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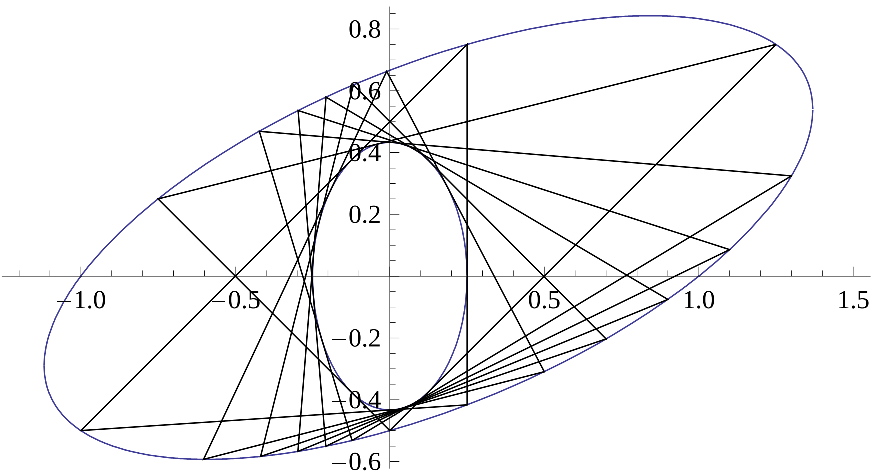
M500 265



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For full details and a booking form see m500.org.uk/may.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details see m500.org.uk/winter.htm.

Advice to authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to the Editor, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation.

M500 Winter Weekend 2016

The **thirty-fifth M500 Society Winter Weekend** will be held at

Florence Boot Hall, **Nottingham University**

Friday 8th – Sunday 10th January 2016.

Details, pricing and a booking form will be available nearer the time. Please refer to the M500 Web site.

http://www.m500.org.uk/winter.htm

Solution 262.6 – Tans

Establish the following result:

$$\frac{\tan\dfrac{\pi}{20}}{\tan^3\dfrac{3\pi}{20}} = \frac{10+\sqrt{50-22\sqrt{5}}}{10-\sqrt{50-22\sqrt{5}}}.$$

Bryan Orman

Let $t = \tan(\pi/20)$; then

$$\tan\frac{3\pi}{20} = \tan\left(\frac{\pi}{4}-\frac{2\pi}{20}\right) = \frac{1-\dfrac{2t}{1-t^2}}{1+\dfrac{2t}{1-t^2}} = \frac{3t-t^3}{1-3t^2}$$

using the $\tan 3A$ formula. Thus

$$(t-1)^5 = 20t^2(t-1), \qquad \text{or} \qquad (1-t)^2 = 2t\sqrt{5}.$$

Since $t \neq 1$ put $t = (1-\epsilon)/(1+\epsilon)$; then $\epsilon\sqrt{5+2\sqrt{5}} = \sqrt{5}$, giving

$$\tan\frac{\pi}{20} = \frac{\sqrt{5+2\sqrt{5}}-\sqrt{5}}{\sqrt{5+2\sqrt{5}}+\sqrt{5}} \qquad \text{and} \qquad \tan\frac{3\pi}{20} = \frac{\sqrt{5+2\sqrt{5}}-1}{\sqrt{5+2\sqrt{5}}+1}.$$

Now write

$$\frac{\tan\dfrac{\pi}{20}}{\tan^3\dfrac{3\pi}{20}} = \left(\frac{\sqrt{5+2\sqrt{5}}+1}{\sqrt{5+2\sqrt{5}}-1}\right)^3 \cdot \frac{\sqrt{5+2\sqrt{5}}-\sqrt{5}}{\sqrt{5+2\sqrt{5}}+\sqrt{5}}.$$

Then setting this equal to $(a+b)/(a-b)$, say, we have

$$5b(3+\sqrt{5}) = a(3-\sqrt{5})\sqrt{5+2\sqrt{5}}$$

after considerable simplification! As $\dfrac{3-\sqrt{5}}{3+\sqrt{5}} = \dfrac{7-3\sqrt{5}}{2}$ we have

$$10b = a\sqrt{(7-3\sqrt{5})^2(5+2\sqrt{5})},$$

which reduces to $10b = a\sqrt{50-22\sqrt{5}}$. Finally take

$$a = 10 \qquad \text{and} \qquad b = \sqrt{50-22\sqrt{5}}.$$

O Botafumeiro

An example of parametric pumping

Bryan Orman

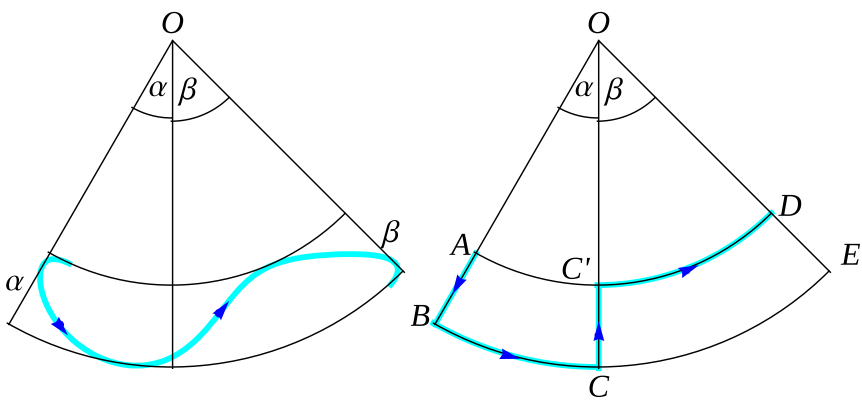
One of the oldest examples of parametric pumping is the swinging of the famous thurible, the giant censer, O Botafumeiro, in the cathedral of St James in Santiago de Compostela, in the north west of Spain.

The censer itself weighs about 57 kg and hangs in the transept, it swings on a rope about 21 m long with an eventual amplitude of about 80° in about 17 cycles.

To appreciate the spectacle, and to put the modelling of the system into context, a visit to the website www.catedraldesantiago.es/es/node/315 would be extremely helpful before reading too far into this article. Furthermore, the public performances have been well recorded by members of the congregation over the years and, to commence a search for demonstrations of the parametric pumping in action, just Google 'Botafumeiro video' and then 'Videos about "botafumeiro" on Vimeo', to view some good examples of the parametric pumping of the censer.

