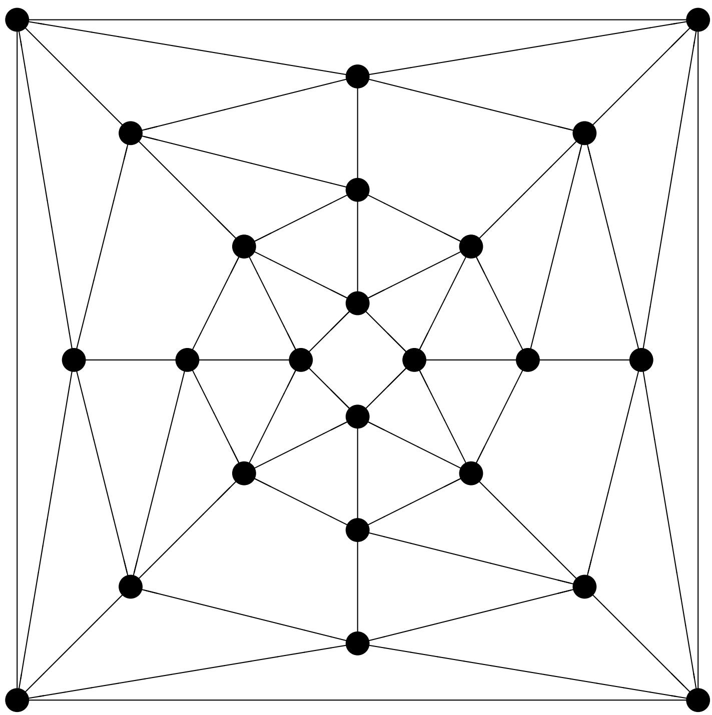
*

ISSN 1350-8539

M500 225



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file to tony@m500.org.uk.

Upon rotation

Dennis Morris

The nature of a spatial rotation depends upon the nature of the space within which the rotation takes place.

In 2-dimensional euclidean space, rotation is a repetitive thing whereby one gets back to the starting point every 360 degrees. Mathematically, a position in 2-dimensional euclidean space is represented by a euclidean complex number matrix; the position (x, y) is represented as the matrix

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \in \mathbb{C}, \qquad \text{where} \quad x, y \in \mathbb{R}.$$

Rotation through the angle θ is accomplished by multiplying this position matrix by the 2-dimensional euclidean rotation matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Since $\cos 0 = \cos 2n\pi$ & $\sin 0 = \sin 2n\pi$ and these functions are periodic, the magnitude of the elements in the rotation matrix repeats every 2π. Thus, when a position matrix is multiplied by a rotation matrix whose angle is $2n\pi$, it moves the position matrix to where it was prior to the rotation. To put it another way, as θ varies, the rotation matrix repeatedly becomes the identity matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

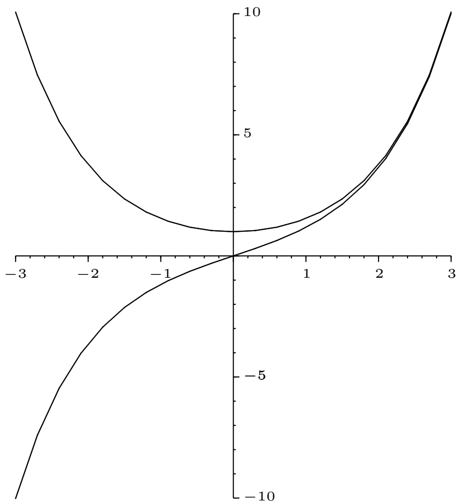
It is important that the trigonometric functions have the same period. If it were the case that the cosine function repeats every 2π and the sine function repeats every 3π, then rotation through neither 2π nor 3π would return the position matrix to its original coordinates; rotation through 6π would do it. If it were the case that the ratio of the repeating periods of the trigonometric functions was irrational, then rotation would never return a position matrix to its original position because, as θ varies, the rotation matrix would never become the identity matrix. A rotation in the positive direction can be undone by a rotation through the same angle in the negative direction because

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos -\theta & \sin -\theta \\ -\sin -\theta & \cos -\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The key to this is the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ and the identities: $\cos -\theta = \cos\theta$, $\sin -\theta = -\sin\theta$. The first of these arises from the fact that the determinant of the rotation matrix is unity; the second arises from the symmetry and anti-symmetry of the cosine and sine functions with respect the sign. In 2-dimensional hyperbolic space (Minkowski space–time), the position matrix and the rotation matrix are

$$\begin{bmatrix} x & y \\ y & x \end{bmatrix}, \text{ where } \ x, y \in \mathbb{R}, \quad \text{and} \quad \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}.$$

The graphs of these hyperbolic trigonometric functions are as follows.



These trigonometric functions are not periodic. At no point is the value of the sinh function equal to zero except $\theta = 0$. Thus, the rotation matrix can never be the identity matrix except at $\theta = 0$. Hence, multiplication by the rotation matrix will never rotate a position matrix in 2-dimensional hyperbolic space back to its own position.

These trigonometric functions are symmetric and anti-symmetric with respect to the sign of the angle and, of course, the determinant of the rotation matrix is unity:

$$\sinh\theta = -\sinh -\theta, \quad \cosh\theta = \cosh -\theta, \quad \cosh^2\theta - \sinh^2\theta = 1.$$

We have

$$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} \cosh -\theta & \sinh -\theta \\ \sinh -\theta & \cosh -\theta \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This means that rotation in the positive direction can be undone by a rotation in the negative direction through the same angle. If it were the case that $\sinh x = -\sinh -2x$ and $\cosh x = \cosh -2x$, then, to undo a rotation in the positive direction through angle χ would require a rotation through 2χ in the negative direction. If it were the case that $\sinh x = -\sinh -3x$ and $\cosh x = \cosh -2x$, then a rotation in the positive direction through angle χ could not be undone by a rotation in the negative direction alone; rotation in the negative direction through 6χ followed by rotation in the positive direction through 2χ would do it. If the ratio were irrational instead of $2:3$, then no combination of rotations would ever return to the original position.

