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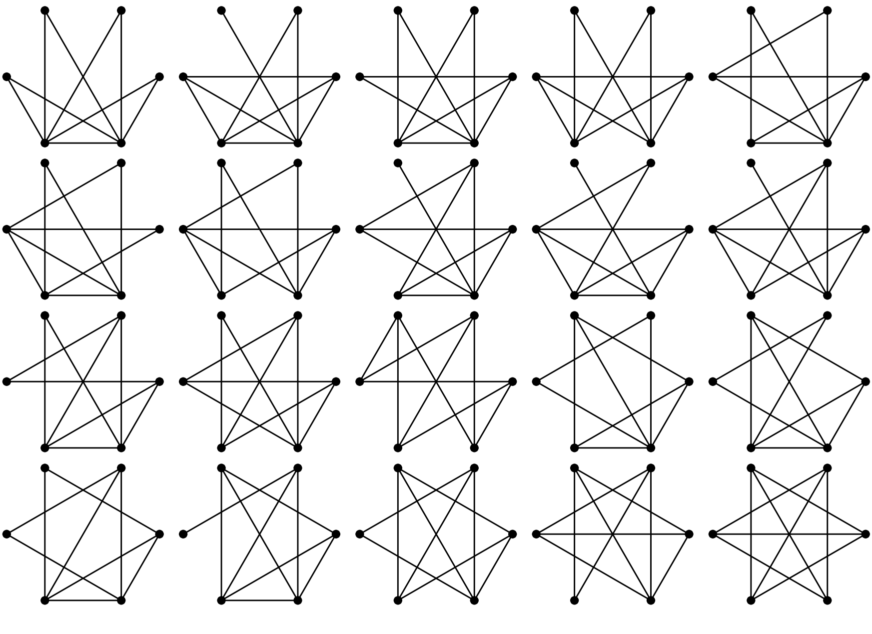

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Bernoulli numbers and prime generating fractions

Roger Thompson

Introduction

Consider the following: Let $a = \dfrac{7293}{7234}$, $n = 8$, and let $b = \dfrac{2^n+1}{2^n-1}$. Calculate

$$d = \left(\frac{a+b}{a-b}\right)^{1/n} = 2.99332690878459\ldots.$$

Round up to the nearest odd integer $c = 3$. Now multiply b by $\dfrac{c^n+1}{c^n-1}$.

Repeating the calculation for d and c using the new value for b, we get $d = 4.9578065406566\ldots$, so $c = 5$. Repeating further, the next few values are as follows.

d	c
6.96977...	7
10.63524...	11
12.73590...	13

d	c
16.12527...	17
18.42182...	19
22.27250...	23

d	c
26.81206...	27

Apart from the last entry, the c are the primes. Starting from $\dfrac{3393195750}{3392780147}$ and $n = 14$, we get these values.

d	c
2.99983...	3
4.99679...	5
6.99901...	7
10.92592...	11

d	c
12.97377...	13
16.75430...	17
18.90488...	19
22.91053...	23

d	c
28.25517...	29
30.75413...	31
36.11388...	37
39.47415...	41

d	c
42.05552...	43
46.21246...	47
51.65032...	53
56.24722...	57

Apart from the last entry, the c are the primes.

So where have these fractions come from, and how many primes can they generate? They are derived from Bernoulli numbers B_n, which crop up all over the place, but particularly in the context of the famous zeta function, defined by $\zeta(s) = \sum_{n=1}^{\infty} \dfrac{1}{n^s}$. It can be shown that for even n,

$$\zeta(n) = \frac{|B_n|\,(2\pi)^n}{2\,n!},$$

where $|x|$ is the absolute value of x.

Bernoulli numbers are rationals, and are zero for odd $n > 1$. The first few non-zero Bernoulli numbers are

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},$$

$$B_{12} = -\frac{691}{2730},\ B_{14} = \frac{7}{6},\ B_{16} = -\frac{3617}{510},\ B_{18} = \frac{43867}{798},\ B_{20} = -\frac{174611}{330}.$$

To understand where the prime generating fractions come from, we need to derive Euler's so-called 'golden key' which links the primes to the zeta function. We can rewrite

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \dots$$

as

$$\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \dots + \frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \dots$$

because all the terms are of the same sign; so

$$\zeta(s) = \frac{1}{2^s}\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\dots\right)+\frac{1}{1^s}+\frac{1}{3^s}+\frac{1}{5^s}+\dots = \frac{1}{2^s}\zeta(s)+\frac{1}{1^s}+\frac{1}{3^s}+\frac{1}{5^s}+\dots;$$

so

$$\begin{aligned}
\zeta(s)\left(1 - \frac{1}{2^s}\right) &= \frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \dots \\
&= \frac{1}{3^s}\left(\frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \dots\right) + \frac{1}{1^s} + \frac{1}{5^s} + \frac{1}{7^s} + \dots \\
&= \frac{1}{3^s}\zeta(s)\left(1 - \frac{1}{2^s}\right) + \frac{1}{1^s} + \frac{1}{5^s} + \frac{1}{7^s} + \dots;
\end{aligned}$$

so

$$\zeta(s)\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right) = \frac{1}{1^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \dots.$$

Continuing in this way, we get

$$\zeta(s) \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right) = 1.$$

Rearranging, we get

$$\zeta(s) \;=\; \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \;=\; \prod_{p \text{ prime}} \frac{p^s}{p^s-1};$$

so

$$\zeta(2s) \;=\; \prod_{p \text{ prime}} \frac{p^{2s}}{(p^s+1)(p^s-1)} \;=\; \zeta(s)^2 \prod_{p \text{ prime}} \frac{p^s-1}{p^s+1};$$

so for even s, we can define

$$R(s) \;=\; \prod_{p \text{ prime}} \frac{p^s+1}{p^s-1} \;=\; \frac{\zeta(s)^2}{\zeta(2s)} \;=\; \frac{\left[\dfrac{|B_s|\,(2\pi)^s}{2\;s!}\right]^2}{\dfrac{|B_{2s}|\,(2\pi)^{2s}}{2(2s)!}} \;=\; \frac{(2s)!B_s^2}{2(s!)^2\,|B_{2s}|},$$

which is rational. Examples:

$$R(2) = \frac{5}{2},\; R(4) = \frac{7}{6},\; R(6) = \frac{715}{691},\; R(8) = \frac{7293}{7234},\; R(10) = \frac{524875}{523833},$$

$$R(12) = \frac{3547206349}{3545461365},\; R(14) = \frac{3393195750}{3392780147},\; R(16) = \frac{15419113345821}{15418642082434}.$$

As you can see, the examples we used initially are the fractions for 8 and 14.