The initial position for the smooth pumping of the censer is achieved by moving the censer off the vertical, then a team of eight men, called tiraboleiros, pull on ropes attached to the main rope of the system in a cyclical manner in order to increase and to decrease the rope's length as the censer passes through the lowest and highest points of the motion.

The left-hand figure, below, not to scale, illustrates the actual trajectory of the censer through one complete cycle to the right. An oscillation would consist of a cycle to the right followed by one to the left, returning the censer to its starting side, albeit with a larger amplitude.



This article will show how a reasonably good mathematical model can be achieved by starting with the simple but unrealistic modelling assumption that the pumping takes place *instantaneously* at the lowest and highest points of the trajectory. The right-hand figure on the previous page represents this situation.

On the upper and lower circular arcs we assume that the system behaves as a simple pendulum of constant energy and so the usual modelling assumptions have to apply, namely that the mass of the rope is neglected, the censer is a particle, and air resistance on the censer is neglected. Consider the single cycle from A to D. Let the censer mass be M and let the maximum and minimum lengths of the rope be L and $L - \Delta L$ so that $OC = L$, $OC' = L - \Delta L$, and therefore $CC' = \Delta L$. By the principle of conservation of energy we have $E(B) = E(C)$ and $E(D) = E(C')$, and taking the datum for gravitational potential energy to be at the lowest point C, these equations become

$$
\begin{aligned}
MgL(1 - \cos\alpha) &= \frac{1}{2} ML^2 \dot{\alpha}^2, \\
Mg(L - \Delta L)(1 - \cos\beta) &= \frac{1}{2} M(L - \Delta L)^2 \dot{\beta}^2,
\end{aligned}
$$

where $\dot{\alpha}$ and $\dot{\beta}$ are the angular speeds at C and C' respectively. Angular momentum is also conserved at the vertical OC since the two forces acting on the particle, the rope tension and the weight of the particle, are both perpendicular to the direction of motion of the particle and therefore do no work on the particle. Thus

$$
ML^2 \dot{\alpha} = M(L - \Delta L)^2 \dot{\beta}.
$$

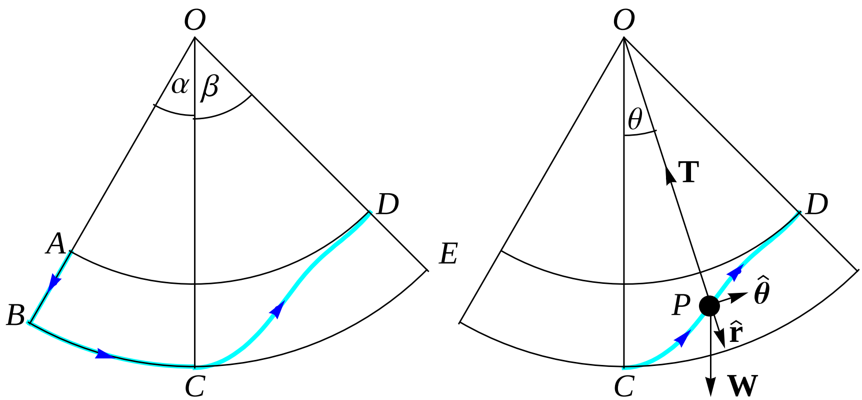
By eliminating the angular speeds from these three equations we have

$$
\sin\frac{\beta}{2} = \left(\frac{L}{L - \Delta L}\right)^{3/2} \sin\frac{\alpha}{2}.
$$

This formula enables us to calculate successive amplitudes during the parametric pumping process with $\alpha = \alpha_n$ and $\beta = \alpha_{n+1}$ if the initial amplitude α_0, say, is specified.

Using the data $L = 20.9$ and $\Delta L = 2.9$ with initial angle $\alpha_0 = 13^\circ$ we find that $\alpha_7 = 68^\circ$ and $\alpha_8 = 89^\circ$. This means that the system is pumped to about the maximum amplitude in about half the number of observed cycles.

We now remove the instantaneous pumping assumption and assume that this takes place from C to D, as shown in the following figures.



The left-hand figure shows the trajectory of the particle in this first revision of the model and the right-hand figure shows the force diagram. Here $\mathbf{T}$ is the tension in the rope and $\mathbf{W}$ is the weight of the particle. We note that r is the length of the rope at angle θ, with $r = L$ at C and $r = L - \Delta L$ at D.

From Newton's Second Law: $M\ddot{\mathbf{r}} = \mathbf{T} + \mathbf{W}$. And resolving in the radial direction, we have $T = M(r\,\dot{\theta}^2 + g\cos\theta - \ddot{r})$.

Now energy is supplied to the system through the pumping from C to D and the work done by the force $\mathbf{T}$ is given by $\int_C^D \mathbf{T}\cdot\mathbf{dr} = -\int_C^D T\,dr$ since $\mathbf{T} = -T\hat{\mathbf{r}}$ and $\mathbf{dr} = dr\,\hat{\boldsymbol{\theta}}$, where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are the radial and transverse unit vectors. So the increase in energy is given by

$$\Delta E \;=\; -\,M\int_C^D \left(r\,\dot{\theta}^2 + g\cos\theta - \ddot{r}\right) dr$$

and, from conservation of energy, $\dot{\theta}^2 = 2(g/r)(\cos\theta - \cos\alpha)$ so that ΔE becomes

$$-Mg\int_C^D (3\cos\theta - 2\cos\alpha)\,dr + M\int_C^D \ddot{r}\,dr.$$

Now at C, $\theta = 0$ and $\dot{r} = 0$, and at D, $\theta = \beta$ and $\dot{r} = 0$. Also $dr = \Delta L$. Thus $\Delta E = 3Mg(1 - \cos\beta)\Delta L$ and, to first order, this is

$$\Delta E(\text{pumping}) \;=\; 3Mg(1 - \cos\alpha)\Delta L.$$

The energy balance is then $E(D) = E(A) + \Delta E(\text{pumping})$, which gives

$$Mg(L-\Delta L)(1-\cos\beta) \;=\; Mg(L-\Delta L)(1-\cos\alpha) + 3Mg(1-\cos\alpha)\Delta L.$$

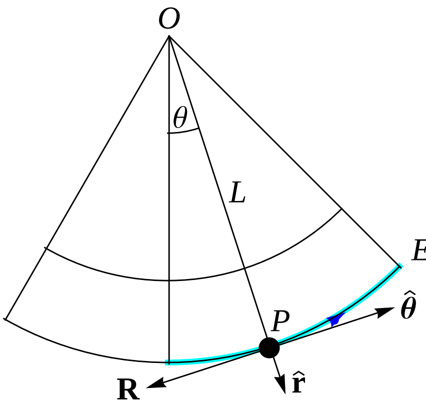
Rearranging, again using half angles, and to first order,