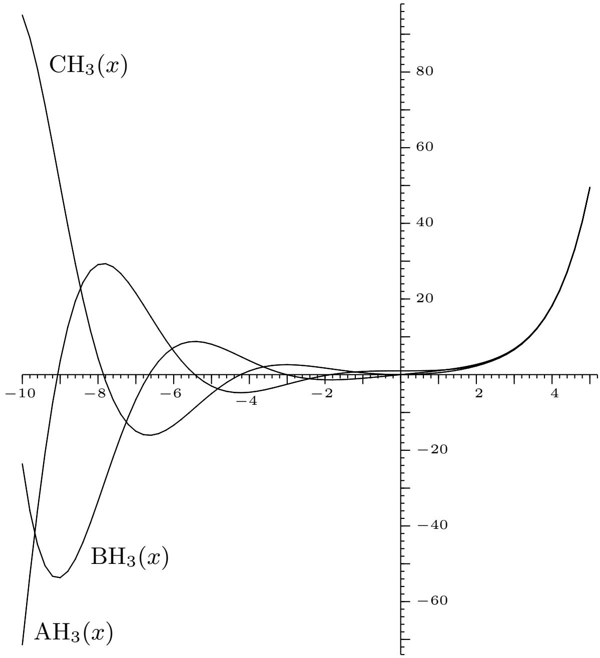
If the only trigonometric functions were the 2-dimensional ones, then all of the above would be nothing more than light conversation. However, there are trigonometric functions in all the higher-dimensional spaces. As hyperbolic angles are very different things from euclidean angles, so the angles of 3-dimensional space are very different things from the angles of 2-dimensional space—ditto rotations. In the 3-dimensional $C_3L^1H^2$-space (the space of the group C_3), a position matrix is

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}.$$

For simplicity, we consider a position with $c = 0$. The rotation matrix from such a position will contain the simple trigonometric functions of this space (which are the compound trigonometric functions when $c = 0$). (Technically, this is an erroneous way to do things and we ought properly to use the compound trigonometric functions; however, to go that way would lead the reader into a quagmire of mathematics when all we want to do is get some simple points over.) We have

$$\exp\left(\begin{bmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{bmatrix}\right) = \begin{bmatrix} e^a & 0 & 0 \\ 0 & e^a & 0 \\ 0 & 0 & e^a \end{bmatrix} . \begin{bmatrix} \mathrm{AH}_3(b) & \mathrm{BH}_3(b) & \mathrm{CH}_3(b) \\ \mathrm{CH}_3(b) & \mathrm{AH}_3(b) & \mathrm{BH}_3(b) \\ \mathrm{BH}_3(b) & \mathrm{CH}_3(b) & \mathrm{AH}_3(b) \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Normalizing this gives the rotation matrix in which $\mathrm{AH}_3()$, $\mathrm{BH}_3()$, $\mathrm{CH}_3()$ are the three simple trigonometric functions of this space. The graphs of these functions are as follows, with details near the origin on the next page.



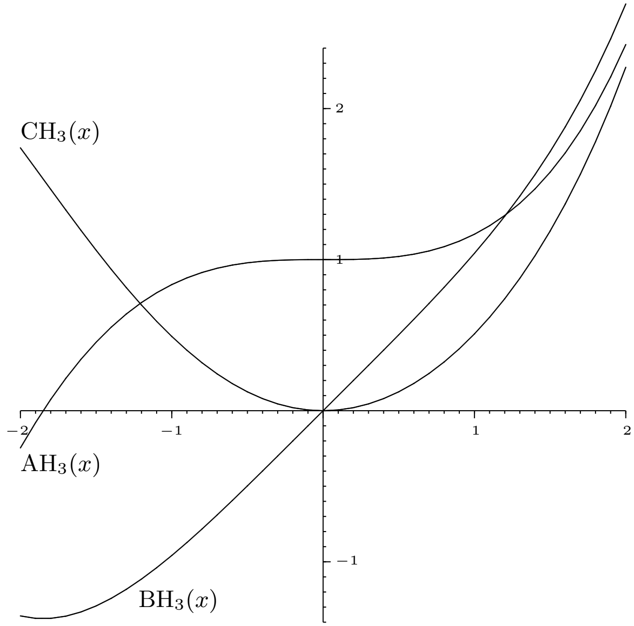
We do not have the identities of the 2-dimensional trigonometric functions. In this case

$$\mathrm{AH}_3(x) \neq \pm\mathrm{AH}_3(-x), \quad \mathrm{BH}_3(x) \neq \pm\mathrm{BH}_3(-x), \quad \mathrm{CH}_3(x) \neq \pm\mathrm{CH}_3(-x),$$

as is obvious from the graphs. This is due to the lack of symmetry with respect to sign. That the determinant is unity leads to the analogous identity

$$\mathrm{AH}_3(x)^3 + \mathrm{BH}_3(x)^3 + \mathrm{CH}_3(x)^3 - 3\mathrm{AH}_3(x)\mathrm{BH}_3(x)\mathrm{CH}_3(x) \;=\; 1.$$

This will be irrelevant when multiplying two rotation matrices together since the terms within it will never appear in the product. Obviously and clearly, a rotation in the positive direction cannot be undone by a rotation through the same angle in the negative direction. Obvious and clear it might be; true it is not. Remarkably:



$$\begin{bmatrix} \mathrm{AH}_3(x) & \mathrm{BH}_3(x) & \mathrm{CH}_3(x) \\ \mathrm{CH}_3(x) & \mathrm{AH}_3(x) & \mathrm{BH}_3(x) \\ \mathrm{BH}_3(x) & \mathrm{CH}_3(x) & \mathrm{AH}_3(x) \end{bmatrix} \begin{bmatrix} \mathrm{AH}_3(-x) & \mathrm{BH}_3(-x) & \mathrm{CH}_3(-x) \\ \mathrm{CH}_3(-x) & \mathrm{AH}_3(-x) & \mathrm{BH}_3(-x) \\ \mathrm{BH}_3(-x) & \mathrm{CH}_3(-x) & \mathrm{AH}_3(-x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

When $x = 3$ these matrices are

$$\begin{bmatrix} 6.57... & 6.82... & 6.69... \\ 6.69... & 6.57... & 6.82... \\ 6.82... & 6.69... & 6.57... \end{bmatrix} \begin{bmatrix} -2.54... & -0.04... & 2.63... \\ 2.63... & -2.54... & -0.04... \\ -0.04... & 2.63... & -2.54... \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

And so, a rotation in the positive direction through angle θ can be undone by a rotation in the negative direction through the same angle—Gobsmacking! We have here a 3-dimensional form of symmetry with respect to sign analogous to the 2-dimensional symmetry but quite different. Since a large rotation can be accomplished by a series of infinitesimally small rotations. The negative rotation follows the same route as the positive rotation (but in reverse).

