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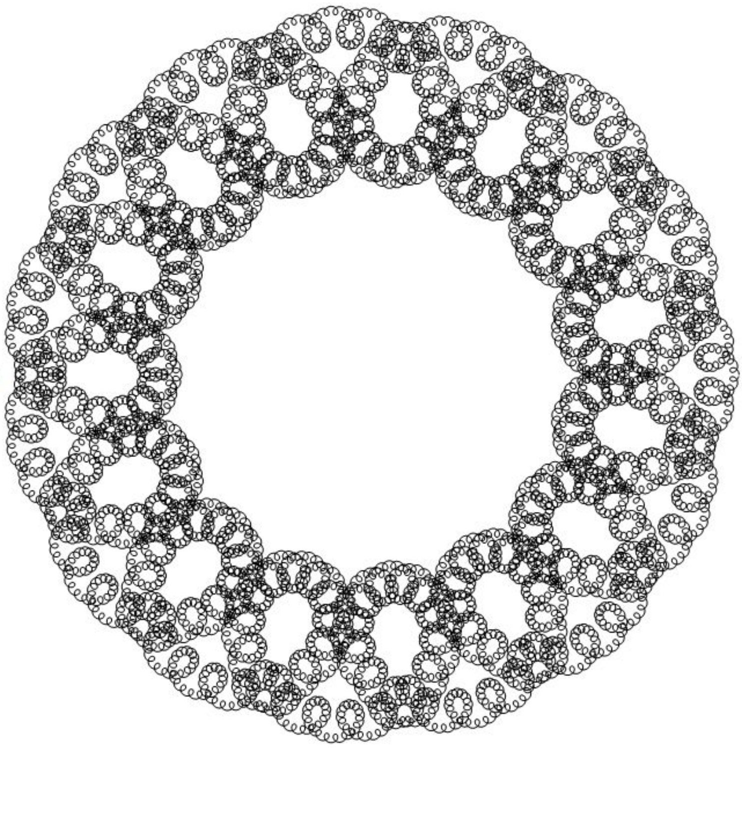

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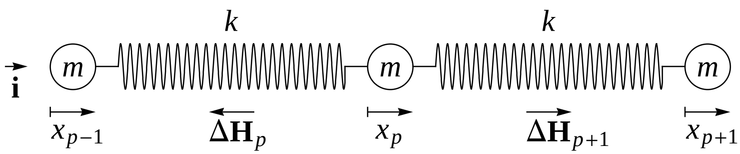
The n-coupled harmonic oscillator

Bryan Orman

Consider the system of n identical particles each with mass m connected linearly by $n+1$ model springs of stiffness k and natural length l_0. Both ends of the system are fixed. There are two scenarios to consider. The first will concern the longitudinal oscillations of the system on a frictionless and horizontal plane and the second will concern the transverse oscillations of the system without the horizontal plane and with the effects of gravity being ignored.

Longitudinal Oscillations

Since the springs are identical the spacing between adjacent particles will be the same when the system is in equilibrium. Being interested in oscillations about the equilibrium positions, the displacements of the particles from equilibrium are the natural coordinates to employ. Let the displacement of the pth particle be x_p ($p = 1$ to n) and consider the motion of this representative particle. The following diagram is appropriate.



By Hooke's Law for a model spring the change in force $\Delta\mathbf{H}$ exerted by the spring is given by $\Delta\mathbf{H} = -k\Delta l\,\mathbf{s}$, where Δl is the change in the spring length from its equilibrium value and $\mathbf{s}$ is a unit vector directed from the spring towards the particle. So for the two forces acting on the pth particle

$$\Delta\mathbf{H}_p = -k(x_p - x_{p-1})\,\mathbf{i}, \quad \Delta\mathbf{H}_{p+1} = -k(x_{p+1} - x_p)(-\mathbf{i}).$$

By Newton's Second Law it follows that, for the pth particle, the equation of motion is given as

$$m\,\ddot{x}_p\,\mathbf{i} = \Delta\mathbf{H}_p + \Delta\mathbf{H}_{p+1}$$

and so

$$m\,\ddot{x}_p = kx_{p+1} - 2kx_p + kx_{p-1},$$

and this is true for $p = 1$ to n provided we note that $x_0 = 0$ and $x_{p+1} = 0$, since the end points are fixed, equivalent to having additional particles at the ends but with zero displacements.

For normal mode oscillations we set $x_p(t) = v_p \cos(\omega t)$, that is, all the particles execute simple harmonic motion with the same angular frequency ω. The equation of motion above now has the form

$$-\frac{\omega^2}{\omega_0^2} v_p \;=\; v_{p+1} - 2v_p + v_{p-1} \qquad \text{for } p = 1 \text{ to } n,$$

where $\omega_0^2 = k/m$, $v_0 = 0$ and $v_{n+1} = 0$. This system can be written in matrix form $A\mathbf{v} = \lambda \mathbf{v}$, where $\lambda = -\omega^2/\omega_0^2$ and

$$A \;=\; \begin{bmatrix} -2 & 1 & 0 & & 0 & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & -2 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & & 0 & 1 & -2 \end{bmatrix}$$

is an $n \times n$ tridiagonal matrix, and the problem is to find the eigenvalues λ_s and the corresponding eigenvectors $\mathbf{v}(s)$, where $s = 1$ to n. It is not possible to apply the standard techniques to find exact solutions for large values of n by this method although an example for small n is instructive.

Four particle system

The eigenvalues are found from the equation $\det(A - \lambda I) = 0$ and, on evaluation, the following quadratic equation $t^2 - 3t + 1 = 0$ for $t = (2 + \lambda)^2$ is obtained. The two solutions for t are

$$t_{\pm} \;=\; \frac{3 \pm \sqrt{5}}{2} \;=\; \left(\frac{\sqrt{5} \pm 1}{2} \right)^2$$

and, from $\lambda = -2 \pm \sqrt{t}$ and $\omega = \omega_0 \sqrt{-\lambda}$, the four normal mode angular frequencies are

$$\omega_1 \;=\; \frac{\omega_0(\sqrt{5} - 1)}{2}, \qquad \omega_2 \;=\; \omega_0 \sqrt{\frac{5 - \sqrt{5}}{2}},$$

$$\omega_3 \;=\; \frac{\omega_0(\sqrt{5} + 1)}{2}, \qquad \omega_4 \;=\; \omega_0 \sqrt{\frac{5 + \sqrt{5}}{2}}.$$

We note that the nested surds for ω_2 and ω_4 cannot be reduced and such irreducible surds are even more complicated for large n, for example, one of the angular frequencies for $n = 9$ is $\omega_0\sqrt{2 + \sqrt{(5 + \sqrt{5})/2}}$. The corresponding eigenvectors, suitably scaled, are are

$$\begin{aligned}
\mathbf{v}(1) &= [1 \quad \alpha \quad \alpha \quad 1], & \mathbf{v}(2) &= [1 \quad \beta \quad -\beta \quad -1],\\
\mathbf{v}(3) &= [1 \quad -\beta \quad -\beta \quad 1], & \mathbf{v}(4) &= [1 \quad -\alpha \quad \alpha \quad -1],
\end{aligned}$$

where $\alpha = (\sqrt{5} + 1)/2$ and $\beta = (\sqrt{5} - 1)/2$. As expected, the in-phase oscillation corresponds to the lowest angular frequency. Note that the above vectors are column vectors; so their transpose should be indicated. Not doing should not lead to confusion!