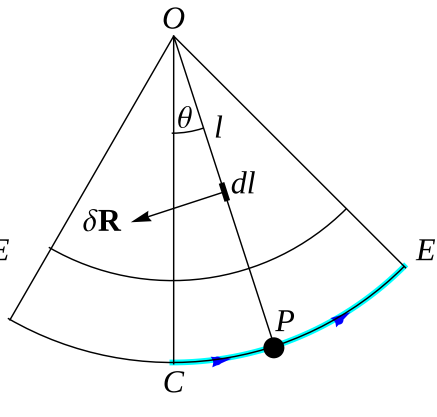
$$\sin\frac{\beta}{2} \;=\; \sqrt{1+\frac{3\Delta L}{L}}\,\sin\frac{\alpha}{2}.$$

With the previous data we find that $\alpha_9 = 67°$ and $\alpha_{10} = 82°$. Again the number of cycles is far less than the expected number of 17. This means that the instantaneous pumping is more efficient than this realistic pumping with the conservative force $\mathbf{T}$. Energy has to be removed from the system during the cycle if we want the number of cycles to achieve the maximum amplitude to increase.

The model has to include air resistance (air drag) on both the rope and the censer, and this will give us our final model. To this end we note that, to first order, equal losses of energy occur during the motion from B to C as from C to D. So we evaluate the energy loss from C to E, the simple pendulum approximation to the C to D motion, and double the answer to obtain the total loss of energy in a cycle.

The air resistance is given by $R = \rho v^2 A c_{\mathrm{d}}/2$, where ρ is the density of the air, v is the speed of the object, A is the cross sectional area of the object and c_{d} is the drag constant. We look at the energy loss from the censer's motion and from the rope's motion separately.





CENSER: The work done by the drag force in this case is $\int_C^E \mathbf{R}\cdot\mathbf{dr} = \int_C^E R\,ds$ since $\mathbf{R} = -R\,\hat{\mathbf{r}}$ and $\mathbf{dr} = ds\,\hat{\boldsymbol{\theta}}$, where s is the arc length from C

along CE. So the increase in energy is given by

$$\Delta E \;=\; -\int_0^{L\beta} \frac{1}{2}\rho v^2 A c_{\mathrm{d}}\, ds.$$

Using $ds = L\, d\theta$ and $v = L\dot{\theta}$ and also $L\dot{\theta}^2 = 2g(\cos\theta - \cos\beta)$ gives

$$\Delta E \;=\; -\rho g A L^2 c_{\mathrm{d}} \int_0^{\beta} (\cos\theta - \cos\beta) d\theta$$

and finally

$$\Delta E(\text{censer}) \;=\; -\rho g A L^2 c_{\mathrm{d}}(\sin\beta - \beta\cos\beta).$$

ROPE: The work done by the drag force in this case is a little more involved since the speed of the rope varies along its length. Consider an element of the rope dl, a distance l from O. Let the thickness of the rope be h; then $A = h\, dl$. Furthermore the speed of the element is lv/L. The force on the element is then

$$\delta R \;=\; \frac{1}{2}\rho \left(\frac{lv}{L}\right)^2 (h\, dl) c_{\mathrm{d}}$$

and, since $ds = l\, d\theta$, the work done on this element is $-\int_0^{\beta} \delta R\, l\, d\theta$, which becomes

$$-\frac{1}{2}\rho h c_{\mathrm{d}} l^3\, dl \int_0^{\beta} \dot{\theta}^2 d\theta.$$

The total increase in energy from the rope is obtained by integrating this over its length, giving

$$-\frac{1}{2}\rho h c_{\mathrm{d}} \frac{1}{4} L^4 \int_0^{\beta} \dot{\theta}^2 d\theta$$

and finally

$$\Delta E(\text{rope}) \;=\; -\rho g h c_{\mathrm{d}} \frac{1}{4} L^3 (\sin\beta - \beta\cos\beta).$$

For the complete cycle we need to add together the contributions from the censer and the rope, and double the answer, thus

$$\Delta E(\text{drag}) \;=\; -2\rho g L^2 \left(A c_{\mathrm{D}} + \frac{1}{4} h L c_{\mathrm{d}}\right)(\sin\alpha - \alpha\cos\alpha),$$

where we have replaced the angle β by α (true to first order), and have designated the censer and rope drag constants by c_{D} and c_{d} respectively.

The energy balance equation is now

$$E(D) \;=\; E(A) + \Delta E(\text{pumping}) + \Delta E(\text{drag}).$$

Inserting the expressions for the four terms into this equation, using half angles as before, and rearranging, we get

$$\sin^2\frac{\beta}{2} \;=\; \left(1+\frac{3\Delta L}{L}\right)\sin^2\frac{\alpha}{2} - K(\sin\alpha - \alpha\cos\alpha),$$

where K is a constant, namely $K = \dfrac{\rho L}{M}\left(Ac_{\mathrm{D}} + \dfrac{hLc_{\mathrm{d}}}{4}\right)$.

To test this final model we employ the following data (in SI units): $L = 20.6$, $\Delta L = 2.90$, $M = 56.5$, $h = 0.045$, $d = 1.2$ (censer height), $A = \pi(d/3)^2$ (notional cross-sectional area of the censer), $\rho = 1.25$, $c_D = 0.59$ and $c_d = 1.15$. Consequently $K = 0.257$ and $1 + 3\Delta L/L = 1.422$. The recurrence formula for the successive amplitudes is then

$$\sin^2\frac{\alpha_{n+1}}{2} \;=\; (1.422)\sin^2\frac{\alpha_n}{2} - (0.257)(\sin\alpha_n - \alpha_n\cos\alpha_n)$$

with $\alpha_0 = 13°$ as before. We find that $\alpha_{20} = 68.3°$ and $\alpha_{21} = 69.2°$, and the amplitude approaches about 73°, asymptotically. The maximum amplitude achieved in practice is about 82° and this occurs after 17 pumping cycles. Our final model will never give this maximum amplitude.

One obvious omission in the model is that of the mass of the rope. It will change the total energy and the angular momentum, but these changes cancel out to first order. Also the rope is not rigid, it is flexible, as can be observed at large amplitudes.