Calculating Bernoulli numbers

How can we calculate B_n for even n? First we will get a rough idea of its size. We have already seen that

$$\zeta(n) = \frac{|B_n|\,(2\pi)^n}{2\;n!};$$

so

$$|B_n| \;=\; \frac{2\;n!\;\zeta(n)}{(2\pi)^n} \;=\; \frac{2\;n!}{(2\pi)^n} \prod_{p \text{ prime}} \left(1 + \frac{1}{p^n-1}\right).$$

Using Stirling's approximation $n! \approx \sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n$, and $\zeta(n) \approx 1$ for large n, we get

$$|B_n| \;=\; 2\sqrt{2\pi n}\left(\frac{n}{2\pi e}\right)^n.$$

Writing

$$B(n,k) \;=\; \frac{2\,n!}{(2\pi)^n} \prod_{p \text{ prime, } k \text{ terms}} \left(1 + \frac{1}{p^n - 1}\right),$$

we see that $B(n,k+1) - B(n,k) \approx 1$ when $q \approx |B_n|$, where q is the $(k+1)$th prime; i.e. when

$$q \;\approx\; \left(2\sqrt{2\pi n}\right)^{1/n} \cdot \frac{n}{2\pi e} \;\approx\; \frac{n}{2\pi e} \;<\; \frac{n}{17}$$

for large n. If $q + c$ is the lowest prime greater than q, then

$$\begin{aligned} B(n,k+2) - B(n,k+1) \;&\approx\; \frac{|B_n|}{(q+c)^n - 1} \;\approx\; \frac{|B_n|}{q^n(1+c/q)^n} \\ &\approx\; \frac{1}{(1+2\pi ec/n)^n} \;\approx\; e^{-17c} \end{aligned}$$

for large n (using $(1+a/n)^n \to e^a$ as $n \to \infty$). In other words, to find the integer part of $|B_n|$, all we have to do is to evaluate $B(n,k+1) - B(n,k)$ until this is less than 1, then iterate once more.

Amazingly, it is possible to calculate the fractional part of B_n, using a theorem of von Staudt and Clausen. This states that:

$$\text{if } n \text{ is even, fractional part}\left[\,|B_{2n}|\,\right] \;=\; \text{fractional part}\left[\sum \frac{1}{p}\right];$$

$$\text{if } n \text{ is odd, fractional part}\left[\,|B_{2n}|\,\right] \;=\; 1 - \text{fractional part}\left[\sum \frac{1}{p}\right],$$

where in both cases the sum is over primes p such that $2n$ is divisible by $p - 1$. For instance,

$$\begin{aligned} \text{fractional part}\left[\,|B_{12}|\,\right] \;&=\; \text{fractional part}\left[\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{13} = \frac{3421}{2730}\right] \\ &=\; \frac{691}{2730}, \qquad \text{and} \quad |B_{12}| \;=\; \frac{691}{2730}. \end{aligned}$$

Also

$$\begin{aligned} \text{fractional part}\left[\,|B_{18}|\,\right] \;&=\; \text{fractional part}\left[\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{19} = \frac{821}{798}\right] \\ &=\; 1 - \frac{23}{798} \;=\; \frac{775}{798}, \\ \text{and} \quad |B_{18}| \;&=\; \frac{43867}{798} \;=\; 54 + \frac{775}{798}. \end{aligned}$$

Properties of the algorithm

Write

$$R(n,k) \ = \prod_{p \text{ prime}, \ k \text{ terms}} \frac{p^n+1}{p^n-1}.$$

If q is such that $R(n) = R(n,k)\dfrac{q^n+1}{q^n-1}$, recalling that $R(s) = \displaystyle\prod_{p \text{ prime}} \frac{p^s+1}{p^s-1}$, then the $(k+1)$th prime must be greater than q. This is where the prime generating properties of $R(n)$ come from.

To find out how far the algorithm presented in the Introduction can generate primes reliably, we examine

$$\frac{R(n)}{\displaystyle\prod_{p \text{ prime}, \, p<q} \frac{p^n+1}{p^n-1}} = \frac{q^n+1}{q^n-1} \prod_{p \text{ prime}, \, p>q} \frac{p^n+1}{p^n-1} = \frac{q^n+1}{q^n-1}(1+\delta_{nq}) = \frac{r^n+1}{r^n-1}$$

say. Then

$$1 + \frac{2}{r^n-1} = 1 + \frac{2}{q^n-1} + \delta_{nq} + \frac{2\delta_{nq}}{q^n-1}; \quad \text{so} \quad \frac{2(q^n-r^n)}{r^n-1} = \delta_{nq}(q^n+1).$$

Then $\dfrac{r}{q} \approx \dfrac{1}{(1+q^n\delta_{nq}/2)^{1/n}}$. Let $r = q - \varepsilon_{nq}$. Then

$$\varepsilon_{nq} \ \approx \ q\left[1 - \frac{1}{(1+q^n\delta_{nq}/2)^{1/n}}\right].$$

If the first primes greater than q are $q+x, q+y, q+z, \ldots$, then for large n,

$$\frac{\delta_{nq}}{2} \ \approx \ \frac{1}{(q+x)^n} + \frac{1}{(q+y)^n} + \frac{1}{(q+z)^n} + \cdots.$$

Then

$$\varepsilon_{nq} \ \approx \ q\left[1 - \frac{1}{\left[1 + \dfrac{1}{(1+x/q)^n} + \dfrac{1}{(1+y/q)^n} + \dfrac{1}{(1+z/q)^n} + \ldots\right]^{1/n}}\right],$$

where n is large, and $q = an$, with a of order of magnitude 1 say, the previous equation becomes

$$\begin{aligned}
\varepsilon_{nq} &\approx an\left[1-\frac{1}{\left[1+e^{-x/a}+e^{-y/a}+e^{-z/a}+\dots\right]^{1/n}}\right]\\
&\approx an\left[1-\frac{1}{1+\dfrac{\log(e^{-x/a}+e^{-y/a}+e^{-z/a}+\dots)}{n}}\right]\\
&\approx an\left[1-\left[1-\frac{\log(e^{-x/a}+e^{-y/a}+e^{-z/a}+\dots)}{n}\right]\right]\\
&= a\log(e^{-x/a}+e^{-y/a}+e^{-z/a}+\dots).
\end{aligned}$$