The general solution of the eigenvalue problem can be achieved using a different approach. The original equation for v_p is just a second order, constant coefficient and linear recurrence relation. Assuming a solution of the form $v_p = \alpha^p$, then $\alpha^2 - (2 + \lambda)\alpha + 1 = 0$, giving the two solutions $\alpha_\pm = \left(2 + \lambda \pm \sqrt{\lambda(\lambda + 4)}\right)/2$. Thus the general solution is

$$v_p \;=\; a\alpha_+^p + b\alpha_-^p \qquad \text{for } p = 0 \text{ to } n + 1.$$

Now $v_0 = 0$ gives $a + b = 0$ and so $v_p = a\left(\alpha_+^p - \alpha_-^p\right)$ and $v_{n+1} = 0$ gives $a\left(\alpha_+^{n+1} - \alpha_-^{n+1}\right) = 0$; that is $(\alpha_+/\alpha_-)^{n+1} = 1$. Furthermore, from the product of the roots, $\alpha_+\alpha_- = 1$; so $\alpha_+^{2(n+1)} = 1 = e^{2\pi si}$, $s = 1$ to n and thus $\alpha_\pm = e^{\pm i\pi s/(n+1)}$. And, from the sum of the roots, $\lambda + 2 = \alpha_+ + \alpha_-$, we obtain the eigenvalues

$$\lambda_s \;=\; -4\sin^2\frac{\pi s}{2(n+1)} \;=\; -\frac{\omega^2}{\omega_0^2},$$

so that

$$\omega_s \;=\; 2\omega_0\sin\frac{\pi s}{2(n+1)}, \qquad s = 1 \text{ to } n.$$

The corresponding eigenvectors are

$$\mathbf{v}(s) \;=\; \left[\sin\frac{\pi ps}{n+1}, \qquad p = 1 \text{ to } n\right].$$

Checking this for the four particle system, for ω_1 we have

$$\mathbf{v}(1) \;=\; \left[\sin\frac{\pi}{5} \quad \sin\frac{2\pi}{5} \quad \sin\frac{3\pi}{5} \quad \sin\frac{4\pi}{5}\right]$$

and scaling down by $\sin\pi/5$, and noting that

$$\sin\frac{3\pi}{5} \;=\; \sin\frac{2\pi}{5} \;=\; 2\left(\sin\frac{\pi}{5}\right)\left(\cos\frac{\pi}{5}\right) \quad\text{and}\quad \sin\frac{4\pi}{5} \;=\; \sin\frac{\pi}{5},$$

we have

$$\mathbf{v}(1) \;=\; [1 \;\; \gamma \;\; \gamma \;\; 1], \qquad \text{with } \gamma = 2\cos\frac{\pi}{5}.$$

The other three eigenvectors are easily calculated as

$$\mathbf{v}(2) \;=\; [1 \;\; \delta \;\; -\delta \;\; -1], \quad \mathbf{v}(3) \;=\; [1 \;\; -\delta \;\; -\delta \;\; 1], \quad \mathbf{v}(4) \;=\; [1 \;\; -\gamma \;\; \gamma \;\; -1],$$

where $\delta = 2\cos 2\pi/5$.

We now have two exact solutions for the longitudinal oscillations of the system, the first cannot be extended to large values of n whereas it can be done for the second. To complete this analysis we need to show that $\alpha = \gamma$ and $\beta = \delta$, equivalently that $\cos\pi/5 = (\sqrt{5}+1)/4$. The following identity is the starting point:

$$\cos n\theta \;=\; \det\begin{bmatrix} \cos\theta & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2\cos\theta & 1 & \dots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \dots & 1 & 2\cos\theta & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 2\cos\theta \end{bmatrix}.$$

To show this let the $n\times n$ trigdiagonal determinant be D_n. Then by expanding about the last row the following second order recurrence relation is obtained,

$$D_n \;=\; 2(\cos\theta)D_{n-1} - D_{n-2}.$$

Solve this for D_n by assuming $D_n = \lambda^n$; then the resultant auxiliary equation $\lambda^2 - 2(\cos\theta)\lambda + 1$ has the two roots $\lambda = e^{\pm i\theta}$ and so $D_n = \alpha e^{ia\theta} + \beta e^{i\theta}$. And since $D_1 = 2\cos^2\theta$ we have $D_n = \cos n\theta$ as required.

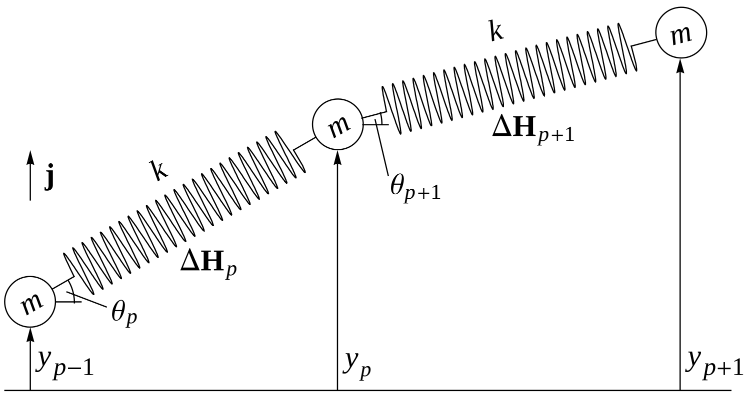
Now if $\cos n\theta = 0$, then $\theta_p = (2p-1)\pi/(2n)$, $p = 1$ to n and $\cos\theta_p$ can be found from $D_n(\cos\theta) = 0$, the polynomial of degree in $\cos\theta$. Since we need $c = \cos\pi/5$ we set $n = 5$ and evaluate D_5 by successive substitution in the recurrence relation. Eventually $D_5 = c\,(16c^4 - 20c^2 + 5)$, and $D_5 = 0$ yields the five roots $c = 0$ and $c^2 = 1/(8(5\pm\sqrt{5}))$.

Evidently the largest value of $\cos\theta$ corresponds to the smallest value of θ, so that $\cos\pi/10 = (5+\sqrt{5})/\sqrt{8}$, and using the double angle formula twice we arrive at $\cos\pi/5 = (1+\sqrt{5})/4$ and $\cos 2\pi/5 = (\sqrt{5}-1)/4$. These are just the results we require; $\alpha = \gamma$ and $\beta = \delta$.

