; = \; 0^3 \text{ ?"}$$

There was silence and the professor hosting the event stood up and explained to the audience that Dr Singh had not been precise in his definitions and that Fermat's Last Theorem applies to positive integers.

The lecture resumed with Dr Singh explaining that he was a physicist and that mathematicians were much more careful when they spoke about their subject. He also spoke about how much he enjoyed mathematical proof and gave (without proof!) the example that "26 is the only number to be sandwiched between a square, 5^2, and a cube, 3^3." The same heckler shouted out, "What about zero? It is sandwiched between 1^2 and $(-1)^3$." Dr Singh looked embarrassed and asked, "Have I made the same mistake?" to be greeted with the entire audience replying "Yes" with one voice.

In Simon Singh's latest book about the maths in *The Simpsons* he gives the following interesting 'equation':

$$3987^{12} + 4365^{12} \; = \; 4472^{12}.$$

Now if you check this on your calculator,

$$\sqrt[12]{3987^{12} + 4365^{12}},$$

you may get a little shock, for your device probably can't cope with such large numbers.

It doesn't take too much to see it is not an equation—but it stands up to a little checking before falling apart. First I checked mod 2. Well, clearly the two odd numbers add to an even one; so still it is working.

Next I thought I would check the last digits. For 7 we have the cycle

$$7 \;\rightarrow\; 9 \;\rightarrow\; 3 \;\rightarrow\; 1 \;\rightarrow\; 7.$$

So 3987^{12} must end in a one. For 5 we have the cycle $5 \to 5$; so 4365^{12} ends in a five. For 2 we have the cycle $2 \to 4 \to 8 \to 6 \to 2$ and therefore 4462^{12} ends in a six. So our equation still holds up for at least the last digit of its more than forty.

It finally falls apart mod 3 (you could check its digital roots too!):

$$3987 \equiv 0 \pmod 3, \quad 1455 \equiv 0 \pmod 3, \quad \text{but} \quad 4472 \equiv 2 \pmod 3).$$

So where the calculator fails simple mathematics does not.

PS. I reminded Simon of the heckling many years ago at a BSHM meeting and he said he had "got over the embarrassment."

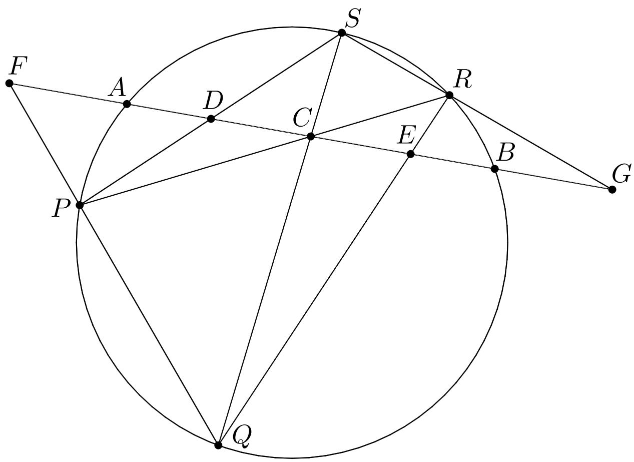
References *Fermat's Last Theorem*, Simon Singh; *The Mathematics of* The Simpsons, also by Simon Singh.

Problem 258.2 – Sandwiched

Find all solutions in integers x and y of the equation $x^3 = y^2 \pm 2$.

Problem 258.3 – Cyclic quadrilateral and chord

Look at the diagram. The diagonals of the cyclic quadrilateral $PQRS$ meet at C, the midpoint of the chord AB. Pairs of opposite sides (extended if necessary) of the quadrilateral meet the extended chord at D and E, and at F and G. Show that C also bisects DE and FG.