Furthermore our model treats the system as a pendulum of variable length whereas in practice it is a compound double pendulum, hinged at the censer. Incorporating this into the model is necessary, but really quite involved. It will however improve the model. Likewise, the pumping and the drag have been considered separately, as can be seen in the energy balance equation. They are interdependent since the trajectory CD is dependant on both the pumping and the drag. Again too complicated for us to consider but it will improve the model.

And finally, the transitions A to B and D to E are not instantaneous. They can be considered as motion under gravity since the real pumping appears to switch off before the highest points A and D are reached.

Solution 261.7 – Integrals involving roots

Let a and b be positive integers. Compute

$$I = \int_0^1 \left(1 - x^{1/a}\right)^{1/b} dx$$

and hence show that it is rational.

Reinhardt Messerschmidt

We will show that the function $x \mapsto (1 - x^{1/a})^{1/b}$ has an antiderivative of the form

$$\left(1 - x^{1/a}\right)^{1/b+1} \left(\sum_{k=0}^{a-1} c_k x^{k/a}\right),$$

where $c_0, c_1, \ldots, c_{a-1}$ are rational numbers that will be determined.

We have

$$\begin{aligned}
&\frac{d}{dx}\left(1 - x^{1/a}\right)^{1/b+1} \left(\sum_{k=0}^{a-1} c_k x^{k/a}\right) \\
&= \left(\frac{b+1}{b}\right)\left(1 - x^{1/a}\right)^{1/b}\left(\frac{-x^{1/a-1}}{a}\right)\left(\sum_{k=0}^{a-1} c_k x^{k/a}\right) \\
&\qquad + \left(1 - x^{1/a}\right)^{1/b+1}\left(\sum_{k=0}^{a-1} \frac{kc_k}{a} x^{k/a-1}\right) \\
&= \left(1 - x^{1/a}\right)^{1/b}\left\{\sum_{k=0}^{a-1} \frac{-(b+1)c_k}{ab} x^{(k+1)/a-1}\right. \\
&\qquad \left. + \sum_{k=0}^{a-1} \frac{kc_k}{a} x^{k/a-1} + \sum_{k=0}^{a-1} \frac{-kc_k}{a} x^{(k+1)/a-1}\right\}.
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{k=0}^{a-1} \frac{-(b+1)c_k}{ab} x^{(k+1)/a-1} &= \frac{-(b+1)}{ab} c_{a-1} + \sum_{k=0}^{a-2} \frac{-(b+1)c_k}{ab} x^{(k+1)/a-1}, \\
\sum_{k=0}^{a-1} \frac{kc_k}{a} x^{k/a-1} &= \sum_{k=1}^{a-1} \frac{kc_k}{a} x^{k/a-1} = \sum_{k=0}^{a-2} \frac{(k+1)c_{k+1}}{a} x^{(k+1)/a-1}, \\
\sum_{k=0}^{a-1} \frac{-kc_k}{a} x^{(k+1)/a-1} &= \frac{-(a-1)}{a} c_{a-1} + \sum_{k=0}^{a-2} \frac{-kc_k}{a} x^{(k+1)/a-1},
\end{aligned}$$

therefore

$$\begin{aligned}
&\frac{d}{dx}\Big(1-x^{1/a}\Big)^{1/b+1}\left(\sum_{k=0}^{a-1} c_k x^{k/a}\right) \\
&= \Big(1-x^{1/a}\Big)^{1/b}\Bigg\{\frac{-(b+1)}{ab}c_{a-1} + \frac{-(ab-b)}{ab}c_{a-1} \\
&\qquad + \sum_{k=0}^{a-2}\left(\frac{-(b+1)}{ab}c_k + \frac{-kb}{ab}c_k + \frac{(k+1)b}{ab}c_{k+1}\right)x^{(k+1)/a-1}\Bigg\}.
\end{aligned}$$

If we let

$$\begin{aligned}
\frac{-(b+1)}{ab}c_{a-1} + \frac{-(ab-b)}{ab}c_{a-1} &= 1, \\
\frac{-(b+1)}{ab}c_k + \frac{-kb}{ab}c_k + \frac{(k+1)b}{ab}c_{k+1} &= 0 \quad \text{for } k = 0, 1, \ldots, a-2,
\end{aligned}$$

in other words,

$$c_{a-1} = \frac{-ab}{ab+1}, \qquad c_k = \frac{(k+1)b}{(k+1)b+1}c_{k+1} \quad \text{for } k = 0, 1, \ldots, a-2, \quad (*)$$

then

$$\frac{d}{dx}\Big(1-x^{1/a}\Big)^{1/b+1}\left(\sum_{k=0}^{a-1} c_k x^{k/a}\right) = \Big(1-x^{1/a}\Big)^{1/b}.$$

It follows from $(*)$ that

$$c_0 = -\prod_{k=1}^{a}\frac{kb}{kb+1},$$

therefore

$$\int_0^1 \Big(1-x^{1/a}\Big)^{1/b}dx = \Big(1-x^{1/a}\Big)^{1/b+1}\left(\sum_{k=0}^{a-1} c_k x^{k/a}\right)\Bigg|_0^1 = -c_0 = \prod_{k=1}^{a}\frac{kb}{kb+1}.$$

For example,

$$\int_0^1 \Big(1-x^{1/3}\Big)^{1/5}dx = \frac{5}{6}\cdot\frac{10}{11}\cdot\frac{15}{16} = \frac{125}{176}.$$

Steve Moon

First make the substitution $y^b = 1 - x^{1/a}$, so that

$$x^{1/a} \;=\; 1 - y^b, \quad x \;=\; (1-y^b)^a, \quad dx \;=\; -\,aby^{b-1}(1-y^b)^{a-1}dy,$$

$$x = 0 \;\rightarrow\; y = 1, \quad x = 1 \;\rightarrow\; y = 0.$$

Then

$$I \;=\; -\int_1^0 yaby^{b-1}(1-y^b)^{a-1}dy \;=\; \int_0^1 aby^b(1-y^b)^{a-1}dy.$$

Now integrate by parts,

$$u \;=\; y, \quad du \;=\; dy,$$

$$dv \;=\; aby^{b-1}(1-y^b)^{a-1}, \quad v \;=\; -\,(1-y^b)^a;$$

$$I \;=\; \big[-y(1-y^b)^a\big]_0^1 + \int_0^1 (1-y^b)^a dy \;=\; \int_0^1 (1-y^b)^a dy.$$

The integrand has a finite binomial expansion with final term in y^{ab}. So

$$I \;=\; \int_0^1 \left(1 - ay^b + \frac{a(a-1)}{2}y^{2b} - \cdots + \frac{(-1)^k a!}{k!(a-k)!}y^{kb} + \cdots + (-1)^a y^{ab}\right) dy,$$

$(k = 0,\, 1,\, \ldots,\, a)$, and we integrate term by term:

$$\begin{aligned} I \;&=\; \left[y - \frac{ay^{b+1}}{b+1} + \cdots + \frac{(-1)^k a! y^{kb+1}}{k!(a-k)!(kb+1)} + \cdots + \frac{(-1)^a y^{ab+1}}{ab+1}\right]_0^1 \\ &=\; \sum_{k=0}^{a} \frac{(-1)^k a!}{k!(a-k)!} \cdot \frac{1}{kb+1}, \end{aligned}$$

which is rational since each term of the sum is rational.