Rotations in the positive direction are similar to the case in hyperbolic 2-dimensional space.

Consider rotations in the negative direction. We have

$$
\begin{aligned}
\mathrm{AH}_3(x) &= \frac{1}{3}e^{x} + \frac{2}{3}e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right),\\
\mathrm{BH}_3(x) &= \frac{1}{3}e^{x} + \frac{1}{3}e^{-x/2}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}x}{2}\right) - \cos\left(\frac{\sqrt{3}x}{2}\right)\right),\\
\mathrm{CH}_3(x) &= \frac{1}{3}e^{x} - \frac{1}{3}e^{-x/2}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}x}{2}\right) + \cos\left(\frac{\sqrt{3}x}{2}\right)\right).
\end{aligned}
$$

These functions are not periodic. The varying amplitude of the periodic bits as x increases in magnitude and the presence of the non-periodic term prevents them from being so. The functions BH_3 and CH_3 are never simultaneously zero except at $x = 0$. Thus, in spite of the wavy character of these functions when $x < 0$, a rotation in the negative direction can never return to its initial position in all three dimensions. It can return to its initial position in any one of the dimensions, but not all at the same time. It does this with approximately the periodicity of the periodic term in the respective trigonometric function. This periodicity is approximately $2\pi/\sqrt{3}$. Continued rotation in both the positive and the negative directions maintains the distance from the origin—which it would since the rotation matrix has determinant unity.

The same does not happen with the compound trigonometric functions. We have

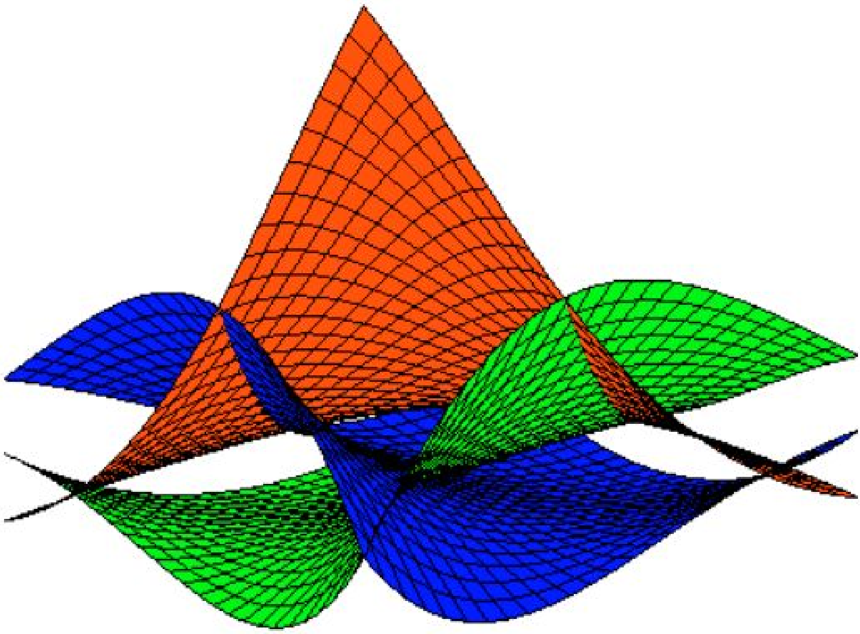
$$
\begin{aligned}
\nu A(x,y) &= \frac{1}{3}e^{x+y} + \frac{2}{3}e^{-(x+y)/2}\cos\frac{\sqrt{3}(x-y)}{2},\\
\nu B(x,y) &= \frac{1}{3}e^{x+y} + \frac{1}{3}e^{-(x+y)/2}\left(\sqrt{3}\sin\frac{\sqrt{3}(x-y)}{2} - \cos\frac{\sqrt{3}(x-y)}{2}\right),\\
\nu C(x,y) &= \frac{1}{3}e^{x+y} - \frac{1}{3}e^{-(x+y)/2}\left(\sqrt{3}\sin\frac{\sqrt{3}(x-y)}{2} + \cos\frac{\sqrt{3}(x-y)}{2}\right),
\end{aligned}
$$

$$
\begin{bmatrix}
\nu A(x,y) & \nu B(x,y) & \nu C(x,y)\\
\nu C(x,y) & \nu A(x,y) & \nu B(x,y)\\
\nu B(x,y) & \nu C(x,y) & \nu A(x,y)
\end{bmatrix}
\begin{bmatrix}
\nu A(-x,-y) & \nu B(-x,-y) & \nu C(-x,-y)\\
\nu C(-x,-y) & \nu A(-x,-y) & \nu B(-x,-y)\\
\nu B(-x,-y) & \nu C(-x,-y) & \nu A(-x,-y)
\end{bmatrix}
= \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}.
$$

When $x = 3$ and $y = 3$, these matrices are

$$\begin{bmatrix} 134.5095 & 134.4597 & 134.4597 \\ 134.4597 & 134.5095 & 134.4597 \\ 134.4597 & 134.4597 & 134.5095 \end{bmatrix} \begin{bmatrix} 13.3912 & -6.6944 & -6.6944 \\ -6.6944 & 13.3912 & -6.6944 \\ -6.6944 & -6.6944 & 13.3912 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

But rotation through $6\pi/\sqrt{3}$ does return to the starting point The graphs of these functions form a 3-dimensional version of the graphs of the simple trigonometric functions above.