The alternative (trigonometric) analysis producing exact values for the normal mode angular frequencies and the corresponding eigenvectors for the longitudinal oscillations of the n-coupled harmonic oscillator holds for all values of n. Finally we consider the transverse oscillations of the n-coupled harmonic oscillator.

Transverse oscillations

The modeling of the transverse oscillations proceeds in much the same way as that for the longitudinal oscillations, but now the particles are displaced perpendicular to the equilibrium position. Let the displacement of the pth particle be y_p ($p = 1$ to n) and consider the motion of this representative particle. The following diagram is appropriate.



Since the motion of the particle is entirely transverse, the horizontal components of the forces acting on the particle will cancel so only the components in the $\mathbf{j}$ direction are relevant here. By Hooke's Law we have

$$\Delta\mathbf{H}_p \;=\; -\,k(l_p - l_0)(\sin\theta_p)\,\mathbf{j}, \qquad \Delta\mathbf{H}_{p+1} \;=\; -\,k(l_{p+1} - l_0)(\sin\theta_{p+1})\,(-\mathbf{j}),$$

where l_p and l_{p+1} are the lengths of the two springs and l_0 are their natural lengths. It is best to use the equilibrium tension instead of the stiffness and applying Hooke's Law in equilibrium, $T_{\mathrm{eq}} = k(l_{\mathrm{eq}} - l_0)$. Since the oscillations are small we can write $l_p = l_{\mathrm{eq}}$ and $l_{p+1} = l_{\mathrm{eq}}$ and

$$\sin\theta_p \;=\; \frac{y_p - y_{p-1}}{l_{\mathrm{eq}}}, \qquad \sin\theta_{p+1} \;=\; \frac{y_{p+1} - y_p}{l_{\mathrm{eq}}}.$$

Applying Newton's Second Law to the pth particle:

$$m\,\ddot{y}_p\;\mathbf{j} \;=\; \Delta\mathbf{H}_p + \Delta\mathbf{H}_{p+1}$$

and, using the above approximations, the equation of motion becomes

$$m\,\ddot{y}_p \;=\; \frac{T_{\mathrm{eq}}}{l_{\mathrm{eq}}}\,\big(y_{p+1} - 2y_p + y_{p-1}\big).$$

We note that this is the same equation as previously encountered for the longitudinal oscillations, where now we have $T_{\mathrm{eq}}/l_{\mathrm{eq}}$ instead of k. If we write $y_p = u_p\cos(\Omega t)$ for the normal mode oscillations, then

$$-\frac{\Omega^2}{\Omega_0^2}\,u_p \;=\; u_{p+1} - 2u_p + u_{p-1} \qquad \text{with } \Omega^2 = \frac{T_{\mathrm{eq}}}{ml_{\mathrm{eq}}}.$$

The normal mode angular frequencies are then

$$\Omega_s \;=\; 2\Omega_0 \sin\left(\frac{\pi s}{2(n+1)}\right), \qquad s = 1 \text{ to } n,$$

and the corresponding eigenvectors $u(s)$ are the same as the $v(s)$ found previously.

The transverse oscillations of the n-coupled harmonic oscillator will approximate the transverse vibration of a taut string for large values of n. Indeed, if we introduce a Cartesian coordinate system with origin at the fixed left hand side then the pth particle will have coordinates $x = pl_{\mathrm{eq}}$ and $y = \sin(\pi ps/(n+1))$. If the string has length L, then $L = (n+1)l_{\mathrm{eq}}$, so that $x = pL/(n+1)$ and eliminating p gives $y = \sin(s\pi x/L)$ for the angular frequency Ω_s. For the taut string we need to let n tend to infinity and to avoid an infinite mass string we put $m = M/n$, where M is the mass of the taut string.

The *fundamental* angular frequency corresponds to $s = 1$ and this frequency is

$$\Omega_1(n) \;=\; 2\sqrt{\frac{n(n+1)T_{\mathrm{eq}}}{ML}}\,\sin\left(\frac{\pi}{2(n+1)}\right)$$

and the final step is to take the limit as n tends to infinity. To this end it is convenient to let $n = 1/\epsilon$ and let ϵ tend to zero, so that

$$\Omega_1(\epsilon) \;=\; 2\sqrt{\frac{T_{\mathrm{eq}}}{ML}}\,\frac{\sqrt{1+\epsilon}}{\epsilon}\,\sin\left(\frac{\pi}{2}\,\frac{\epsilon}{1+\epsilon}\right).$$

In the limit this becomes $\Omega_1 = \pi\sqrt{T_{\mathrm{eq}}/(ML)}$, which is the fundamental angular frequency of a taut string and the profile is the well known sinusoidal $y = \sin(\pi x/L)$ with $0 < x < L$.

The fundamental frequency is $f = \Omega_1/(2\pi)$ and so $f = 1/2\sqrt{T_{\mathrm{eq}}/(ML)}$. We can check that this gives the correct frequency by employing the data appearing in *MST209*, Block 5, page 79 for the E string of a guitar, whose fundamental frequency is 323 Hz. In *MST209* a *computer program* was written to compute the fundamental frequency of the corresponding n-particle system, for any positive integer n, and then the results were plotted on a graph, frequency against n. It was seen that the predicted values *approached* the experimental value of 323Hz.

Data: $k = 4000\,\mathrm{Nm}^{-1}$, $M = 0.25\,\mathrm{g}$, $l_0 = 0.633\,\mathrm{m}$ and $L = 0.650\,\mathrm{m}$ so that $T_{\mathrm{eq}} = k(L - l_0) = 68\,\mathrm{N}$. When these are inserted in the *exact* expression for the fundamental frequency the value 323 Hz is obtained.

Problem 264.1 – Lift-off

Tony Forbes

As is well known, the energy required to lift a fixed mass M well away from the surface of a planet of mean density ρ and radius R is approximately

$$\frac{4\pi G\rho R^3 M}{3}\int_R^\infty \frac{dr}{r^2} \;=\; \frac{4\pi G\rho M}{3}R^2.$$

Thus, on the assumption that ρ is the same for any Earth-like planet from which one might want to escape, we can say that the energy is proportional to R^2.

Assuming the traditional method of providing the energy, find a simple function of R that gives approximately the amount of rocket fuel required.

Bear in mind that the fuel, its container and the engine are not massless, and I expect the main function of the rocket will be to lift itself as well as quite a lot of unburnt fuel off the planet. Let us assume for definiteness that M_0 is the mass of the empty rocket and M_1 is the mass of fuel required. It might be reasonable to assume that M_0 and M_1 are proportional to M, at least approximately; so we could fix M at some specific value, say 1000 kg. However, the situation will be further complicated by staging—necessary to avoid having to lift the entire mass M_0 all the way up. At the end of the exercise there should be no fuel remaining. Also you might want to choose a fuel with a specific operational efficiency for converting itself into lifting energy.