Since the algorithm rounds up to an odd integer, the maximum allowable ε is 2; so the algorithm fails if $a\log(e^{-x/a}+e^{-y/a}+e^{-z/a}+\dots)>2$, i.e.

$$e^{-0/a}+e^{-x/a}+e^{-y/a}+e^{-z/a}+\dots \;>\; e^{2/a}.$$

In the context of the above sum of exponentials, the densest set of values for $0, x, y, z, \dots$ are

$$0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56, 62, 68, 72, 78, \dots.$$

This is the so-called 'greedy sequence of prime offsets' (have a look at http://oeis.org/search?q=A135311&sort=&language=english). In this case,

$$e^{2/a} \;=\; 1+e^{-2/a}+e^{-6/a}+e^{-8/a}+e^{-12/a}+e^{-18/a}+e^{-20/a}+e^{-26/a}+\dots,$$

giving

$$a \approx 3.30521982918534468219198281147726292090922844812331352586960658810308837694304793608818322\,1$$

(see http://oeis.org/search?q=A217552&sort=&language=english). As a sanity check, for $n=8$, primes should be generated reliably to at least 26.44, and for $n=14$, to 46.27, verifying what we found in the Introduction.

If the algorithm rounds up to the nearest integer, rather than the nearest odd integer, we get

$$a \approx 2.3253739112563394412883668000066830981796655240427536297\,85302384641340990.$$

We now need a definition. An admissible sequence of length k is a set of k integers in ascending order with its first entry zero, such that for each prime $p \le k$, there is at least one integer smaller than k which does not correspond to any member of the set modulo p. For instance, the sequence $0, 2, 6$ is admissible since:

the sequence modulo 2 is 0, 0, 0 (i.e. 1 is not present);

the sequence modulo 3 is 0, 2, 0 (i.e. 1 is not present);

the sequence modulo 5 is 0, 2, 1 (i.e. 3, 4 are not present).

The first k entries of the greedy sequence are admissible for any k. The greedy sequence also has the property that for any k, the $(k+1)$th entry is the lowest possible that makes the first $k+1$ entries admissible.

It is thought, though not proven, that primes p exist such that p plus each member of an admissible sequence are all primes. As an example, we use the first 15 terms of the greedy sequence:

$$
\begin{aligned}
&4436064611739178930 1 + 0,\\
&44360646117391789301 + 2,\\
&44360646117391789301 + 6,\\
&\ldots,\\
&44360646117391789301 + 56
\end{aligned}
$$

are all primes.

Admissible sequences are called dense if their highest entry is the smallest possible. The first k entries of the greedy sequence form a dense admissible sequence for $k \le 15$; 0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56, for example. However, the first k entries of the greedy sequence are not dense for $k > 15$. For example, dense admissible sequences of length 16 have their highest entry 60, whereas the next entry in the greedy sequence is 62. For the purposes of the algorithm, the greedy sequence is densest in the sense that the earlier values in the sequence dominate in the exponential sum derived above.

In conclusion, for a given n, the algorithm generates primes reliably up to at least $3.3052\,n$ for sufficiently large n.

Problem 267.1 – Trigonometric integral

Compute $\displaystyle\int_0^1 \arcsin\cos\sin\arccos x\,dx$.

Towers: Hanoi, Saigon and beyond

Tony Forbes

For integers $p \geq 2$ and $n \geq 1$, we define the function $T_p(n)$ by

$$T_2(n) \;=\; \infty \quad \text{for } n \geq 2, \qquad T_p(1) \;=\; 1 \quad \text{for } p \geq 2, \tag{1}$$

$$T_p(n) \;=\; \min_{1 \leq k \leq n-1} \Big\{ 2T_p(k) + T_{p-1}(n-k) \Big\}. \tag{2}$$

So $T_2(n)$ is completely defined by (1) and it is not difficult to compute $T_3(n)$. Observe that when $p = 3$ the expression inside the curly brackets on the right of (2) is finite only when $k = n - 1$. Hence

$$T_3(n) \;=\; 2T_3(n-1) + T_2(1) \;=\; 2T_3(n-1) + 1,$$

which when combined with (1) gives $T_3(n) = 2^n - 1$. For $p \geq 4$ things are not so easy, at least for large n, and I don't immediately see any alternative to evaluating a large number of minima over k. The results of some computations are tabulated on page 9. To save space I did not include a bulky column for $T_3(n)$.

As you can see from the table, $T_p(n)$ seems to become constant when p gets large, and indeed one might conjecture that

$$T_n(n) \;=\; 2n + 1 \quad \text{and} \quad T_p(n) \;=\; 2n - 1 \;\text{ whenever }\; p > n. \tag{3}$$

At this point it is appropriate to describe a generalization of a familiar puzzle involving the transfer of a tower of discs.

There are p vertical pegs lined up in a row and n discs of distinct radii. The discs have holes in their centres so that they can be threaded on to the pegs. Initially, all n discs are placed on the left-hand peg in descending order of size to form a conical tower, as shown on the left in the picture on page 10. The object of the game is to transfer the entire tower to the right-hand peg. Discs are moved from peg to peg, one at a time according to the following rules:

(i) only a disc at the top of a tower may be moved;

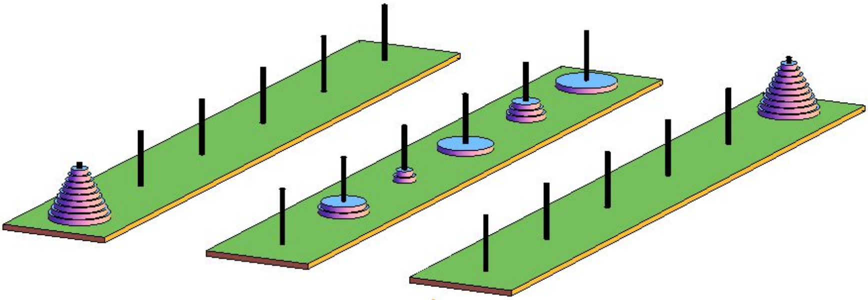
(ii) you must never put a disc on top of a smaller disc.