Tony Forbes

Another problem is suggested. Prove that the three given answers are the same (see next page for the 3rd):

$$\prod_{k=1}^{a} \frac{kb}{kb+1} \;=\; \sum_{k=0}^{a} \frac{(-1)^k a!}{k!(a-k)!} \cdot \frac{1}{kb+1} \;=\; \frac{\Gamma(a+1)\Gamma(1+1/b)}{\Gamma(a+1+1/b)}.$$

Tommy Moorhouse

We make use of the elementary fact (easy to prove if you're not convinced) that the product of two rational numbers is rational.

We also use the binomial expansion:

$$(1+z)^{\beta} \;=\; 1+\beta z+\frac{\beta(\beta-1)}{2!}z^2+\cdots+\frac{\beta(\beta-1)\cdots(\beta-k)}{(k+1)!}z^{k+1}+\cdots.$$

Here $z=-x^{1/a}$ and $\beta=1/b$ and the integration term by term gives

$$x-\frac{\beta x^{1+1/a}}{(1+1/a)}+\cdots+(-1)^{k+1}\frac{\beta(\beta-1)\cdots(\beta-k)x^{1+(k+1)/a}}{(k+1)!(1+(k+1)/a)}+\cdots.$$

The substitution $x=1$ gives us an infinite sum

$$I \;=\; 1+\sum_{k=0}^{\infty}(-1)^{k+1}\frac{a\beta(\beta-1)(\beta-2)\cdots(\beta-k)}{(k+1)!(a+(k+1))}.$$

We recognize this (at least if we have a standard textbook to refer to) as

$$F(a,-\beta,a+1,1),$$

the hypergeometric function. This evaluates to (see, for example, Whittaker and Watson, p. 282)

$$\frac{\Gamma(a+1)\Gamma(1+\beta)}{\Gamma(1)\Gamma(a+1+\beta)}.$$

Since a is an integer we can expand the Γ functions, e.g.

$$\Gamma(a+1+\beta) \;=\; (a+\beta)\Gamma(a+\beta) \;=\; \cdots \;=\; (a+\beta)(a+\beta-1)\cdots\beta\Gamma(\beta).$$

The $\Gamma(\beta)$s cancel and the remaining expression is rational.

Reference

E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge, 1927 (reprinted 1992).

Problem 265.1 – Three circles

There are three concentric circles with radii $a \le b \le c$. Show that it is possible to draw an equilateral triangle with one vertex on each circle if $c \le a+b$ but not if $c > a+b$.

Solution 188.1 – Ones

> Throw n dice. The total score is s. What is the expected number of ones?

Robin Whitty

The number of ways to put s indistinguishable balls into n distinguished boxes, s and n positive, is equal to $\binom{s+n-1}{n-1}$. If our boxes are dice then we must make sure every box has at least one ball in it by subtracting n from the s balls at our disposal, reducing our count to

$$\binom{s-n+n-1}{n-1} = \binom{s-1}{n-1}.$$

To make sure that no box gets more than six balls is more work and gives the following inclusion–exclusion formula:

$$\#\text{ ways to get a total of } s \text{ using } n \text{ dice} = \sum_{i\geq 0}(-1)^i\binom{n}{i}\binom{s-6i-1}{n-1}.$$

We will denote this count by $F(s,n)$.

Now let $G(s,n,k)$ denote the number of ways to get a total of s using n dice of which k dice show a score of 1. Then the expected number of ones on throwing n dice and getting a total score of s is given by $E(s,n) = \left(\sum_{k=0}^{n} kG(s,n,k)\right)/F(s,n)$.

To calculate the value of $G(s,n,k)$ we choose k dice to be equal to 1, increase the minimum score of the remaining dice to 2 and apply inclusion–exclusion as before, but to $n-k$ dice:

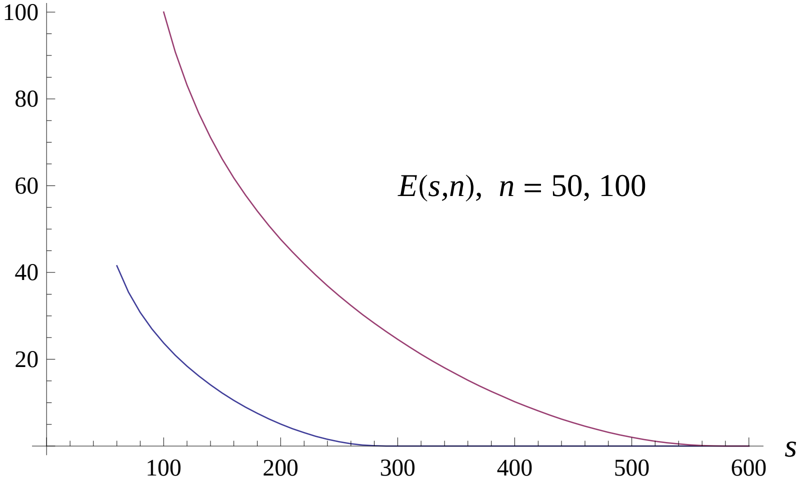
$$G(s,n,k) = \binom{n}{k}\sum_{i\geq 0}(-1)^i\binom{n-k}{i}\binom{s-n-5i-1}{n-k-1}.$$

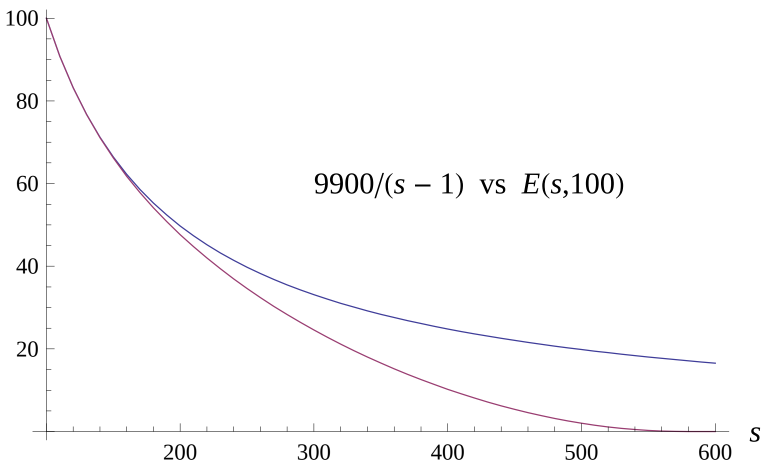
This is valid unless $s=n=k$ in which case $G(s,n,k)=1$ but the inclusion–exclusion does not apply since the number of dice, $n-k$, is no longer positive.