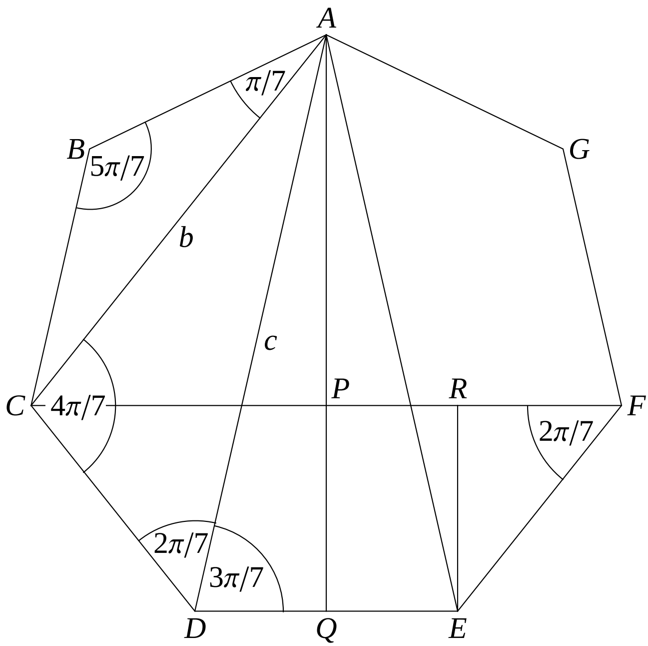
Anyway, we look forward to your analysis of this interesting problem.

Solution 221.3 – Six tans

Show that

$$\mathcal{S} = \frac{1}{\tan^2 \frac{1}{7}\pi + \tan^2 \frac{3}{7}\pi} + \frac{1}{\tan^2 \frac{3}{7}\pi + \tan^2 \frac{5}{7}\pi} + \frac{1}{\tan^2 \frac{5}{7}\pi + \tan^2 \frac{1}{7}\pi} = \frac{17}{26}.$$

Steve Moon



First we need to derive some of the properties of a regular heptagon to generate formulae for trigonometric functions of $\pi/7$, $3\pi/7$ and $5\pi/7$ in terms of the diagonal b as shown. The heptagon is $ABCDEFG$ with sides of unit length. Its internal angles are each $5\pi/7$. Since the vertices lie on a circle, the angle subtended at a vertex v by a side not adjacent to v is $\pi/7$. Let the length of the short and long diagonals be b and c respectively.

Triangles EFR and APC are similar; so

$$\frac{c}{2} - \frac{1}{2} = \frac{|RF|}{|FE|} = \frac{|CP|}{|AC|} = \frac{c}{2b} \quad \text{and hence} \quad c = \frac{b}{b-1}, \quad \text{or} \quad \frac{1}{b} + \frac{1}{c} = 1.$$

From $\triangle ABC$, we have $\cos\pi/7 = b/2$ and by the sine rule on $\triangle ACD$, $b/(\sin 2\pi/7) = c/(\sin 4\pi/7)$; hence

$$c = b\,\frac{\sin 4\pi/7}{\sin 2\pi/7} = 2b\cos\frac{2\pi}{7} = 2b\left(2\cos^2\frac{\pi}{7} - 1\right) = b^3 - 2b.$$

Therefore, recalling that $c = b/(b-1)$, we generate a cubic,

$$b^3 - b^2 - 2b + 1 = 0. \tag{1}$$

From $\cos\pi/7 = b/2$ we have

$$\tan^2\frac{\pi}{7} = \frac{1 - b^2/4}{b^2/4} = \frac{4 - b^2}{b^2}, \tag{2}$$

and using the sine rule on $\triangle ADE$,

$$\sin\frac{3\pi}{7} = c\,\sin\frac{\pi}{7} = \frac{b}{b-1}\sin\frac{\pi}{7};$$

hence

$$\tan^2\frac{3\pi}{7} = \frac{\dfrac{b^2}{(b-1)^2}\sin^2\dfrac{\pi}{7}}{1 - \dfrac{b^2}{(b-1)^2}\sin^2\dfrac{\pi}{7}} = \frac{b^2(4-b^2)}{b^4 - 8b + 4}. \tag{3}$$

Also

$$\tan^2\frac{5\pi}{7} = \tan^2\frac{2\pi}{7} = \left(\frac{2\tan\pi/7}{1 - \tan^2\pi/7}\right)^2 = \frac{b^2(4-b^2)}{b^4 - 4b^2 + 4}. \tag{4}$$

The algebra involved in solving the cubic (1) for b, substituting into expressions (2), (3), (4) and then evaluating $\mathcal{S}$ is extremely onerous, and not recommended. Instead we aim for a method not requiring us to solve (1). From (1), we derive

$$b^3 = b^2 + 2b - 1 \quad\text{and}\quad b^4 = b^3 + 2b^2 - b = 3b^2 + b - 1, \tag{5}$$

and expect that repeated use of these will ultimately result in an expression for $\mathcal{S}$ that is at worst a quadratic in b. First we use (5) to derive new expressions for (3) and (4):

$$\tan^2\frac{3\pi}{7} = \frac{b^2 - b + 1}{3b^2 - 7b + 3}, \qquad \tan^2\frac{5\pi}{7} = \frac{b^2 - b + 1}{-b^2 + b + 3}. \tag{6}$$

Now we use (5) again on the terms of S using (2) and (6):

$$\frac{1}{\tan^2\frac{1}{7}\pi+\tan^2\frac{3}{7}\pi}=\frac{-3b^4+7b^3-3b^2}{2b^4-6b^3-10b^2+28b-12}=\frac{5b^2-11b+4}{10b^2-18b+8},\tag{7}$$

$$\frac{1}{\tan^2\frac{1}{7}\pi+\tan^2\frac{5}{7}\pi}=\frac{-b^4+b^3+3b^2}{2b^4-2b^3-6b^2+4b+12}=\frac{b^2+b}{-2b^2+2b+12},\tag{8}$$

$$\frac{1}{\tan^2\frac{3}{7}\pi+\tan^2\frac{5}{7}\pi}=\frac{-3b^4+10b^3-b^2-18b+9}{2b^4-8b^3+14b^2-12b+6}=\frac{-b+2}{12b^2-26b+12}.\tag{9}$$

In the hope of some cancellation we add (7), (8), (9) to provide an expression for $\mathcal{S}$, again using the relations (5). Thus

$$\mathcal{S} \;=\; \frac{595b^2-1309b+476}{910b^2-2002b+728} \;=\; \frac{17(35b^2-77b+28)}{26(35b^2-77b+28)} \;=\; \frac{17}{26},\tag{10}$$

as required. Expressions (6) to (10) were done using Maple. Clearly you could do it in one step but here I am trying to find an 'analytic' method, and it's all too easy to make errors on the way by hand.