How many moves are needed?

If $p = 2$, there is not a lot you can do. You are stuck unless $n = 1$ and then it takes only one move to transfer the single disc.

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
4	9	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
5	13	11	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
6	17	15	13	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
7	25	19	17	15	13	13	13	13	13	13	13	13	13	13	13	13	13	13
8	33	23	21	19	17	15	15	15	15	15	15	15	15	15	15	15	15	15
9	41	27	25	23	21	19	17	17	17	17	17	17	17	17	17	17	17	17
10	49	31	29	27	25	23	21	19	19	19	19	19	19	19	19	19	19	19
11	65	39	33	31	29	27	25	23	21	21	21	21	21	21	21	21	21	21
12	81	47	37	35	33	31	29	27	25	23	23	23	23	23	23	23	23	23
13	97	55	41	39	37	35	33	31	29	27	25	25	25	25	25	25	25	25
14	113	63	45	43	41	39	37	35	33	31	29	27	27	27	27	27	27	27
15	129	71	49	47	45	43	41	39	37	35	33	31	29	29	29	29	29	29
16	161	79	57	51	49	47	45	43	41	39	37	35	33	31	31	31	31	31
17	193	87	65	55	53	51	49	47	45	43	41	39	37	35	33	33	33	33
18	225	95	73	59	57	55	53	51	49	47	45	43	41	39	37	35	35	35
19	257	103	81	63	61	59	57	55	53	51	49	47	45	43	41	39	37	37
20	289	111	89	67	65	63	61	59	57	55	53	51	49	47	45	43	41	39
21	321	127	97	71	69	67	65	63	61	59	57	55	53	51	49	47	45	43
22	385	143	105	79	73	71	69	67	65	63	61	59	57	55	53	51	49	47
23	449	159	113	87	77	75	73	71	69	67	65	63	61	59	57	55	53	51
24	513	175	121	95	81	79	77	75	73	71	69	67	65	63	61	59	57	55
25	577	191	129	103	85	83	81	79	77	75	73	71	69	67	65	63	61	59
26	641	207	137	111	89	87	85	83	81	79	77	75	73	71	69	67	65	63
27	705	223	145	119	93	91	89	87	85	83	81	79	77	75	73	71	69	67
28	769	239	153	127	97	95	93	91	89	87	85	83	81	79	77	75	73	71
29	897	255	161	135	105	99	97	95	93	91	89	87	85	83	81	79	77	75
30	1025	271	169	143	113	103	101	99	97	95	93	91	89	87	85	83	81	79
31	1153	287	177	151	121	107	105	103	101	99	97	95	93	91	89	87	85	83
32	1281	303	185	159	129	111	109	107	105	103	101	99	97	95	93	91	89	87
33	1409	319	193	167	137	115	113	111	109	107	105	103	101	99	97	95	93	91
34	1537	335	201	175	145	119	117	115	113	111	109	107	105	103	101	99	97	95
35	1665	351	209	183	153	123	121	119	117	115	113	111	109	107	105	103	101	99
36	1793	383	225	191	161	127	125	123	121	119	117	115	113	111	109	107	105	103
37	2049	415	241	199	169	135	129	127	125	123	121	119	117	115	113	111	109	107
38	2305	447	257	207	177	143	133	131	129	127	125	123	121	119	117	115	113	111
39	2561	479	273	215	185	151	137	135	133	131	129	127	125	123	121	119	117	115
40	2817	511	289	223	193	159	141	139	137	135	133	131	129	127	125	123	121	119
41	3073	543	305	231	201	167	145	143	141	139	137	135	133	131	129	127	125	123
42	3329	575	321	239	209	175	149	147	145	143	141	139	137	135	133	131	129	127
43	3585	607	337	247	217	183	153	151	149	147	145	143	141	139	137	135	133	131
44	3841	639	353	255	225	191	157	155	153	151	149	147	145	143	141	139	137	135
45	4097	671	369	263	233	199	161	159	157	155	153	151	149	147	145	143	141	139
46	4609	703	385	271	241	207	169	163	161	159	157	155	153	151	149	147	145	143
47	5121	735	401	279	249	215	177	167	165	163	161	159	157	155	153	151	149	147
48	5633	767	417	287	257	223	185	171	169	167	165	163	161	159	157	155	153	151
49	6145	799	433	295	265	231	193	175	173	171	169	167	165	163	161	159	157	155
50	6657	831	449	303	273	239	201	179	177	175	173	171	169	167	165	163	161	159

The case $p = 3$ is *The Tower of Hanoi*, and the solution, $H(n)$, is well known. First transfer the top $n-1$ discs from peg 1 to peg 2 ($H(n-1)$ moves) and then transfer them to peg 3 (another $H(n-1)$ moves) after placing the largest disc on to peg 3 (1 move). Therefore $H(n) = 2H(n-1)+1$, which when combined with $H(1) = 1$ gives $H(n) = 2^n - 1$. The puzzle is solved and $H(n) = T_3(n)$.



We discussed $p = 4$ in M500 **163**, where it was called *The Tower of Saigon*. There we described the following strategy. (i) Transfer the top k discs to a spare peg using *Tower of Saigon* moves; (ii) put the remaining $n-k$ discs on the destination peg using *Tower of Hanoi* moves (because the peg that now contains the first k discs is not available); (iii) transfer the k smallest discs to the destination peg (*Tower of Saigon* moves). We choose k to minimize the number of moves, $S(n)$. Thus $S(1) = 1$ and

$$S(n) \;=\; \min_{1 \le k \le n-1} \left\{ 2S(k) + H(n-k) \right\};$$

hence $S(n) = T_4(n)$. This is the *Frame–Stewart algorithm* [1]. However, it is has never been proved that $T_4(n)$ represents the best solution. The Tower of Saigon problem is therefore still open—although it is known that $T_4(n)$ is optimal for $n \le 30$.