Problem 258.4 – Poly-Bernoulli numbers

Tony Forbes

First we need some background—in fact quite a lot of it. The poly-Bernoulli numbers in the title generalize the (more) familiar Bernoulli numbers, B_n, the coefficients of $x^n/n!$ in the Taylor expansion of $x/(e^x-1)$:

$$\frac{x}{e^x-1} \;=\; \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \qquad (1)$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$	0	$\frac{7}{6}$

In a similar manner we use the *poly-logarithm* function,

$$\mathrm{Li}_k(x) \;=\; \sum_{m=1}^{\infty} \frac{x^m}{m^k}, \qquad (2)$$

which generalizes the function $-\log(1-x) = \mathrm{Li}_1(x)$, to define the poly-Bernoulli numbers $B_n^{(k)}$ and the generating function $\mathcal{B}_k(x)$ by

$$\mathcal{B}_k(x) \;=\; \frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} \;=\; \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}. \qquad (3)$$

This is a lot to hold in the mind all at once; so a few remarks are in order.

We would like to think that the $B_n^{(1)}$ are the same as the ordinary Bernoulli numbers, and this is indeed nearly always true. If you put $k=1$ in (3), then $\mathrm{Li}_1(1-e^{-x}) = x$ and we get something like (1) but with $1-e^{-x}$ in the denominator of the right-hand side instead of e^x-1. The effect of this difference is that $B_1^{(1)} = -B_1 = 1/2$ whereas $B_n^{(1)} = B_n$ for all other n.

Putting $k=0$ gives $\mathrm{Li}_0(x) = x/(1-x)$, $\mathcal{B}_0(x) = e^x$ and $B_n^{(0)} = 1$. Also, by integrating and differentiating (2) term by term,

$$\mathrm{Li}_{k+1}(x) \;=\; \int_0^x \frac{\mathrm{Li}_k(t)}{t}\,dt \quad \text{and} \quad \mathrm{Li}_{k-1}(x) \;=\; x\,\frac{d\,\mathrm{Li}_k(x)}{dx}. \qquad (4)$$

We are particularly interested in negative values of k. So, using the second equality in (4), we can derive a recursive formula for the generating function:

$$\mathcal{B}_{k-1}(x) \;=\; \mathcal{B}_k(x) + (e^x-1)\mathcal{B}'_k(x).$$

Starting from $\mathcal{B}_0(x) = e^x$ we obtain $\mathcal{B}_{-1}(x) = e^{2x}$,

$$\begin{aligned}
\mathcal{B}_{-2}(x) &= -e^{2x} + 2e^{3x}, \\
\mathcal{B}_{-3}(x) &= e^{2x} - 6e^{3x} + 6e^{4x}, \\
\mathcal{B}_{-4}(x) &= -e^{2x} + 14e^{3x} - 36e^{4x} + 24e^{5x}, \\
\mathcal{B}_{-5}(x) &= e^{2x} - 30e^{3x} + 150e^{4x} - 240e^{5x} + 120e^{6x}, \\
\mathcal{B}_{-6}(x) &= -e^{2x} + 62e^{3x} - 540e^{4x} + 1560e^{5x} - 1800e^{6x} + 720e^{7x},
\end{aligned}$$

and in general $\mathcal{B}_{-r}(x)$ is a polynomial of degree $r+1$ in e^x with leading coefficient $r!$. Hence

$$B_n^{(-r)} = \sum_{j=1}^{r+1} b_{r,j}\, j^n,$$

where $b_{r,j}$ is the coefficient of e^{jx} in $\mathcal{B}_{-r}(x)$. Thus we can compute $B_n^{(k)}$ for small $k \le 0$ and $n \ge 0$.

	n							
k	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
−1	1	2	4	8	16	32	64	128
−2	1	4	14	46	146	454	1394	4246
−3	1	8	46	230	1066	4718	20266	85310
−4	1	16	146	1066	6902	41506	237686	1315666
−5	1	32	454	4718	41506	329462	2441314	17234438
−6	1	64	1394	20266	237686	2441314	22934774	202229266
−7	1	128	4246	85310	1315666	17234438	202229266	2193664790
−8	1	256	12866	354106	7107302	117437746	1701740006	22447207906

Upon looking at the table one cannot help noticing a striking similarity between the rows and the columns. On the basis of this observation we are finally ready to state the problem.

Prove that $B_r^{(-s)} = B_s^{(-r)}$ for non-negative r and s.

Problem 258.5 – Integral

Suppose $a, b > 0$. Show that

$$\int_0^\infty \frac{\cos ax - \cos bx}{x}\, dx = \log \frac{b}{a}.$$

Solution 255.1 – Elementary trigonometry

Show that $\cot\left(\frac{\pi}{6} - \frac{1}{2}\arccos\frac{11}{14}\right) = 3\sqrt{3}$.