Solutions to Page 21: {*Broken Arrow, The Third Man, The Killing Fields, The Fifth Element, Dr Strangelove, Twelve Monkeys, North by Northwest, Monsters Inc., Fahrenheit 911, Raiders of the Lost Ark, Short Circuit, Titanic, The Sphere, Bridge Over Troubled Water, The Simpsons, Fahrenheit 451, A Walk in the Black Forest, Route 66, Love Minus Zero: No Limit, The Two Towers, The Matrix Revolutions, Separate Tables, Prime Suspect, 20000 Leagues Under the Sea*}

Solution to Colours (page 14): Mensa's answer: 75—count 15 for vowels, 10 for consonants. Of course, the correct answer is 60—by mixing paints; purple = blue + red, green = blue + yellow, orange = yellow + red.

Solution 223.3 – Factorization

For which integer values of d does $x^4 - x - d$ factorize?

Dick Boardman

All quartics have four roots and can be factorized into four linear factors. However, the roots may be integer, rational, irrational or complex. I will consider first the case where one of the roots is integer.

When $d = 0$, $x^4 - x = x(x-1)(x^2+x+1)$.

Let $g(x) = x^4 - x$. If x is less than 0 or greater than 1, $g(x) > 0$. For $0 \le x \le 1$, $g(x)$ is negative and has a minimum with a value about 0.227. Differentiating $g(x)$ shows that it has no other local minimum. Hence $g(x) = d$ where $d > 0$ always has two real roots and two complex roots. The two real roots will have opposite signs.

Consider the set of integers $S = \{g(n)\}$, where n is an integer. Then $g(x) = g(n)$ will always have one integer root at $x = n$. Furthermore

$$\frac{g(x) - g(n)}{x - n} = -1 + n^3 + n^2x + nx^2 + x^3.$$

Thus when d is a member of the set S, $g(x) - d$ will always factorize into a linear term and a cubic polynomial with integer coefficients.

Suppose d is an integer not in S, $d > 1$. Let SP be the set of integers $\{g(n) : g(n) \ge 1\}$. If d is not in SP, it must lie between two consecutive members of SP. That is, $g(m) < d < g(m+1)$. However, $g(x) - d$ is negative for $x = m$ and positive for $x = m+1$ and is strictly increasing for $m < x < m+1$. Hence $g(x) = d$ has a positive root between m and $m+1$ and it cannot be an integer. Similarly, by defining a set of integers SM $= \{g(n) : g(n) < 0, n \in \mathbb{Z}\}$, it can be shown that $g(x) - d$ cannot have an integer negative root. Thus $g(x) = d$ can be factorized into a linear factor and a cubic polynomial with integer coefficients if $d \in S$ and not otherwise.

We now consider the case where d is a rational number a/b, where $-M < a < M$ and $0 < b < N$ and the fraction (possibly improper) is in its lowest terms. Call the set of these numbers R. This is a finite set with less than MN members and no duplicates. Arrange the members in increasing order and number them $1, \ldots, K$, $K < MN$.

Now consider the set $G = \{g(n) : n \in R\}$. By the same argument, $g(x) - d$ can be factorized into a linear and a cubic polynomial with rational coefficients which are members of R if d is a member of G. If d is rational and its numerator and denominator are less than M^4 and N^4 but is not a member of G it cannot be so factorized.

Mathematics in the kitchen – V
Tony Forbes

Following the trend we started in M500 **188** and continued in one or two further M500s since then, here's another experiment that you can perform with materials available in any reasonably well-equipped kitchen.

You will need a heavy saucepan lid, the heavier the better. It should have a generous rim (the part that fits inside the saucepan) the bottom of which forms a perfect circle. It must have rotational symmetry, including the handle, and, ideally, be made out of stainless steel. To give you some idea of what I have in mind, go to your local department store and inspect the largest item in the STELLAR range of cookware.

You will also need a perfectly flat kitchen worktop. If yours is of the currently fashionable type, where there is an outer covering of ceramic tiles, the experiment won't work very well, and you will need to get it replaced by a basic worktop made of plastic-covered chipboard. Failing that, a solid $1\,\text{m} \times 1\,\text{m} \times 5\,\text{cm}$ block of platinum will do.

Now for the experiment. This is really simple. Hold the saucepan lid right-way up but at an angle of about 30 degrees to the horizontal. Give it a spin and let it drop on to the worktop.

Under the right conditions you might observe the following interesting behaviour. At first the saucepan lid performs some ungainly spinning/rolling motion around the worktop. But after a few seconds, the thing *reverses its direction* and then settles down to a steady slow rotation (about $\frac{1}{3}\pi/\text{sec}$ in my case) on which is superimposed a kind of wobbly vibration component. This continues for some time during which, as gravity exerts its influence, the amplitude of the vibration decreases but its frequency increases. While all of this is going on the saucepan lid continues to rotate at the same steady rate in the direction opposite to its original spin. Eventually a climax is reached and the noise becomes unbearable to anyone else who happens to be in the kitchen. The saucepan lid then comes to a sudden halt and all is quiet again.

Needless to say, I am utterly amazed. Please write to us if you can explain what is going on.

You are advised to issue yourself and any spectators with suitable ear protection. Also you should clear the area of anything breakable. And, as usual, *we ask you not to perform the experiment if you are unwilling to take responsibility for accidents.*

Pascal's triangle revisited revisited

Sebastian Hayes

Thanks to Martin Hansen (M500 **221**) for a 'back to basics' article 'Revisiting Pascal's triangle' and the charming formula.

The problem with Pascal's triangle—known centuries earlier to the melancholy hedonist Omar Khayyàm and earlier still to the Chinese—is to connect up the formula for a particular entry with the binomial coefficients. As Martin Hansen says, it is most annoying that the r in ${}^nC_r$ does not, in the customary notation, refer to the row but to the column. Also, the stylish $n!/((n-r)!r!)$, which has now almost invariably replaced $n(n-1)(n-2)\ldots(n-r+1)/r!$ is offputting to the beginner and does not demonstrate the relation between *permutations* and *combinations*.