Solution 260.2 – Right-angled triangle

Show that for $|a|<1$,

$$\arctan a \;\approx\; \frac{3a}{1+2\sqrt{1+a^2}} \;\approx\; \frac{172^\circ a}{1+2\sqrt{1+a^2}}$$

and that the last expression is a better approximation to $\tan^{-1} a$.

Steve Moon

First note the infinite Taylor series expansion about 0:

$$\arctan a = a-\frac{a^3}{3}+\frac{a^5}{5}-\frac{a^7}{7}+\dots,\qquad |a|<1.\tag{1}$$

Now consider the approximation

$$\begin{aligned}\frac{3a}{1+2\sqrt{1+a^2}} &= \frac{3a}{1+2(1+a^2/2-a^4/4+a^6/16+O(a^8))}\\ &= a\left(1+\frac{a^2}{3}\left(1-\frac{a^2}{4}+\frac{a^4}{8}+O(a^6)\right)\right)^{-1}\\ &= a\left(1-\frac{a^2}{3}\left(1-\frac{a^2}{4}+\frac{a^4}{8}\right)+\frac{a^4}{9}\left(1-\frac{a^2}{4}\right)^2-\frac{a^6}{27}+O(a^8)\right)\\ &= a-\frac{a^3}{3}+\frac{7a^5}{36}-\frac{29a^7}{216}+O(a^9).\end{aligned}\tag{2}$$

Thus (2) agrees with (1) up to the term in a^3. The approximation term in a^5 is $a^5/180$ less than the corresponding term in (1). If we go to a^7 the approximation is improved since $-1/7 < -29/216$.

If we convert to degrees, we must multiply by $180/\pi$. Now $3 \cdot 180/\pi \approx 171.89$. Because (2) the Taylor series by $a^5/180$ up to a^5, we might improve the approximation by rounding up. Hence

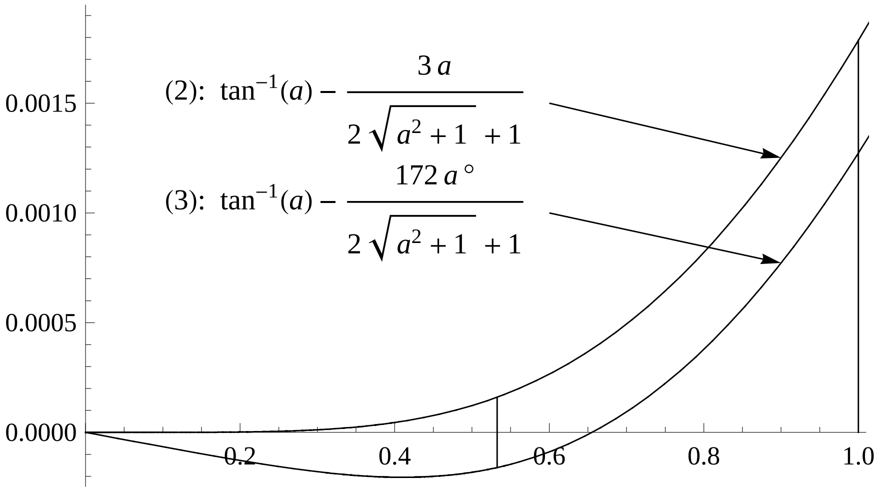
$$\arctan a \;\approx\; \frac{172^\circ a}{1 + 2\sqrt{1 + a^2}}. \qquad (3)$$

Rather than recalculate the series expansion, we can demonstrate the improvement by considering $a = 1$. At this extremal value we conjecture that the errors in the approximations are maximal. Thus we have

$$\arctan a \;=\; 45^\circ, \qquad \frac{3}{1 + 2\sqrt{2}} \;\approx\; 44.8976^\circ, \qquad \frac{172}{1 + 2\sqrt{2}} \;\approx\; 44.9271.$$

So the error in using (2) is about 0.228 per cent compared with 0.162 per cent using (3).

To test the conjecture on error size we offer the following graphs.



Although not a rigorous treatment of the errors, we see that on average (3) is better than (2) (i.e. the area under the graph of the absolute value of (3) is less than the area under (2)) and that the magnitude is at its greatest when $a = 1$, the limit of the validity of the series expansions used. However, formula (2) is more accurate when a is in the interval $(0, 0.53247)$.

Problem 262.2 – Tutte's Golden Identity

Tony Forbes

Let $\phi = (\sqrt{5}+1)/2$ denote the golden ratio, and for positive integer n and real x, define the polynomial $f_n(x)$ by

$$f_n(x) \;=\; x(x-1)(x-2)\ldots(x-(n-1)).$$

Let

$$\Delta_n \;=\; f_n(\phi+2) - (\phi+2)\phi^{3n-10} f_n(\phi+1)^2.$$

Show that

$$\Delta_n \;\equiv\; 0 \quad \big(\mathrm{mod}\ 5^{\lfloor (n+4)/10 \rfloor}\big) \tag{1}$$

by which we mean that the left-hand side is a multiple of $5^{\lfloor (n+4)/10 \rfloor}(a+b\sqrt{5})$ with integers a and b.

Thanks to Robin Whitty for the idea behind this problem.

The congruence (1) involving the rather complicated expression Δ_n resembles *Tutte's Golden Identity*,

$$P_G(\phi+2) \;=\; (\phi+2)\phi^{3v-10} P_G(\phi+1)^2, \tag{2}$$

which holds when G is a graph of v vertices that has a planar triangulation and $P_G(x)$ is its chromatic polynomial.

A graph has a planar triangulation if the thing can be drawn on a plane or a sphere in such a manner that no edges cross and all of its faces (the vertex-free regions bounded by edges) are triangles.

The chromatic polynomial as a function of x is the number of ways the vertices of G can be coloured with x colours such that no two adjacent vertices have the same colour. For example, with a little experimentation one can discover that there are $x(x-1)(x-2)$ ways of colouring the three vertices of the triangle graph when x colours are available.

More generally, observe that $f_n(x)$ is the chromatic polynomial of the complete graph K_n. Therefore, since both K_3 and K_4 have planar triangulations, we have (2) and hence also (1). In fact (1) is trivial when $n \le 5$ because the exponent of 5 on the right is zero. On the other hand, K_6, for example, does not have a planar triangulation. So when we compute

$$\Delta_6 \;=\; f_6(\phi+2) - (\phi+2)\phi^8 f_6(\phi+1)^2 \;=\; -5(184+82\sqrt{5})$$

we find that (1) holds but (2) is false. Also

$$\frac{\Delta_7}{5} = -8921 - 3993\sqrt{5}, \qquad \frac{\Delta_8}{5} = -725888 - 324582\sqrt{5},$$
$$\frac{\Delta_9}{5} = -89060071 - 39829573\sqrt{5},$$
$$\frac{\Delta_{10}}{5} = -15365937476 - 6871843140\sqrt{5},$$
$$\frac{\Delta_{11}}{5} = -3547036167180 - 1586283089468\sqrt{5},$$
$$\frac{\Delta_{12}}{5} = -1055651020495940 - 472101480617932\sqrt{5},$$
$$\frac{\Delta_{13}}{5} = -393614418974726400 - 176029719801518066\sqrt{5},$$
$$\frac{\Delta_{14}}{5} = -179718640965541380340 - 80372619595110800782\sqrt{5},$$
$$\frac{\Delta_{15}}{5} = -98625817325560092452445 - 44106806375699891289037\sqrt{5};$$

but when we get to $n = 16$ the denominator on the left can have another 5:

$$\frac{\Delta_{16}}{25} = -128104016555980826632413 56 - 57289857841944538880786 42\sqrt{5}.$$

For more information about Bill Tutte and his amazing chromatic polynomial identity, see http://www.theoremoftheday.org/, Number 223 in 'The Whole Jolly Lot'.