Vincent Lynch

Writing θ for $\arccos(11/14)$ and using $\cos\theta = 2\cos^2(\theta/2) - 1$, this gives $\cos(\theta/2) = 5/(2\sqrt{7})$, so that $\tan(\theta/2) = \sqrt{3}/5$. Now we can use

$$\cot\left(\frac{\pi}{6} - \frac{\theta}{2}\right) \;=\; \frac{1 + \tan(\pi/6)\tan(\theta/2)}{\tan(\pi/6) - \tan(\theta/2)} \;=\; \frac{1 + 1/\sqrt{3}\cdot\sqrt{3}/5}{1/\sqrt{3} - \sqrt{3}/5}$$

and this simplifies to give $3\sqrt{3}$.

More generally, whenever $\theta = \arccos\frac{3k^2 + 6k - 1}{6k^2 + 2}$, k rational,

(a) $\tan(\theta/2)$ will be a rational multiple of $\sqrt{3}$,

(b) $\cos\theta$ will be rational since $\cos\theta = \frac{1 - t^2}{1 + t^2}$, where $t = \tan(\theta/2)$,

and $\cot\left(\frac{\pi}{6} - \frac{\theta}{2}\right) = k\sqrt{3}$. In the problem as stated, $k = 3$.

Problem 258.6 – Kolmogorov Distance

Mike Lewis

Kolmogorov Distance is a non-parametric test of the similarity of two statistical distributions. Kolmogorov Distance is the maximum separation in the y axis between the CDFs under test. Prove that the x coordinate of this maximum separation is where the PDFs of the two distributions cross.

Problem 258.7 – Counting primes

As usual, $\pi(x)$ denotes the number of primes $\le x$. Show that

$$\pi(x) \;=\; \left\lfloor \sum_{n=1}^{x-1} \sin^2\left(\frac{n! + 1}{n + 1}\cdot\frac{\pi}{2}\right)\right\rfloor .$$

Thanks to Robin Whitty for suggestions relating to this problem. (The π on the right is the usual approximation to 22/7.)

Letter

Education for all

Dear Members,

I believe that there should be scholarships which would allow the best students to earn university tuition fees plus money towards living expenses. The snag is that our current 'one size fits all' wouldn't allow teachers to give the necessary tuition. To get round this I suggest that courses be made available on the internet. These courses would start at just below 'O' level and would take pupils to about the old 'scholarship' level. There would be on-line discussion and mutual help groups. The courses should use top grade teachers, making a high standard of tuition available to everybody. The cost of this would not be significant in terms of the national education budget while the benefits, in terms of well qualified students starting their degree courses would be considerable. The courses themselves would be reusable year after year while no extra load would be placed on existing schools and teachers. The courses should eventually cover at least STEM subjects. The OU could well take a lead in providing these courses. How the exams would be administered and who would be allowed to sit them would be for discussion.

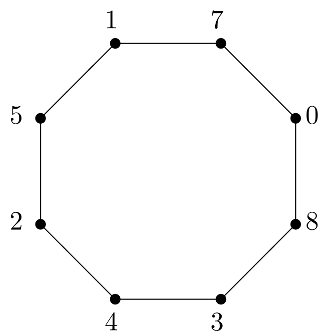
I would be interested to read other members' views on this idea.

R. M. Boardman

Problem 258.8 – Cycle graphs

Tony Forbes

The vertices of an n-cycle graph are labelled with distinct integers from the set $\{0, 1, \ldots, n\}$ in such a manner that each positive integer from 1 to n occurs exactly once as the difference between the labels of two adjacent vertices. For which n is this possible?



In the example on the right $n = 8$. The labels are 0, 1, 2, 3, 4, 5, not 6, 7 and 8. All integers from 1 to 8 occur as differences.

Clearly both 0 and n must be present as adjacent labels (otherwise there is no way to create difference n). Must 1 and $n - 1$ both occur as labels?

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Front cover: A 16-bipyramid.

M500 Society Committee – call for applications

The M500 committee invites applications from members to join the Committee. Please apply to the Secretary by 1st October 2014.