We can now calculate $E(s,n)$; a plot is given below for $n=50$ (lower curve) and $n=100$ (upper curve). We do not get much information from this calculation! However, we can try and simplify things under the assumption that s is not much larger than n. In this case the chance of more than six balls being put into any dice box is small unless k is large. So we can try just taking the first term in each inclusion–exclusion summation (ignoring the 'error terms'). The resulting approximation to $E(s,n)$ simplifies very satisfactorily:

$$E(s,n) \;\approx\; \frac{\displaystyle\sum_{k=0}^{n} k\binom{n}{k}\binom{s-n-1}{n-k-1}}{\displaystyle\binom{s-1}{n-1}} \;=\; \frac{n(n-1)}{s-1}.$$

This approximation is compared for $n = 100$ in the plot below. It appears to be quite effective up to about $s = 200$.





Solution 261.4 – Projectile

> A projectile is fired from a cannon in a uniformly distributed random direction above the ground. Show that the probability of it exceeding a fraction α of its maximum range is $\sqrt{1-\alpha}$. As usual, air resistance is non-existent, the ground is flat and gravity acts vertically downwards.

Reinhardt Messerschmidt

Step 1. Suppose the cannon is at the origin of (x, y, z)-space, with the (x, y)-plane being the ground and the positive z-axis pointing to the sky. Let $\theta \in [0, 2\pi]$ be the azimuthal angle at which the projectile is fired, i.e. the angle between the positive x-axis and the projection of the line of fire onto the (x, y)-plane. Let $\phi \in [0, \pi/2]$ be the elevation angle at which the projectile is fired, i.e. the angle between the line of fire and its projection onto the (x, y)-plane. We will find the maximum range of the projectile and the values of (θ, ϕ) for which the projectile exceeds the fraction α of its maximum range.

After it leaves the cannon, the projectile's acceleration is

$$(0,\ 0, -g),$$

where $g > 0$ is the gravitational constant. The projectile's initial velocity is

$$(\sigma \cos\phi \cos\theta,\ \sigma \cos\phi \sin\theta,\ \sigma \sin\phi),$$

where $\sigma > 0$ is the speed at which it leaves the cannon. The projectile's velocity at time t is

$$(\sigma \cos\phi \cos\theta,\ \sigma \cos\phi \sin\theta,\ \sigma \sin\phi - gt),$$

and its position at time t is

$$(\sigma t \cos\phi \cos\theta,\ \sigma t \cos\phi \sin\theta,\ \sigma t \sin\phi - gt^2/2).$$

The equation $\sigma t \sin\phi - gt^2/2 = 0$ implies $t = 0$ or $t = 2\sigma g^{-1} \sin\phi$, therefore the projectile hits the ground at time $2\sigma g^{-1} \sin\phi$. Its distance from the cannon at this time is

$$\begin{aligned}
&\sqrt{\big(\sigma(2\sigma g^{-1}\sin\phi)\cos\phi\cos\theta\big)^2 + \big(\sigma(2\sigma g^{-1}\sin\phi)\cos\phi\sin\theta\big)^2} \\
&= (2\sigma^2 g^{-1}\sin\phi\cos\phi)\sqrt{\cos^2\theta + \sin^2\theta} \\
&= \sigma^2 g^{-1}\sin(2\phi),
\end{aligned}$$

which can be viewed as a function of ϕ. The first derivative is

$$2\sigma^2 g^{-1}\cos(2\phi),$$

and the equation $2\sigma^2 g^{-1}\cos(2\phi) = 0$ implies $\phi = \pi/4$. The second derivative is

$$-4\sigma^2 g^{-1}\sin(2\phi),$$

which is negative at $\phi = \pi/4$, therefore the maximum range is attained if and only if $\phi = \pi/4$. The maximum range is

$$\sigma^2 g^{-1}\sin(2(\pi/4)) = \sigma^2 g^{-1}.$$

The projectile exceeds the fraction α of its maximum range if and only if

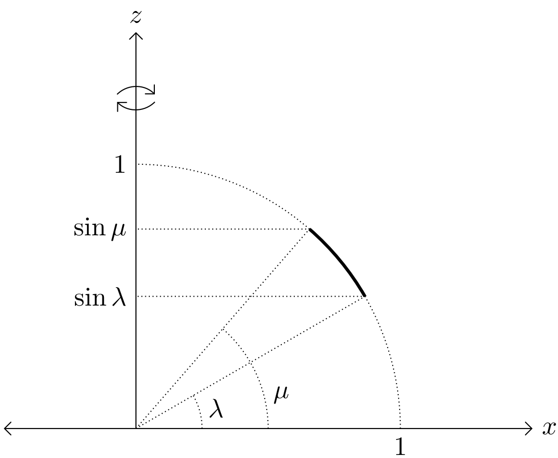
$$\sigma^2 g^{-1}\sin(2\phi) > \alpha\sigma^2 g^{-1},$$

i.e. if and only if

$$\beta < \phi < \pi/2 - \beta, \tag{$*$}$$

where $\beta = \frac{1}{2}\arcsin\alpha$.

Step 2. For every λ, μ such that $0 \le \lambda \le \mu \le \pi/2$, let $E(\lambda,\mu)$ be the set of all points on the unit sphere with an elevation angle between λ and μ. We will find the area of $E(\lambda,\mu)$ by revolving the unit circle in the (x,z)-plane around the z-axis.