From the table we see that the differences $\Delta T_4(n) = T_4(n) - T_4(n-1)$ look like this:

$$\Delta T_4(n) \;=\; 2, 2, 4, 4, 4, 8, 8, 8, 8, 16, 16, 16, 16, 16, 32, 32, 32, 32, 32, 32, 64, \ldots .$$

In [2] it is proved that the pattern continues indefinitely, and the closed formula (also in [2]) was obtained by Peter Fletcher in M500 **166**:

$$T_4(n) \;=\; 2^r \left(n - \frac{r(r-1)}{2} - 1 \right) + 1, \quad \text{where } r \;=\; \left\lceil \frac{\sqrt{8n+1} - 3}{2} \right\rceil. \qquad (4)$$

Applying the same strategy to larger puzzles provides the best known solutions: $T_p(n)$ for p pegs, $p \geq 5$. But since they all involve the unproven $T_4(n)$ for four pegs we cannot be sure that they are optimal for large n. It occurred to me that the procedure for 5 pegs, say, might be improved by having two parameters j and k. We transfer the top k discs to a spare peg using 5-peg moves, transfer the next j discs to another spare peg using *Saigon* moves and the final $n-j-k$ discs to the destination peg with *Hanoi* moves. The new solution satisfies

$$\overline{T}_5(n) \;=\; \min_{1\leq k\leq n-1,\ 1\leq j\leq n-k} \left\{2\overline{T}_5(k) + 2S(j) + H(n-k-j)\right\};$$

but $2S(j) + H(n-k-j) \geq S(n-k)$ by (2) and we have gained nothing.

For $p \geq 5$, the differences appear to exhibit the same kind of pattern involving powers of 2. Thus, writing 2, 2, 2 as $2:3$, for example, we have

$$\begin{aligned}
\Delta T_5(n) \;&=\; 2:3,\ 4:6,\ 8:10,\ 16:15,\ 32:21,\ 64:28,\ \ldots,\\
\Delta T_6(n) \;&=\; 2:4,\ 4:10,\ 8:20,\ 16:35,\ 32:56,\ 64:84,\ \ldots,\\
&\ldots,\\
\Delta T_p(n) \;&=\; 2:p-2,\ 4:\binom{p-1}{2},\ 8:\binom{p}{3},\ 16:\binom{p+1}{4},\ \ldots
\end{aligned}$$

and in general 2^r occurs $\binom{p+r-3}{r} = \binom{p+r-3}{p-3}$ times in the sequence. This also holds for $p = 3$ since $T_3(n) - T_3(n-1) = 2^{n-1}$ and the binomial coefficient is always 1.

Unfortunately I have no proof that when $p \geq 5$ the differences $\Delta T_p(n)$ actually have the stated properties. A gap which the reader might like to fill. In view of this ignorance, let us define a new function $U_p(n)$.

Let $U_2(n) = T_2(n)$. For $p \geq 3$, let $U_p(1) = 1$ and when $n \geq 2$ we assume that the differences $\Delta U_p(n) = U_p(n) - U_p(n-1)$ have the properties stated above. Thus for $p \geq 3$ and $n \geq 2$,

$$\Delta U_p(n) \;=\; 2:\binom{p-2}{1},\ 4:\binom{p-1}{2},\ 8:\binom{p}{3},\ \ldots,\ 2^r:\binom{p+r-3}{p-3},\ \ldots.$$

Then we know that $U_3(n) = T_3(n)$ and $U_4(n) = T_4(n)$, and in the absence of a proof we conjecture that $U_p(n) = T_p(n)$ when $p \geq 5$. To obtain a closed formula for $U_p(n)$ we split n into two parts, m and $n-m$, where

$$m \;=\; 1 + \sum_{i=1}^{r-1} \binom{p-3+i}{p-3} \;=\; \binom{p-3+r}{p-2},$$

and r is the positive integer satisfying

$$\binom{p-3+r}{p-2} \;<\; n \;\le\; \binom{p-2+r}{p-2}. \tag{5}$$

Then by starting with 1 and summing the differences we have

$$U_p(n) \;=\; 1+\sum_{j=1}^{r-1} 2^j \binom{p-3+j}{p-3} + 2^r\left(n-\binom{p-3+r}{p-2}\right), \quad p\ge 3,\; n\ge 2,$$

where r satisfies (5). The main advantage of course is that we avoid all those computations to determine the minima in (2).

To see how it works, let $p=3$. The inequalities (5) defining r reduce to $r=n-1$ and so

$$U_3(n) \;=\; 1+\sum_{j=1}^{n-2} 2^j + 2^{n-1} \;=\; 2^n-1 \;=\; T_3(n) \;=\; H(n).$$

When $p=4$ it is a little more complicated. Now the condition (5) becomes $(r+1)r < 2n \le (r+2)(r+1)$, and to get an explicit value for r we solve $2n=(r+2)(r+1)$ and round up. So $r=\left\lceil(\sqrt{8n+1}-3)/2\right\rceil$ and

$$U_4(n) \;=\; 1+\sum_{j=1}^{r-1} 2^j(j+1) + 2^r\left(n-\frac{r(r+1)}{2}\right).$$

Since the sum evaluates to $2^r(r-1)$, we obtain the same formula as in (4):

$$U_4(n) \;=\; 2^r\left(n-\frac{r(r-1)}{2}-1\right)+1, \quad \text{where } r \;=\; \left\lceil\frac{\sqrt{8n+1}-3}{2}\right\rceil.$$

By a similar kind of argument one can obtain formulae for further values of p:

$$\begin{aligned}
U_5(n) &= 2^r\left(n-\frac{(r-1)(r^2+r+6)}{6}\right)-1,\\
U_6(n) &= 2^r\left(n-\frac{(r-1)r(r^2+3r+14)}{24}-1\right)+1,\\
U_7(n) &= 2^r\left(n-\frac{(r-1)(r^4+6r^3+31r^2+26r+120)}{120}\right)-1,\\
U_8(n) &= 2^r\left(n-\frac{(r-1)r(r^4+10r^3+65r^2+140r+444)}{720}-1\right)+1
\end{aligned}$$

and so on, with r defined by

$$r \;=\; \lceil x \rceil, \qquad \text{where} \quad \binom{p-2+x}{p-2} = n \quad \text{and} \quad x \ge 0.$$