In my experience as a teacher, it is fairly readily grasped by the beginner that, if we are selecting r (distinguishable) objects from a store of n objects, there are n ways to choose the first object, only $n-1$ ways of choosing the second object (since one object has already been selected), and so on. One is tempted to end with $(n-r)$ but we must have r items and the first slot can be filled in $n = n-0$ ways, so the last bracketed expression is $(n-(r-1)) = (n-r+1)$. Thus the formula $n(n-1)(n-2)\ldots(n-r+1)$ for the total permutations of r objects taken from a pool of n objects with $1 \le r \le n$. This can be checked for simple cases, for example when we take the whole lot giving $n(n-1)(n-2)\ldots(n-n+1) = n!$ permutations as expected and when we only take a single object with $n\ldots(n-1+1) = n$ ways of doing just this.

The next step is to argue that, for any r objects taking them all, there are $r!$ permutations but only one selection (since the same objects are present each time). Confusingly, mathematicians call this a combination—even though in the case of a 'combination' lock the order of the digits is important. We conclude that the rapport *total number of permutations : total number of combinations* is $r! : 1$. So to reduce the number of permutations to the number of combinations we divide by $r!$ giving us the time-honoured formula $n(n-1)(n-2)\ldots(n-r+1)/r!$. This is how old-fashioned algebra books, such as those by Crystal or Hall & Knight introduce the topic of combinations. The relation to the expansion of

$$(a+b)^n \;=\; (a+b)(a+b)(a+b)\ldots(a+b) \qquad (n \text{ brackets})$$

becomes clear once we realize that the number of times a^r comes up is the same as the number of ways of selecting r things from n.

The triangular representation of the binomial coefficients is picturesque and so ingrained now that it will probably never be changed. But it does not generalize readily to formulae for the triangular, tetrahedral and higher-order numbers because each set starts one step behind the previous one, as it were, and this throws out the reckoning. A much more logical representation is the following.

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	3	6	10	15	21
4	1	4	10	20	35	56
5	1	5	15	35	70	126
6	1	6	21	56	126	252

Thus ${}^{\text{column}}X_{\text{row}} = {}^{c}X_r$ with $c = 1, 2 \ldots$ and $r = 1, 2, \ldots$ fixes any entry. The Rule of Formation is given by

$$ {}^{c}X_{r+1} \;=\; {}^{c}X_r + {}^{c-1}X_{r+1}; \qquad {}^{c}X_1 = 1. $$

Take 56 in the bottom row above. It is in the fourth column and sixth row. We have $56 = {}^{4}X_6 = {}^{4}X_5 + {}^{3}X_6 = 35 + 21$.

Applying this across the board means that to sum any column up to and including the entry in the rth row we simply move across one space. (This is the case because all entries in the first row are the same, namely 1.) For example,

$$
\begin{aligned}
56 &= 35 + 21 \\
&= 20 + 15 + 21 \\
&= 10 + 10 + 15 + 21 \\
&= 4 + 6 + 10 + 15 + 21 \\
&= 1 + 3 + 6 + 10 + 15 + 21 \;=\; \sum {}^{3}X_r .
\end{aligned}
$$

"Hey, how many people do you need to have in a room to guarantee the chance that at least two of them would have the same birthday?"

"I don't know. Three hundred sixty-four."

"Ha! Twenty-three. At least two out of every twenty-three people will have exactly the same birthday. Statistical odds. A lot of life is coincidence."

—*The Kills* by Linda Fairstein [sent by JRH]

The class number and Wilson's theorem

Paul Jackson

Using Wilson's theorem we have $(p-1)! \equiv -1 \pmod{p}$, and there are an even number of elements in the group $\mathbb{Z}_p^*$ of integers modulo p under multiplication; so we can rewrite the left-hand side as a product of squares, $1^2 \cdot 2^2 \cdot 3^2 \cdot \dots \cdot ((p-1)/2)^2 s$, by pairing up each element with its additive inverse, where $s = (-1)^{(p-1)/2}$. Then, equating with -1, we have in general $(((p-1)/2)!)^2 \equiv -s \pmod{p}$; thus when $p = 4n+1$, $-s = -1$, and when $p = 4n-1$, $-s = 1$.

Now for the former case we already know that square roots of -1 exist, there will be two of course, and so we can evaluate $((p-1)/2)!$ for odd primes $p = 4n+1$. For the other case, however from looking at small values it appears that there is no obvious pattern to the value of $((p-1)/2)!$ when the prime has the form $4n-1$.

In general, $(((p-1)/2)!)^2$ is the product of non zero quadratic residues, and for a prime of the form $4n-1$, which we call q, this is congruent to 1. We reduce these squares modulo q to lie in the interval $[1, q-1]$ and call this set Q, and we know that these all have distinct magnitudes; that is, they do not form pairs that sum to zero, as we have for the quadratic residues of primes of the form $4n+1$. So if in Q we replace all those elements x that exceed $(q-1)/2$ by $x-q$ we have the set Q'. But by group properties these will have the same magnitudes as the complete set $1, 2, \dots, (q-1)/2$, the product of which we wish to find.

For example, working modulo 11, we have $Q = \{1, 4, 9, 5, 3\}$, and $Q' = \{1, 4, -2, 5, 3\}$, and we see that each element in Q' takes a magnitude the range 1 to 5. Now of course in general Q is a group under multiplication, hence each element has an inverse; so the product of the elements is congruent to 1, so our problem is equivalent to counting sign changes, or counting the number of elements of Q that exceed $(q-1)/2$. Also we only need the information about whether there is an odd or even number of negative signs. How are we to do this? In the above example, to transform Q into Q' we only need to subtract 11, and in the general case we will need to subtract q an appropriate number of times. But we know that the sum of the elements of Q and hence Q' is congruent to zero modulo q.

So if we denote the sum of the elements of Q, as $\Sigma(Q)$, and the sum of the elements of Q' as $\Sigma(Q')$ then, putting $\Sigma(Q) = mp$ and $\Sigma(Q') = np$, we suspect that if we knew the parity of m, the parity of $m-n$, would follow, and the problem would be solved!