By the formula for the area of a surface of revolution,

$$\text{Area of } E(\lambda,\mu) \;=\; \int_{\sin\lambda}^{\sin\mu} 2\pi x\sqrt{1+\left(\frac{dx}{dz}\right)^2}\,dz,$$

where $x=\sqrt{1-z^2}$. We have

$$\frac{dx}{dz} \;=\; \frac{-z}{\sqrt{1-z^2}}, \qquad \sqrt{1+\left(\frac{dx}{dz}\right)^2} \;=\; \frac{1}{\sqrt{1-z^2}},$$

therefore

$$\text{Area of } E(\lambda,\mu) \;=\; 2\pi\int_{\sin\lambda}^{\sin\mu} dz \;=\; 2\pi(\sin\mu-\sin\lambda). \qquad (**)$$

Step 3. By $(*)$ and $(**)$, the probability that a projectile fired in a uniformly distributed random direction exceeds the fraction α of its maximum range is

$$\frac{\text{Area of } E(\beta,\pi/2-\beta)}{\text{Area of } E(0,\pi/2)} \;=\; \frac{2\pi(\sin(\pi/2-\beta)-\sin\beta)}{2\pi(\sin(\pi/2)-\sin 0)} \;=\; \cos\beta-\sin\beta.$$

We will show that $\cos\beta-\sin\beta=\sqrt{1-\alpha}$.

By the half-angle identities for cos and sin,

$$\begin{aligned}\cos\beta &= \cos(\tfrac12\arcsin\alpha) = \sqrt{\tfrac12(1+\cos(\arcsin\alpha))},\\ \sin\beta &= \sin(\tfrac12\arcsin\alpha) = \sqrt{\tfrac12(1-\cos(\arcsin\alpha))}.\end{aligned}$$

Note that

$$1 \;=\; \cos^2(\arcsin\alpha)+\sin^2(\arcsin\alpha) \;=\; \cos^2(\arcsin\alpha)+\alpha^2;$$

therefore $\cos(\arcsin\alpha)=\sqrt{1-\alpha^2}$ and so

$$\cos\beta-\sin\beta \;=\; \sqrt{\tfrac12\Big(1+\sqrt{1-\alpha^2}\Big)}-\sqrt{\tfrac12\Big(1-\sqrt{1-\alpha^2}\Big)}.$$

Hence

$$\begin{aligned}(\cos\beta-\sin\beta)^2 &= \frac{1+\sqrt{1-\alpha^2}}{2}-2\sqrt{\tfrac14\Big(1-(1-\alpha^2)\Big)}+\frac{1-\sqrt{1-\alpha^2}}{2}\\ &= 1-\alpha.\end{aligned}$$

$\square$

Solution 260.1 – Iterated trigonometric integral

For positive integer n, define $F_n(x)$ by

$$F_1(x) \;=\; \sin(\arctan x), \qquad F_{n+1}(x) \;=\; \sin(\arctan F_n(x)).$$

Show that for $a \geq 0$,

$$\int_0^a F_n(x)\,dx \;=\; \frac{\sqrt{na^2+1}-1}{n}.$$

Reinhardt Messerschmidt

We will show that if $x > 0$ and $n \in \{0, 1, \ldots\}$ then

$$\sin\left(\arctan\left(\frac{x}{\sqrt{nx^2+1}}\right)\right) \;=\; \frac{x}{\sqrt{(n+1)x^2+1}}. \tag{$*$}$$

Let

$$y \;=\; \sin\left(\arctan\left(\frac{x}{\sqrt{nx^2+1}}\right)\right).$$

Applying $(\tan\circ\arcsin)$ on both sides,

$$\frac{x}{\sqrt{nx^2+1}} \;=\; \tan(\arcsin y) \;=\; \frac{\sin(\arcsin y)}{\cos(\arcsin y)} \;=\; \frac{y}{\cos(\arcsin y)}.$$

Squaring both sides,

$$\frac{x^2}{nx^2+1} \;=\; \frac{y^2}{\cos^2(\arcsin y)} \;=\; \frac{y^2}{1-\sin^2(\arcsin y)} \;=\; \frac{y^2}{1-y^2}.$$

Rearranging,

$$y^2 \;=\; \frac{x^2}{(n+1)x^2+1}.$$

Since $x > 0$ we have $y > 0$, therefore

$$y \;=\; \frac{x}{\sqrt{(n+1)x^2+1}}.$$

It follows from $(*)$ and induction that for every $n \in \{1, 2, \ldots\}$,

$$F_n(x) \;=\; \frac{x}{\sqrt{nx^2+1}},$$

therefore

$$\int_0^a F_n(x)\,dx \;=\; \frac{1}{2n}\int_0^a \frac{2nx}{\sqrt{nx^2+1}}\,dx \;=\; \left.\frac{\sqrt{nx^2+1}}{n}\right|_0^a \;=\; \frac{\sqrt{na^2+1}-1}{n}.$$

$\square$

Quadratum magia

Eddie Kent

Whenever talk turns to magic squares most naturally one thinks of Albrecht Dürer and his engraving Melancholia I. This shows a 4×4 magic square with magic constant 34 – the number that all the columns, rows, diagonals, corner 4s and many more groups add up to. In addition the second and third squares of the last row contain the numbers 15 and 14, giving the date of the engraving. Incidentally this is the only Dürer work that contains its own title: Melencolia·I; it has been the subject of numerous works of interpretation, including two volumes by Peter–Klaus Schuster (*Melencolia I—Dürers Denkbild*, Gebr. Mann 1991).

The subject of magic squares has itself been enthusiastically attacked over the years: Wikipedia gives a bewildering array of examples. My own small contribution to the genre came about through reading a puzzle in *Mathematical Spectrum* when I sent in a curiosity devised by a pensioned Moravian officer named Wenzclides. This is an 8×8 array giving a knight's-move magic square from 1 to 64 in which each row and column adds to 260. But that's all. No diagonals, no nothing.

47	10	23	64	49	2	59	6
22	63	48	9	60	5	50	3
11	46	61	24	1	52	7	58
62	21	12	45	8	57	4	51
19	36	25	40	13	44	53	30
26	39	20	33	56	29	14	43
35	18	37	28	41	16	31	54
38	27	34	17	32	55	42	15

At the time I submitted this it had not been beaten, and in 2003 it was shown (by Stertenbrink, Meyrignac and Mackay) that it never will be; and I have no idea who Wenzclides was. I found the array in a very old and fairly useless mathematics book which has long since disappeared.

In 1770 Euler devised a magic square of squares with a magic constant of 8515: rows, columns, diagonals and many symmetric sets of four cells. I believe this last makes it 'panmagic'.