However, apart from $p = 3$, 4, 5, 6, 8 and 10, I do not know of any case where is possible to get a closed formula for r in terms of n. Cases $p = 8$ and 10 involve polynomials with messy solutions but for $p = 6$ the quartic is quite easy to solve:

$$\binom{x+4}{4} = n \;\text{ and }\; x \ge 0 \quad \Rightarrow \quad x = \frac{\sqrt{5+4\sqrt{24n+1}}-5}{2}.$$

So for computing $U_6(n)$ we set $r = \left\lceil \frac{1}{2}\left(\sqrt{5+4\sqrt{24n+1}}-5\right)\right\rceil$. And by solving $(x+3)(x+2)(x+1) = 6n$ we obtain a slightly more complicated r for computing $U_5(n)$:

$$r \;=\; \left\lceil \sqrt[3]{3n+\sqrt{9n^2-\frac{1}{27}}}+\sqrt[3]{3n-\sqrt{9n^2-\frac{1}{27}}}-2 \right\rceil.$$

Finally, we can offer a simple explanation of (3) in terms of the puzzle. If $p > n$, there are enough spare pegs to store $n-1$ discs, one per peg. So we do $n-1$ moves to get rid of the top $n-1$ discs, one more to transfer the nth disc and another $n-1$ to pile the rest on top of it; $2n-1$ altogether. This corresponds to making the choice $k = 1$ at each stage of the recursion formula (2):

$$\begin{aligned} T_p(n) \;&=\; 2T_p(1) + T_{p-1}(n-1) \;=\; 2 + T_{p-1}(n-1) \;=\; 4 + T_{p-2}(n-2) \\ &=\; \ldots \;=\; 2(n-2) + T_{p-n+2}(2) \;=\; 2(n-1) + T_{p-n+1}(1) \;=\; 2n-1 \end{aligned}$$

since $T_{p-n+1}(1) = 1$. Similarly but with a bit more work we can derive

$$T_p(n) \;=\; 4n - 2p + 1 \qquad \text{for} \;\; p \le n \le \frac{p(p-1)}{2}.$$

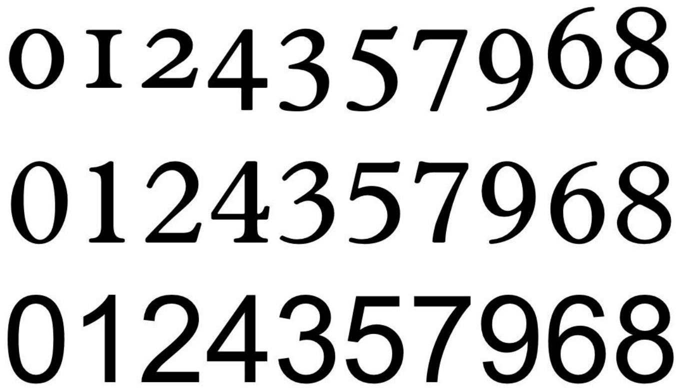
[1] B. M. Stewart and J. S. Frame, Solution to advanced problem 3819. *American Mathematical Monthly* **48** 3 (March 1941), 2169.

[2] Paul K. Stockmeyer, Variations on the Four-Post Tower of Hanoi Puzzle, *Congressus Numerantium* **102** (1994), 3–12.

Legibility of numerals

Ralph Hancock

When I was at school, an unimaginably long time ago, we had books of ‘log tables’ – not just logarithms, but also trigonometrical ratios – and consulted them extensively. These tables were printed with what I now know are called ‘old style figures’, extending above and below the line as in the first line of this illustration.

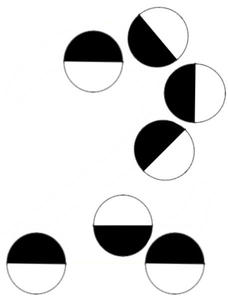


Legibility is particularly important in tables of figures, whose grey, uniform look makes it easy to misread them. It’s obvious that the way some numerals go above and some below the line makes it easier to recognise them once you have accustomed yourself to their arbitrary conventional arrangement. Try putting your copy of M500 at the far end of the room and looking at it; I think you will agree that in the old style line the digits are more distinguishable than in the other two.

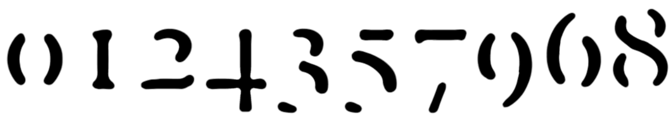
While you are doing this, also compare the second and third lines, which use standard ‘lining’ numerals. Which is more legible? In particular look at the last three in the line. Do you find that in the third example it’s easy to confuse 9, 6 and 8?

This brings up another factor in legibility, one that goes right to the core of the phenomenon of visual perception. When we look at any object, whether it is a digit or a rhinoceros, our eyes scanning over it pick up only a hasty and partial impression. Most of what we see is an image built up by the brain, and this image is built up in stages over a fraction of a second.

The first and fastest stage is the detection of edges between light and dark areas at various angles, and there are specialised cells in the visual cortex which do just this. If an object has strongly contrasted light and dark areas of adequate width, it takes only a few of these cells to produce an impression of its shape. Here is an example of how only seven cells can give absolutely unambiguous information that the figure is a 2.



The 'serif' fonts in the first two lines, with their clearly defined thick black regions whose positions are unique to that numeral, stimulate these cells much more strongly than the outlines of even thickness in the 'sans serif' font in the third line. The thin parts of the numeral are less well picked up by this mechanism, but that is is unimportant: only the thick parts have to be registered for the number to be read. The brain very quickly perceives something like this for the whole line, and this is enough to make it legible.

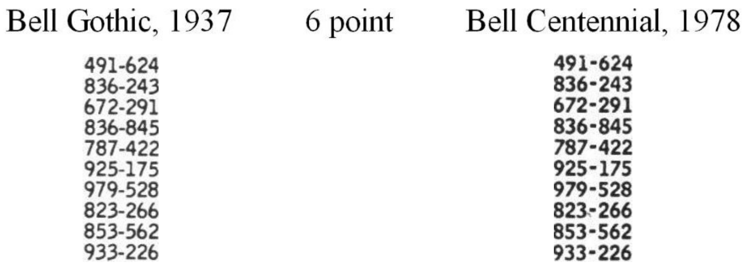


Sans serif numerals need further brain processing in greater detail for their shape to be read, so recognition of such fonts is slower and legibility is reduced. The difference is particularly important with numbers. Words are recognised as much by their general shape as by the details of the letters in them, but numerals can occur in any order and there is no such thing as a 'familiar' number.