Problem 264.3 – Determinant

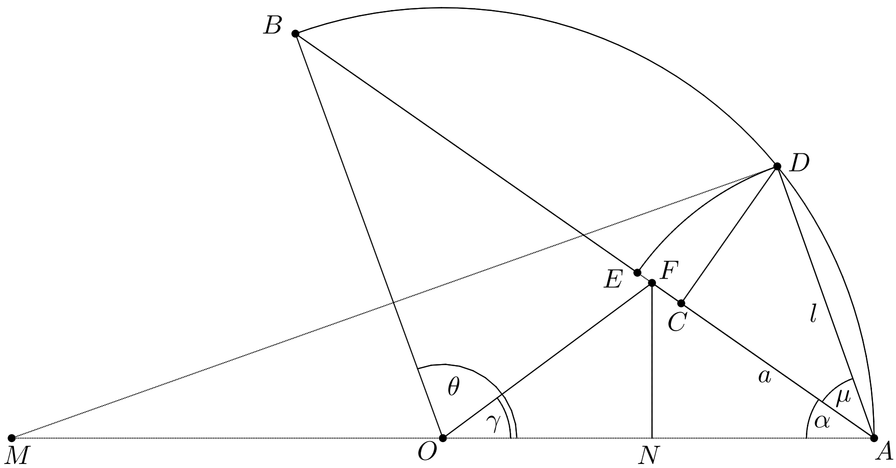
An $n \times n$ matrix has a in each entry on the diagonal and b everywhere else. What is its determinant? For example, when $a = 6$ and $b = 5$ we get

$$\det \begin{bmatrix} 6 & 5 & 5 & 5 & 5 & 5 \\ 5 & 6 & 5 & 5 & 5 & 5 \\ 5 & 5 & 6 & 5 & 5 & 5 \\ 5 & 5 & 5 & 6 & 5 & 5 \\ 5 & 5 & 5 & 5 & 6 & 5 \\ 5 & 5 & 5 & 5 & 5 & 6 \end{bmatrix} = 31.$$

These objects occur in design theory. If a projective plane of order n has incidence matrix A, then $A^{\mathrm{T}}A$ is of this form with $a = n + 1$ and $b = 1$.

Solution 261.5 – Angle trisection

The diagram represents a Euclidean construction due to Albrecht Dürer for trisecting an angle—at least approximately—if the angle is not too large. Then $AOB = \theta$, $|AC| = |AB|/3$, angle $ACD = 90°$, $|AE| = |AD|$ and $|EF| = |EC|/3$. Find an exact expression for $\angle AOF$ as a function of θ.



Tommy Moorhouse

Construct point M as the intersection of the circle through A centred on O with the extended line AO. Angle $\angle MDA$ is a right angle. Call the angle $\angle CAD$ μ and denote by l the length $|AD|$. Call $\angle OAC$ α and write $|AC|$ as a. Denote by γ the angle we wish to find, namely $\angle AOF$. Choose the length $|AO|$ to be 1, so that $|AB| = 2\cos\alpha$. By construction $a = 2(\cos\alpha)/3$. Then $l = 2\cos(\alpha+\mu)$ and $\cos\mu = a/l$. For the moment we will eliminate l to find an expression for μ.

$$2\cos\mu\cos(\alpha+\mu) \;=\; a \;=\; \frac{2\cos\alpha}{3}.$$

The expression on the left is $\cos(\alpha+2\mu)+\cos\alpha$, so

$$\cos(\alpha+2\mu) \;=\; -\frac{\cos\alpha}{3}$$

giving

$$2\mu \;=\; \arccos\left(-\frac{\cos\alpha}{3}\right) - \alpha.$$

We can now write l in terms of μ, but we postpone doing this for clarity. Length $|AF|$ is, by construction, $a + 2(l-a)/3 = a/3 + 2l/3$ which is

$$\frac{2\cos\alpha}{9} + \frac{2l}{3}.$$

The final step is to raise a perpendicular from AO at N passing through F. Then $|ON| = x$ say, and $|NA| = 1-x$. Then $\cos\alpha = (1-x)/|AF|$ so that $x = 1 - |AF|\cos\alpha$ while $|NF| = |AF|\sin\alpha$ and

$$\tan\gamma \;=\; |NF|/x \;=\; \frac{|AF|\sin\alpha}{1-|AF|\cos\alpha}.$$

The substitutions required to obtain an explicit formula for the 'trisected' angle (γ, say) are a little tedious but perfectly possible, and I used MAPLE to find

$$\gamma \;=\; \arctan\left(\frac{\left(\frac{2}{9}\cos\alpha + \frac{4}{3}\cos\left(\frac{\alpha}{2}+\frac{1}{2}\arccos\left(-\frac{\cos\alpha}{3}\right)\right)\right)\sin\alpha}{1-\left(\frac{2}{9}\cos\alpha + \frac{4}{3}\cos\left(\frac{\alpha}{2}+\frac{1}{2}\arccos\left(-\frac{\cos\alpha}{3}\right)\right)\right)\cos\alpha}\right).$$

If for example the angle to be trisected is 120 degrees then the value of the 'trisected' angle is around 39.947 degrees, so the error in this case is around 0.13%.

Problem 264.4 – Polynomial

Tony Forbes

Let m be a positive integer, let $r > 1$, and consider the polynomial

$$P(x) \;=\; (x+1)(x+2)\ldots(x+m) - r\,m!.$$

(i) Show that $P(x)$ has a positive root.

(ii) Show that when m is even there also is a negative root.

(iii) Show that all other roots (if any) of $P(x)$ are non-real.

For example, $(x+4)(x+3)(x+2)(x+1) - 24r$ has roots

$$x \;=\; \frac{-5+\epsilon_1\sqrt{5+4\epsilon_2\sqrt{24r+1}}}{2}, \qquad \epsilon_1,\ \epsilon_2 = \pm 1,$$

which is real only when $\epsilon_2 = 1$ and then the negative and positive roots correspond to $\epsilon_1 = -1$ and $\epsilon_1 = 1$ respectively.