4624	841	1681	1369
289	961	6241	1024
3481	784	529	3721
121	5929	64	2401

The square roots of these numbers are not magic and there is no reason why they should be, for any squares. But over time this construction has led to many similar being devised, not only 4×4 but also, in particular by C. Boyer (see *Math. Intel.* 27, 2005), 5×5, 6×6 and 7×7 magic squares of squares. Boyer published his results in reply to Martin Gardner, who in an update to his column in *Scientific American* offered a prize of \$100 to anyone who can find a 3×3 example with nine distinct square numbers. This prize has never been claimed and I doubt if the 1996 offer still exists but the challenge is out there. In 1998 Gardner wrote 'So far no one has come forward with a "square of squares" – but no one has proved its impossibility either. If it exists, its numbers would be huge, perhaps beyond the reach of today's fastest computers.' Well, maybe.

Thus in the interests of pure stupidity and time wasting and because I am virtually retired from M500, after decades of fun, and am in search of immortality I am prepared to offer one hundred American dollars to any M500 member who comes up with a 3×3 magic square of 9 distinct square numbers arranged so that the rows, columns, and both diagonals each add up to the same number. Making it pandiagonal will not increase the value of the prize. Whether my daughter will honour this pledge after my imminent death I've no idea. You'll have to ask her.

Problem 265.2 – Seven squares

Find integers n, p, and q such that the seven numbers

$$n^2,\quad n^2+p,\quad n^2-p,\quad n^2+q,\quad n^2-q,\quad n^2+p+q \quad\text{and}\quad n^2-p-q$$

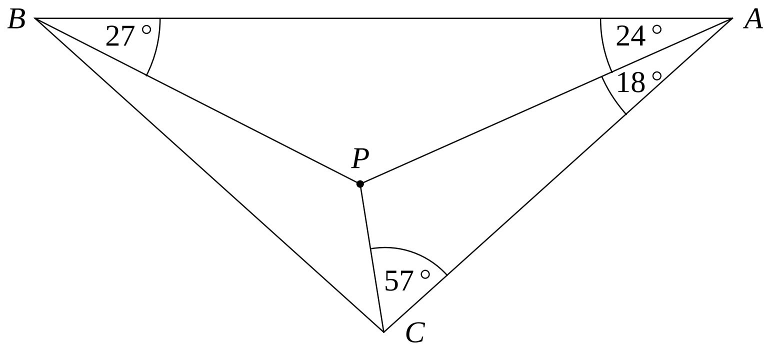
are distinct integer squares. Unfortunately there is no prize—unless you can augment the list with two more, n^2+p-q and n^2-p+q, and then arrange them into the much sought after magic square.

Problem 265.3 – Isosceles triangle

Dick Boardman

Triangle ABC has an interior point P such that $\angle PAB = 24°$, $\angle PAC = 18°$, $\angle PCA = 57°$ and $\angle PBA = 27°$.

In number 136 of *Nick's Mathematical Puzzles*, which you can find at http://www.qbyte.org/puzzles/, you are asked to prove that the triangle is isosceles. When you have done that, perhaps by clicking the 'Solution' button, we then want you to devise a way of finding similar sets of angles which are whole numbers of degrees.



Problem 265.4 – Stopping time

A casino operates a simple mechanism for increasing its revenue at your expense. You pay £10 and choose an integer, x. The casino chooses a number y, $1 \le y \le 10$, uniformly at random and proceeds as follows.

Step 1. If $x \equiv y \pmod{10}$, stop.

Step 2. Otherwise replace x by $2x + e$, where $e = 0$ or 1, chosen at random with equal probability (by tossing a coin, for example), pay you £1 and go to Step 1.

As you can imagine, the prospect of almost infinite wealth makes the game attractive. Show that the casino has the advantage, nevertheless, and that your expected loss is the same whatever number x you started with.

The problem was inspired by a lecture on Kemeny's constant by Robin Whitty, based on entry 236 in his *Theorem of the Day*, which you can look up at http://www.theoremoftheday.org/.

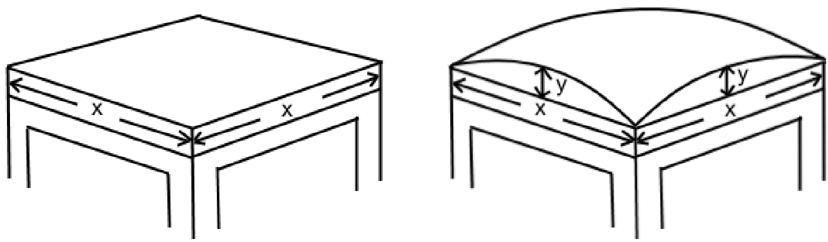
Problem 265.5 – Telephone box

Ralph Hancock

The new red telephone boxes used to have flat top panels, but the design was criticised for being boring and they were fitted with domed tops in the style of the old cast-iron ones designed by Giles Gilbert Scott, and based on Sir John Soane's self-designed mausoleum in St Pancras Old Church: http://goo.gl/OOd8jk. The shape of the new roof is, to put it simply, what you get when you apply a square cookie cutter to a sphere.

The new top for the boxes replaces a square flat panel measuring x by x, and the corners of the dome are at the places where the corners of the original square were. It is impossible to measure the diameter of the curved surface directly, but you can measure the height of the four identical sectors of circles around the edge, and they are y high.

I have to paint the new panels for all these boxes red. It was easy to work out how much paint I needed for the old flat panels whose area was simply x^2. But what is the area of the new panels, including the four sides?



Problem 265.6 – Triples

Tony Forbes

Show that the number of integer triples (a, b, c), where $1 \le a \le b \le c$ and $a + b + c = n$, is $\lfloor n^2/12 + 1/2 \rfloor$; i.e. the nearest integer to $n^2/12$.

Problem 265.7 – Population control

In an overpopulated country the law restricts a woman to one child if it is a boy and at most two if her first child is a girl. How is the population affected?

STUDENT: "What is x?"

TEACHER: "42."

ANOTHER STUDENT: "But yesterday you said it was 23."

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04.00

Front cover Two ellipses, $x^2 - 2xy + 3y^2 - y/2 - 1$ and $3x^2 + y^2 - 3/16$, and some triangles with vertices on one ellipse and sides tangent to the other. See http://www.theoremoftheday.org/Theorems.html, number 229, or http://www.maths.qmul.ac.uk/~whitty/LSBU/MathsStudyGroup/ADF PonceletsPorism.pdf.