We next observe using the example above that $\Sigma(Q) = 1+9+3+4+5 = 22$, and $\Sigma(Q') = 1-2+3+4+5 = 1+2+3+4+5-2\cdot 2 = 11$, and it is easy to see that this will work in general. Let $T(n)$ be the nth triangle. We can define the value of $T(n)$ when $n = (q-1)/2$ as $T((q-1)/2)$, and so as we found above for $q = 11$, $T(5) - 2\cdot 2 = 11$ or, generalizing under modulo q, $T((q-1)/2) - 2e = \Sigma(Q')$. But $2e$ is always even; so $\Sigma(Q')$ and $T((q-1)/2)$ must share the same parity. Here, e is defined as the sum of the magnitudes of non-quadratic residues we must replace in Q when we perform the reduction. Further, $\Sigma(Q') \equiv 0 \pmod q$; so $T((q-1)/2)$ and $2e$ are in the same congruency class. Thus we know the parity of n.

Now the value of m is given by the function $L(q) = (q-1-2h(q))/4$, where $h(q)$ is the class number of the quadratic field $\mathbb{Q}(\sqrt{-q})$. This formula is from Kenneth Ribet's article, 'Modular forms and diophantine questions' at http://math.berkeley.edu/~ribet/Articles/icfs.pdf. So the function $L(q)$ for $q \equiv 3 \pmod 4$ depends on the class number, $h(q)$, and so the solution to this problem is connected to uniqueness of factorization of integers in $\mathbb{Q}(\sqrt{-q})$. Also $L(q) = \Sigma(Q)/q = m$, so we would know the parity of m, and as we know the parity of n, given by $T((q-1)/2) - 2e = nq$, we can compute the value of $((q-1)/2)!$ modulo odd primes q.

We could write this explicitly, by observing the following: as q is odd, nq and n share the same parity, which is also the case for $\Sigma(Q')$ and $T((q-1)/2)$. Thus n and $T((q-1)/2)$ have the same parity, and as $T((q-1)/2) = (q^2-1)/8$, we can put

$$(\tfrac{1}{2}(q-1))! \;\equiv\; (-1)^c, \qquad \text{where } c \;=\; \tfrac{1}{8}(q^2-1) + L(q).$$

We further observe that this can be simplified, depending on the quadratic character of 2, as $(2/q) = (-1)^b$, where $b = (q^2-1)/8$. Hence

$$\begin{aligned} (\tfrac{1}{2}(q-1))! \;&\equiv\; (-1)^{L(q)} & \text{if } (2/q) \;&=\; 1, \\ (\tfrac{1}{2}(q-1))! \;&\equiv\; (-1)^{L(q)+1} & \text{if } (2/q) \;&=\; -1. \end{aligned}$$

Thus we have a surprising connection between Wilson's theorem and the class number: even a seemingly trivial problem can have unexpected depths.

Reference

K. A. Ribet, Modular forms and diophantine questions, *Challenges for the 21st Century* (ed. L. H. Y. Chen *et al.*), World Scientific, Singapore 2001.

Problem 225.1 – Toroidal planet
Tony Forbes

There is a planet which, rather than being a spherical object, has the shape of a (solid) torus. You are standing somewhere on its innermost circle. Depending on the parameters of the torus, do you stay attached to the ground, or do you drift upwards, attracted towards the rest of the planet arched out above you?

I was going to add the condition that the material of which the planet is made has uniform density, but I now think this is not necessary. In keeping with real life, as on Earth, for instance, all you can assume about the planet's density is that it is constant under rotation about its main axis of symmetry.

Thanks to **Robin Whitty** for communicating this problem to me.

There's a picture of a torus on the front cover of M500 **204**.

Problem 225.2 – Eighth powers

For what values of m is 16 an eighth power modulo m?

Problem 225.3 – GCD

Compute $\gcd(n! + 1, (n + 1)!)$, where n is a positive integer.

Problem 225.4 – I choose a number

Tony Forbes

I am very generous. I choose an integer X, $1 \le X < \infty$. You choose an integer, Y. If $Y \le X$, I give you £Y. If $Y > X$, I give you £0. The offer is available once only.

How best do you take advantage of my generosity?

Colours

I (TF) found this in a list of problems published by Mensa.

> Purple is worth 70 points, green is worth 60 points and red is worth 35 points. How many points is orange worth?

It shouldn't give you too much trouble. But be surprised when you look up Mensa's answer on page 7 of this issue.

Tarts

Ian Adamson

> There are N tarts, all the same weight except one, which is a little heavier or a little lighter than the others. It is well known that n weighings are sufficient to find the faulty tart when $N = (3^n - 1)/2$, and sometimes it is also possible to tell whether the faulty tart is light or heavy. Recall that ADF asked a good question at the bottom of page 20 in issue **222**: Are there any n for which we can *always* determine the relative weight of the bad tart amongst $(3^n - 1)/2$ tarts in n weighings?

The performance of n weighings appears to give a results space of 3^n possibilities.

However if the weighings are (obviously) designed to give a result then of these 3^n possibilities, two will never occur as they would imply impossible situations (a tart being in more than one state of heaviness, lightness or goodness—or more than one tart being faulty) and one possibility won't occur unless all the tarts are good. Thus all m tarts must be weighed and solution space is $2m = 3^n - 3 \;\Rightarrow\; m = (3^n - 3)/2$.

However if only the faulty tart is required and not whether it be heavy or light then one tart need not be weighed and solution space is $1 + 2(m - 1) = 3^n - 2 \;\Rightarrow\; m = (3^n - 1)/2$. (If the faulty tart is weighed then its heaviness or lightness is given whether or not we want to know.)

Geometers (does not that include all of us?) might like to note that if for each weighing we write $+1,\ -1,\ 0$ as the right pan ascends or descends or neither respectively then we obtain coordinates of the cells of an n-cube.