Solution 262.2 – Digit sum ratio

I (VL) regularly visit the Missouri state university maths website http://people.missouristate.edu/lesreid/Challenge.html and often send solutions. Here is an example of a challenge problem. Let $S(n)$ denote the sum of the (base 10) digits of n. Show that for any positive integer m there is an n such that $m = S(n^2)/S(n)$. In addition, M500 readers might like to investigate bases other than 10. Do all number bases $b = 2, 3, \ldots$ have the stated property?

Dave Wild

If there is a solution to this problem in base $b = 10$, then I would expect there to be a solution in other bases with the possible exception of base 2. In base 2 we can only use the digits 0 and 1; so I shall assume that n only contains these digits. Let us try a few values.

$n = 1$. $n^2 = 1$ in all bases and $m = S(n^2)/S(n) = 1$.

$n = 11$. For base $b > 2$, we have $n^2 = 121_b$ and $m = 2^2/2 = 2$. If $b = 2$, $n^2 = 1001_2$ and $m = 1$. Perhaps adding a zero would help.

$n = 101$. For $b > 2$, $n^2 = 10201_b$ and, as before, $m = 2$. If $b = 2$, $n^2 = 11001_2$ and $m = 3/2$. So adding a zero has increased the value of m for $b = 2$.

$n = 111$. For $b > 3$, $n^2 = 12321_b$ and $m = 3^2/3 = 3$. This is a step backwards as the solution is valid for fewer bases. Add a zero to see if we can get rid of the 3.

$n = 1011$. For $b > 2$, $n^2 = 1022121_b$ and $m = 3^2/3 = 3$. If $b = 2$, $n^2 = 1111001_2$ and $m = 5/3$.

Let us look more closely at what is happening when n contains 3 ones. We can write $n = b^x + b^y + b^z$, where $x > y > z \geq 0$. So squaring n will produce 3^2 terms which we can write as $b^{2x}+b^{2y}+b^{2z}+2b^{x+y}+2b^{x+z}+2b^{y+z}$.

$\boldsymbol{b > 2}$. If the powers of b in each term are distinct, as is the case when $n = 1011$, then n^2 will consist of three ones and $(3^2 - 3)/2$ twos. So $S(n^2) = 3^2$. More generally we can see if n contains M ones and we can chose the powers x, y, z, $\ldots$, so that $2x$, $2y$, $2z$, $\ldots$, $x + y$, $x + z$, $\ldots$ are all distinct, then n^2 will contain M ones and $(M^2 - M)/2$ twos and $m = M^2/M = M$. This looks promising.

$\boldsymbol{b = 2}$. If $n = 1011$, then

$$n^2 = 2^6 + 2^2 + 2^0 + 2 \cdot 2^4 + 2 \cdot 2^3 + 2 \cdot 2^1 = 2^6 + 2^2 + 2^0 + 2^5 + 2^4 + 2^2.$$

The powers of 2 are no longer distinct and we eventually get $2^6 + 2^0 + 2^5 + 2^4 + 2^3$. Can we spread out the powers of two so they remain distinct? If x, y, and z were all even this would mean the first three terms would end up as even powers and the next 3 terms would be odd powers. $n = 1000101$ is a suitable choice as $n^2 = 2^{12} + 2^4 + 2^0 + 2^9 + 2^7 + 2^3$ and this gives $m = 2$.

Based on this evidence here is the proposed solution.

For an integer $M > 0$, let $n = T(M) = b^{a[1]} + b^{a[2]} + \cdots + b^{a[M]}$, where $a[k] = 2(2^{k-1} - 1)$. Clearly $S(n) = M$ as n contains M ones.

$$\begin{aligned} n^2 &= [b^{2a[1]}] + [2b^{a[2]+a[1]} + b^{2a[2]}] + [2b^{a[3]+a[1]} + 2b^{a[3]+a[2]} + b^{2a[3]}] + \ldots \\ &\qquad + [2b^{a[M]+a[1]} + \cdots + 2b^{a[M]+a[M-1]} + b^{2a[M]}]. \end{aligned}$$

All the powers of b of the terms within a pair of bracket are different and are less than those in any following pair of brackets. Therefore all the powers of b are distinct and even.

For $b > 2$ then, when n^2 is written it will contain M ones and $M(M-1)/2$ twos. As their sum is $M + M(M-1) = M^2$, $S(n^2) = M^2$. So $m = M$.

When $b = 2$ the $M(M-1)/2$ terms of the form $2b^{a[i]+a[j]}$ can be written as $b^{a[i]+a[j]+1}$. So the expression for n^2 contains M even powers of 2 and $M(M-1)/2$ odd powers of 2. Therefore $S(n^2) = M + M(M-1)/2 = M(M+1)/2$. So $m = (M+1)/2$.

Therefore, for a given m, the number $n = T(M)$ satisfies $S(n^2)/S(n) = m$, where $M = m$ for bases greater than 2, and $M = 2m - 1$ for base 2.

How to do mathematical research

Here is some useful advice to anyone who is about to undertake any kind of mathematical research involving the significant use of a computer.

First (i) *prove* that a particular mathematical object with specific properties does not exist. Then do not (ii) waste a vast amount of computer time trying to find it.

First (iii) perform a modest amount of routine computing to discover an interesting mathematical object with specific properties. Then do not (iv) waste a lot of time, mental effort and paper trying to prove that it does not exist.

Erase all evidence that you did (ii) before (i).

Erase all evidence that you did (iv) before (iii).

Solution 260.1 – Iterated trigonometric integral

For positive integer n, define $F_n(x)$ by

$$F_1(x) \;=\; \sin(\arctan x), \qquad F_{n+1}(x) \;=\; \sin(\arctan F_n(x)).$$

Show that for $a \geq 0$,

$$\int_0^a F_n(x)\,dx \;=\; \frac{\sqrt{na^2+1}-1}{n}.$$

Steve Moon

For some $\theta = \arctan x$ we have $\theta = \arcsin x/\sqrt{1+x^2}$ by considering a suitable right-angled triangle. Therefore

$$F_1(x) \;=\; \frac{x}{\sqrt{1+x^2}} \quad\text{and}\quad F_2(x) \;=\; \sin\left(\arctan\frac{x}{\sqrt{1+x^2}}\right) \;=\; \frac{x}{\sqrt{1+2x^2}}.$$

We conjecture

$$F_n \;=\; x/\sqrt{1+nx^2} \qquad \text{for } n \in \mathbb{Z}^+ \tag{1}$$

and proceed by induction. If $F_k = x/\sqrt{1+kx^2}$ is true for $k \in \mathbb{Z}^+$, then

$$F_{k+1} \;=\; \sin\left(\arctan\frac{x}{\sqrt{1+kx^2}}\right) \;=\; \frac{x}{\sqrt{1+(k+1)x^2}}.$$

Hence (1) is true by induction and

$$\int_0^a F_n(x)\,dx \;=\; \int_0^a \frac{x}{\sqrt{1+nx^2}} \;=\; \left[\frac{\sqrt{1+nx^2}-1}{n}\right]_0^a \;=\; \frac{\sqrt{1+na^2}-1}{n}.$$

Problem 264.5 – Two planets

Two planets A and B orbit a sun. When does B appear brightest when viewed from A? For simplicity, assume A and B orbit the sun in concentric circles of radii a and b respectively, and that B is a sphere with uniform reflectivity.