There is a problem with old style figures, however. They look fine printed at full size, but when you use them at a small size for exponents their differing height gives a messy look. For the same reason they are unsuitable for use in formulae. The fonts used by professional publishers which include old style figures often have a lining set as well, and the publishing software has a switch that can choose one or the other. The first two lines in the first illustration use two styles of figures from the same font, Monotype Plantin. (The third line is in Arial.)

There is one place where the superior legibility of serif fonts is unavailable, and this is in telephone directories. These are printed at high speed on cheap, lumpy paper at a tiny size. Here you have to use a sans serif font, because the thin lines of serif fonts will simply disappear. But even here there is scope for increasing legibility.

The following example is from the directories printed for the United States telephone company Bell. Until 1937 cities had printed their directories in any font that appealed to them, but in 1937 Bell imposed a uniform style, the Bell Gothic font designed for the purpose by C. H. Griffith. It worked very well while the directories were printed with traditional metal type.



However, in the 1970s the much faster web offset printing method came into use, and the old font was not up to the demands it made. Bits of letters, and even of the bold numerals, disappeared; numbers were poorly legible and dialling mistakes were common. So Bell asked the type designer Matthew Carter to come up with something more suitable, and he designed the Bell Centennial font. You will notice that the bold type used for the numbers is quite a lot heavier, which makes it both more legible and more resistant to rough printing. He also managed to condense the lettering in the light variant of the font used for addresses while retaining the thickness of the lines, which keeps many entries from overrunning on to a second line and gives a small but worthwhile saving of space.

There are a few sans serif fonts with the more legible old style figures, but these would be unsuitable for use in the cramped line spacing of a directory, and the disparity between old shapes and modern styling seems a bit incongruous.

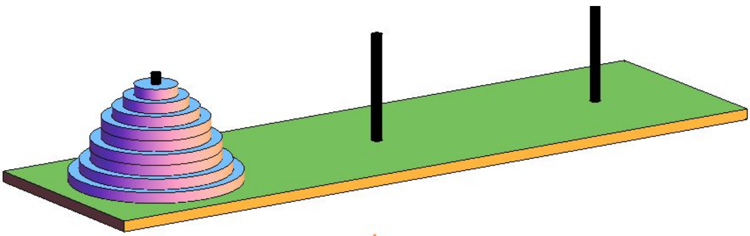
Problem 267.2 – Hanoi revisited

Tony Forbes

This is not quite the same as the traditional *Tower of Hanoi* puzzle. There are three vertical pegs lined up in a row and n discs. They have the properties stated on page 8. The object of the game is to transfer the entire tower to the right-hand peg by moving discs from peg to peg, one at a time according to the usual rules:

only a disc at the top of a tower may be moved;
you must never put a disc on top of a smaller disc.

For definiteness, assume there are t distinct radii, $1, 2, \ldots, t$, and that radius i occurs r_i times, so that $n = r_1 + r_2 + \cdots + r_t$.



(i) How many moves are needed?

(ii) How many distinct types of starting position are there? For example, suppose there are four discs. Then there are eight types, namely 1111, 2111, 2211, 2221, not 2222 because it is essentially the same as 1111, 3211, 3221, 3321 and 4321.

(iii) Depending on the distribution of the radii, the number of moves varies from n (when all discs have the same radius) to $2^n - 1$ (all different). Which numbers in this interval are represented?

Problem 267.3 – Floor and ceiling

Tony Forbes

Show that for positive integer n, these two functions are identical:

$$f(n) = 2^k \left(n - \frac{k(k-1)}{2} - 1\right), \quad \text{where} \quad k = \left\lfloor \frac{\sqrt{8n+1} - 1}{2} \right\rfloor,$$

$$c(n) = 2^r \left(n - \frac{r(r-1)}{2} - 1\right), \quad \text{where} \quad r = \left\lceil \frac{\sqrt{8n+1} - 3}{2} \right\rceil.$$

Euler's Basel Problem

Peter Griffiths

We have

$$\frac{\sin u\pi}{u\pi} = 1 - \frac{(u\pi)^2}{3!} + \frac{(u\pi)^4}{5!} - \frac{(u\pi)^6}{7!} + \cdots.$$

To convert this $(\sin u\pi)/(u\pi)$ series from an alternating summation series into a product series, apply the Newtonian summation formula

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots,$$

then multiply through by u^2 giving

$$\frac{u^2\pi^2}{6} = u^2 + \left(\frac{u}{2}\right)^2 + \left(\frac{u}{3}\right)^2 + \cdots.$$

To convert this summation series to a product series, deduct each term from 1:

$$\frac{\sin u\pi}{u\pi} = \left[1 - \left(\frac{u}{1}\right)^2\right]\left[1 - \left(\frac{u}{2}\right)^2\right]\left[1 - \left(\frac{u}{3}\right)^2\right]\cdots.$$

In the case of the appropriate antilog series, each term should be added to 1. A similar procedure can be applied to cosines except that the appropriate angle should be $u\pi/2$, not $u\pi$, and the Newtonian summation formula should be

$$\frac{u^2\pi^2}{8} = u^2 + \left(\frac{u}{3}\right)^2 + \left(\frac{u}{5}\right)^2 \cdots.$$

To convert this summation series to a product series, deduct each term from 1:

$$\cos\left(\frac{u\pi}{2}\right) = \left[1 - \left(\frac{u}{1}\right)^2\right]\left[1 - \left(\frac{u}{3}\right)^2\right]\left[1 - \left(\frac{u}{5}\right)^2\right]\cdots.$$

In the case of the appropriate antilog series, each term should be added to 1. This is not a proof in the Euclidean sense, since Euler is assuming what his contemporaries thought should be proved, namely the Newtonian formulae. Nevertheless the beneficial consequences of this non-proof are more important than the strict rules of proof.

- 4m x 5m/13ft $1^{31}/_{64}$" x 16ft $4^{27}/_{64}$" (approx)

From the packaging of an inexpensive dust sheet.

Sent by Jeremy Humphries.