If all the weighings are designed in advance then $(0, 0, ..., 0)$ is the cell representing the situation where all tarts are good and $\pm(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ where $\varepsilon_i \in \{-1, +1\}$ are the cells representing the two impossible situations. The remaining $3^n - 3$ cells represent respectively the fact that one of the m or $m - 1$ weighed tarts is faulty; $\pm(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ gives a specific tart and the sign shows its heaviness or lightness.

Useless of course!

An 82-year-old prostitute nicknamed Grandma was freed after being caught soliciting a client on the street in the red light district of Taipei City. Taiwanese police said that she was in good physical shape and with light make-up could easily pass for a 70-year-old. [Sent by Eddie Kent]

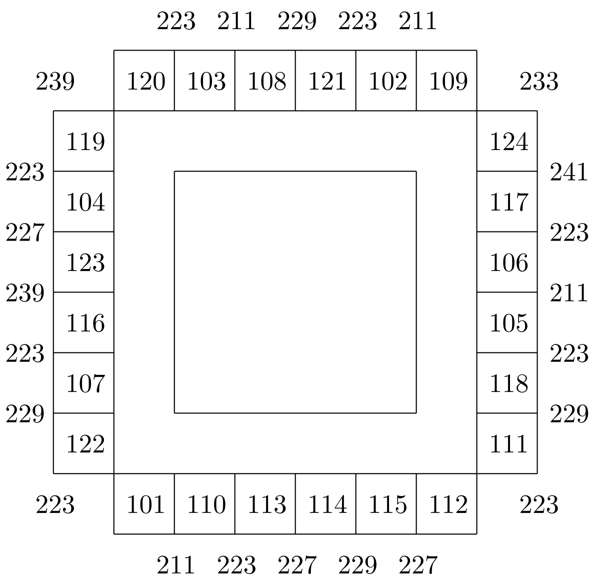
Solution 202.3 – The puzzled hotelier

> A hotelier told me that the rooms on the first floor had consecutive 3-digit numbers, beginning with 1, starting at 101. "They were," he said, "off four corridors forming a square and ordered so that the sums of pairs of numbers of adjacent rooms were all primes." He told me how many rooms there were (on the first floor) and I countered, "There couldn't have been fewer." What number did he say?

[This is one of a number of items that have lain dormant for a long time waiting for right moment for someone to have a go. Indeed, the solution to this one is appearing after an interval of nearly four years. It just goes to show that it's never too late to tackle an M500 problem.]

Steve Moon

This is one solution, for 24 rooms, but I suspect there are others.



Each adjacent pair sums to one of the following primes: 211, 223, 227, 229, 233, 239, 241.

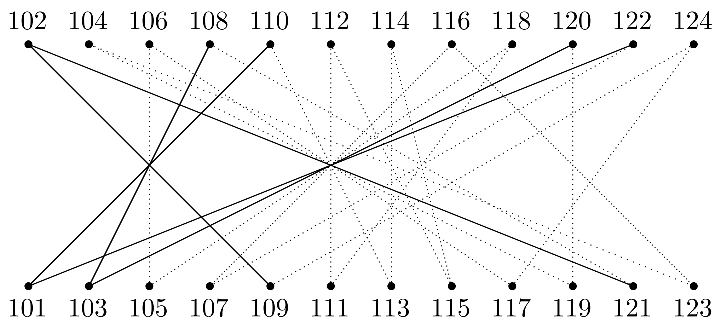
The total number of rooms must be a multiple of four. Twenty rooms cannot work, because the first two rooms only sum to a prime with one other number in the range 101–120:

$$\begin{aligned} 101 + 110 &= 211 \\ 102 + 109 &= 211. \end{aligned}$$

With 24 rooms, there are two combinations with rooms 101, 102, 103 that sum to a prime. The solution must contain the following components.

$$\begin{array}{ccccccc} - & 110 & - & 101 & - & 122 & - \\ - & 109 & - & 102 & - & 121 & - \\ - & 108 & - & 103 & - & 120 & - \end{array}$$

Then I put these fragments into a bipartite graph with odd and even numbers as the two sets, and I tried to construct a cycle.



Having worked out all the combinations of rooms in the range 101–124 which give primes, I thought this would help—but there are too many. Hence my solution by 'trial and error'. I cannot be sure that it is unique.

Problem 225.5 – Pythagorean triangles

Find right-angled triangles with integer sides x, y, z, $z^2 = x^2 + y^2$, such that z and $x + y$ are squares.

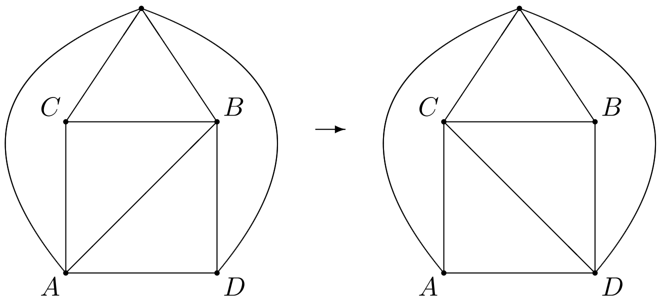
The problem was posed by Fermat to Mersenne in 1643. So it shouldn't give you too much trouble, assuming Fermat did actually have a triangle or two in mind. However, I (TF) do not know whether Mersenne succeeded in finding any solutions.

Problem 225.6 – Triangulations

Given a triangulation $\mathcal{T}$ of a set of points in the plane, show that any other triangulation of the same set of points can be obtained from $\mathcal{T}$ by a sequence of flipping operations.

A *triangulation* of a set of $n \geq 3$ points in the plane is a set of $3n - 6$ lines, where each line joins two points, and the whole plane is partitioned into triangles. The lines do not have to be straight and we are flexible as to what is a triangle: it is just a set of three points joined together by three non-crossing, continuous but not necessarily straight lines. And don't forget that the infinite part of the plane must also lie inside a triangle.

A *flipping operation* works like this. Choose any line. Then it will necessarily be the common border of two triangles, ABC and ABD, say. Remove the line AB and add a new line joining CD. Like this.



Thanks to **Stefanie Gerke** of Royal Holloway College, London for the idea behind this problem.