Q. What does the 'B' in Benoit B Mandelbrot stand for?
A. Benoit B Mandelbrot. —Sent by Eddie Kent

Solution 262.5 – HH or TH

During one of his visits to M500, David Singmaster suggested this interesting opportunity for possible wealth enhancement.

> 'We all know that in tossing a fair coin you are just as likely to get a head followed by a head as a tail followed by a head. So what can the harm be in accepting this offer of a simple game? You will toss a coin repeatedly until you get a head followed by a head or a tail followed by a head. If it's a head followed by a head, I'll give you £2. If it's a tail followed by a head, you give me a £1.'

Was it wise to take up his kind offer?

Tommy Moorhouse

I took a game to end when the criterion (a head followed by a head – HH – or a tail followed by a head – TH) had been met, at which point David Singmaster or I would pay out, and (funds permitting) another game could start.

Suppose the first throw yielded H. Then if the second throw yielded H David would lose £2. The probability of this happening is 1/4. On the other hand, all the other possibilities result in me paying David £1, because any sequence starting with T must eventually be interrupted by a H. HT and TT do not pay out either way.

In the long run, say after 100 games, we could expect that David would pay me approximately £50, while I would lose around £75. This statement is obviously statistical hand-waving, but is indicative of the balance of the game!

Dave Wild

If a sequence of tosses contains a tail then the sequence is terminated when the next head is tossed, and this results in a loss of £1. Therefore the only sequence of tosses which results in winning £2 is HH. The expected gain per game is therefore $2 \cdot 1/4 - 1 \cdot 3/4 = -25\text{p}$. Therefore it would be wise not to accept the kind offer.

The same conclusion was also reached in a similar manner by **Reinhardt Messerschmidt**.

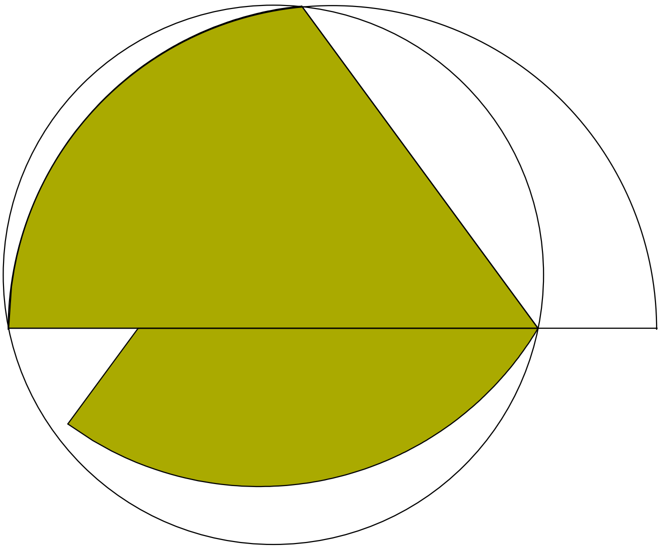
Tony Forbes

I must admit that I find the solutions presented above extremely convincing. However, I also find David Singmaster's original argument extremely convincing. So I played 100000 games, resulting in HH 25114 times and TH 74886 times. A loss of £24658 has finally convinced me.

Problem 264.6 – Semicircle dissection

Tony Forbes

You have a semicircular disc, D, of radius r. Dissect D into two pieces using a single straight cut and try to arrange them so that they fit on a disc of radius 1. For what values of r is this possible?



The problem has an important application. Suppose you have shared a large pizza with a friend. You will obviously want an efficient algorithm for cutting up your semicircular half so that it fits on your plate. This could be difficult if your pizza's area is not much smaller than that of the plate—and impossible if the areas are equal. On the other hand, if the pizza is sufficiently small but with radius greater than 1, only a single straight cut will be needed. The diagram shows one way of doing it when $r = 1.2$. As you can see, there is room for improvement.

Problem 264.7 – Log cot integral

Show that

$$\int_0^{\pi/4} (\tan\theta)(\log\cot\theta)\,d\theta \;=\; \frac{\pi^2}{48}.$$

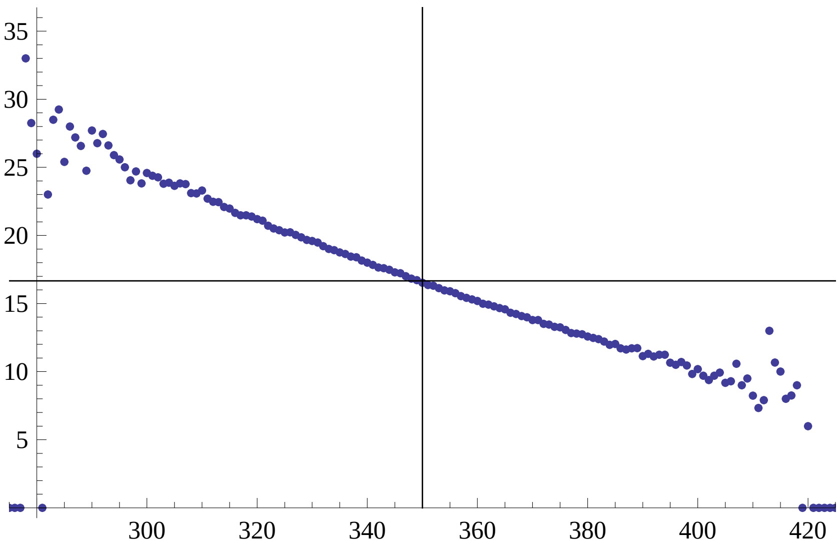
Re: Problem 188.1 – Ones

> Throw n dice. The total score is s. What is the expected number of ones?

Tony Forbes

This is not a solution—except to assert that for large n and $s \approx 7n/2$, the expected number of ones should be approximately $n/6$. As far as I am aware the problem has not been resolved for the benefit of M500 readers.

Some experimentation suggests a linear (or something that looks as if it could possibly be linear) relationship. I threw a hundred dice a quarter of a million times, on each occasion recording the score and the number of ones. The results are illustrated. The horizontal axis is the score, s, the vertical axis is the average number of ones for s. Armed with this hint, can someone supply an exact answer to the problem?



M500 Society Committee – call for applications

The M500 committee invites applications from members to join the Committee. Please apply to Secretary of the M500 Society not later than 1st October 2015.

Contents M500 264 – June 2015

04.00

Front cover: $\displaystyle\sum_{j=0}^{3}\left(4^j\cos 16^j\theta,\ 4^j\sin 16^j\theta\right),\quad \theta\in[0,2\pi].$