Taxicabs and mathematics

Eddie Kent

On page 12 of his book *Ramanujan*, New York 1940, G. H. Hardy recounts an anecdote concerning the Indian mathematician Srinivasa Ramanujan. It is a startling tale and has become world famous, even appearing in *The Simpsons*. While I would hesitate to call Hardy a liar I've always felt that the story requires a considerable suspension of disbelief. Maybe, though, it was just meant to illustrate the way Ramanujan thought.

I was startled indeed when I came to page 65 of *The Cheyne Mystery* by Freeman Wills Crofts, Penguin 1926, reprinted 1960. The hero Maxwell Cheyne wishes to know where his quarry are staying. He knows the number of the taxi they used so he waits for it to return to the rank. At last!: 'Taxi Z 1729 suddenly appeared and drew into position.'

Hardy's anecdote is as follows: 'I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."'

This is now known as the Hardy–Ramanujan number and it is one of the so-called 'taxicab' numbers, those that can be expressed in n different ways as the sum of two cubes. It does appear in a much earlier notebook kept by Ramanujan so he clearly didn't have to do an instant calculation. But on the other hand it shows he cared about such things: Or that Hardy cared about him.

And Freeman Wills Crofts: could this possibly have been a coincidence? In 1926? Ramanujan arrived in England in 1914 and returned to India in 1919, where he soon died. It is possible that the taxicab incident was known and talked about in the 20s. And it got into the crime novel as a neat little insider joke. Surely this is more likely than that Wills Crofts picked the number at random out of the ether. Incidentally, Ramanujan's widow Janakiammal died in 1994.

Problem 267.4 – Bernoulli numbers

Let n be a positive even integer and let $b(n) = (2\,n!)^{1/n}/(2\pi)$. Show that

$$\left\lfloor b(n)^n \sum_{k=1}^{\lfloor b(n) \rfloor} \frac{1}{k^n} \right\rfloor \;=\; \lfloor |B_n| \rfloor \;\text{ or }\; \lfloor |B_n| \rfloor - 1,$$

or find a counter-example. Here, B_n is the nth Bernoulli number. The problem is suggested by Roger Thompson's article in this issue. See page 1 *et seq.* for the definition and properties of B_n.

Issue 264 – How to do mathematical research

J. J. Reynolds

Issue **264** of M500 includes a short section entitled 'How to do mathematical research.' I am not quite sure whether these are serious suggestions. The cost of computer time has been negligible for many years, and a modest home PC will do more arithmetic in an hour than an individual worker could do in a lifetime.

Proving that a particular mathematical object with specific properties does not exist may take many hours of work. To me, it would be far more sensible to spend a few minutes writing a program to try to find an example. While the program is running, you can have a cup of tea, a cake or a biscuit, a meal, take your dog for a walk—whatever. If you find an example, then you will have saved a lot of time. If you don't find one, then is the time to try to prove it doesn't exist.

I worked for about twenty years in aircraft design. The conventional wisdom—which I believed and still believe—was that you could learn more from projects that failed than from those that succeeded. Don't throw away work you have done. Keep it filed away where you can find it—on a computer. If you want to find interesting mathematical objects, I would suggest you look at computer language structures like those which were introduced in C to overcome problems with FORTRAN IV. In FORTRAN IV, it was difficult to link different types of data. An employee would have a name, an age, a gender, a salary and so on. All elements of a FORTRAN array had to be of the same type. In C, the `struct` statement was used to overcome this problem, but there are other potential uses for this which could provide significant entertainment for mathematicians.

Array dimensions were also a problem in FORTRAN IV. It was necessary to specify a maximum size for an array. There were two problems. If the array was too small, a real problem might occur which the program couldn't solve. On the other hand, the array size for the largest problems might not compile on a less powerful computer. The linked list, using the `struct` statement, was introduced so that array sizes could be increased on more powerful computers, without needing separate programs on different computers. A three-dimensional linked list could be used to model the earth's atmosphere—not easily, I agree, but I am sure that properties of such lists would keep pure mathematicians amused for a while.

We all know that computers are manufactured, and therefore in the eyes of some are not virtuous. Pencils and paper are also manufactured. ...

Problem 267.5 – Quadruples

Tony Forbes

Show that the number of integer quadruples (a, b, c, d), where $1 \le a \le b \le c \le d$ and $a + b + c + d = n$, is given by

$$\left\lfloor \frac{n^3}{144} + \frac{n^2}{48} - \frac{n \bmod 2}{16} + \frac{1}{2} \right\rfloor .$$

The expression n mod 2 is to be interpreted as 0 if n is even, 1 if n is odd.

This extension of Problem 265.6 (Show that there are $\lfloor n^2/12 + 1/2 \rfloor$ triples (a, b, c), where $1 \le a \le b \le c$ and $a + b + c = n$) is very interesting. Again, and somewhat surprisingly, there is a closed formula which one can discover just by experimentation. All I did was generate a list of values obtained by counting (a, b, c, d) that satisfy the conditions and then fit two nice-looking cubics to the data. The corresponding formula for pairs is clearly $\lfloor n/2 \rfloor$.

M500 Mathematics Revision Weekend 2016

The M500 Revision Weekend 2016 will be held at

Yarnfield Park Training and Conference Centre,
Yarnfield, Staffordshire ST15 0NL

between Friday 13th to Sunday 15th May 2016.

The standard cost, including accommodation (with en suite facilities) and all meals from dinner on Friday evening to lunch on Sunday is £285. The standard cost for non-residents, including Saturday and Sunday lunch, is £170. There will be an early booking period up to the 16th April with a discount of £20 for both members and non-members.

Members may make a reservation with a £25 deposit, with the balance payable at the end of February. Non-members must pay in full at the time of application and all applications received after the 28th February must be paid in full before the booking is confirmed. Members will be entitled to a discount of £15 for all applications.

A shuttle bus service will be provided between Stone station and Yarnfield Park on Friday and Sunday. This will be free of charge, but seats will be allocated for each service and must be requested before 1st May. There is free on-site parking for those travelling by private transport. For full details and an application form see the Society's web site at www.m500.org.uk.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most undergraduate and postgraduate mathematics OU modules, subject to the availability of tutors and sufficient applications.

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04.03

Front cover The twenty connected 6-vertex graphs with 9 edges.

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