Problem 225.7 – Cubes

Tony Forbes

Find non-negative integer solutions of

$$(a+b)(a+c)(b+c) - a^3 - b^3 - c^3 \;=\; d^9.$$

Here are a few to get you started,

$$\{1, 79, 92, 2\},\ \{4, 17, 31, 2\},\ \{10, 10, 32, 2\},\ \{35, 875, 1143, 3\},$$

but what we really want is a general pattern of some kind.

The speed of dark

Tony Huntington

Eddie Kent dropped another of his 'innocent' questions at the end of his piece on Russell's Attic (M500 **222**) when he asked: "What is the speed of dark?". My first thought was that this was actually a trivial problem:—

Let c (pronounced "see") be the speed of light, then $\bar{c}$ (pronounced "not see"—because it's dark) is the speed of dark. Assuming that dark is the opposite of light then

$$\bar{c} \;=\; -c.$$

Multiplying through by $\bar{c}$:

$$\bar{c}^2 \;=\; -c \cdot \bar{c} \;=\; -0;$$

hence

$$\bar{c} \;=\; \sqrt{-0} \;=\; i \cdot \sqrt{0}.$$

This is a very satisfactory outcome as its physical interpretation is that the speed of dark is purely imaginary and has a magnitude equivalent to the square root of nothing at all!

My smugness at this neat solution was short-lived when someone pointed out that my fundamental assumption (that dark is the opposite of light) was open to challenge. In what way are they opposites? Light has properties such as intensity and colour. Does dark have equivalent inverse properties? What does the equivalent of a beam of light look like (assuming that you can somehow 'see' dark) in the dark world? Does dark have the same wave-particle duality as light? I haven't found satisfactory solutions to any of these questions, but I do have a new gadget which should be of use. It's the latest SATNAV and it has a 'Square One' button. When you push this button, the SATNAV shows you the most direct route back to square one. Follow me ....

And a closing anecdote on the topic of light and dark, which unfortunately is totally true .... I was once in a shop in Dubai attempting to buy a pair of sun glasses. I didn't find anything that I liked, but the salesman was determined not to let me out of the shop without relieving me of the burden of my cash. In desperation he finally showed me a pair of spectacles with clear plastic lenses. When I pointed out that they were not much use as sun glasses, he remarked that they would be ideal for that service at night!

The Ten Commandments of OU mathematics

Martyn Lawrence

Thou shalt answer the questions that are verily set out in the exam paper, and not the ones that thou wouldst prefer to answer.

Whatsoever thou dost to one side of an equation, do thou also to the other side. Dividest thou not by zero, lest the locusts of indefinability are plagued upon thee.

Thou shalt use thy common sense (!) when thou hast derived an answer, else thou wilt have olive trees 9,000 cubits high. Yea, even fathers younger than sons.

Thou shalt not covet thy neighbour's solutions, verily in pain of being called a cheat.

Thou shalt be wary of the teachings of false prophets, particularly past students from the same course who claim to know all about Integration or Eigenvectors.

When thou knowest not, and thou art totally stumped, thou shalt consult with course colleagues on the internet and, if thy search still elude thee, thou shalt consult thine Almighty Tutor—assuming thou hast knowledge of the particular combination number of his personal telephone apparatus (and he's not down the pub!).

The scribing of a correct answer in an assignment proveth not that thou hast solved the problem successfully, as lack of working upon thy tablet convinceth not thy doubting Tutor, and he shall diminish thy marks accordingly.

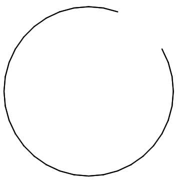
Thou shalt reflect upon thy youth, and recall how easy study was in those days before Adding-Up begat Sums, Sums begat Arithmetic, and Arithmetic begat Mathematics. Fear not the Day of Judgement, nor thy Exam, whichever cometh the sooner.

Should thou fail totally to work out a solution, thou shalt check to see that thy Study Unit bear not false witness; or is possibly even telling porkies. Hast thou checked the errata in the errata?

Thou shalt learn, read, scribe and speak in the many tongues of Mathematics, all the days of thy life, so that Grade 1s and 2s shall surely follow thee unto and beyond graduation.

Page 21

A friend of mine in the legal profession told me to avoid the word 'dingbats' to describe these things—possibly that name is jealously guarded by someone. Anyway, see how many you can get before looking up the answers. Films, TV, books, songs. Answers elsewhere in this issue.



$N = 561$
$3^N \equiv 3 \pmod{N}$

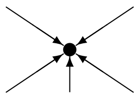
$x^2 + y^2 + z^2 = 1$

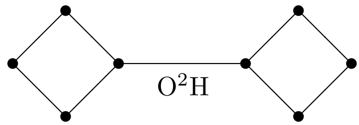
$^{11}\mathrm{B}_5$

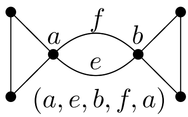
$0^\circ \cdot 315^\circ$

$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$
$\nabla_\mu Y_\nu + \nabla_\nu Y_\mu = 0$

$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$



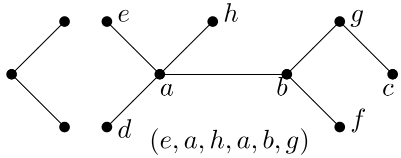




$$\frac{C}{20000\ell}$$

$232\frac{7}{9}{}^\circ\mathrm{C}$

$488\frac{1}{3}{}^\circ\mathrm{C}$



$x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x}}}}}}}}}}$ $y^{y^{y^{y^{y^{y^{y^{y^{y^{y^{y}}}}}}}}}}$

8.124038404635960360...

£500 £500 £500 £500 £500 £500
£500 £500 £500 £500 £500 £500

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

♂ ♂ ♂

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$1 \times 2 = 2,\ 2 \times 2 = 4,\ 3 \times 2 = 6, \ldots$

$1 \times 3 = 3,\ 2 \times 3 = 6,\ 3 \times 3 = 9, \ldots$

u
d
s 0, Ph.D.
c
t
b

:Ti:

$\nexists \lim(\heartsuit - 0)$

Contents M500 225 – December 2008

04.01

Cover: The snub cube graph.