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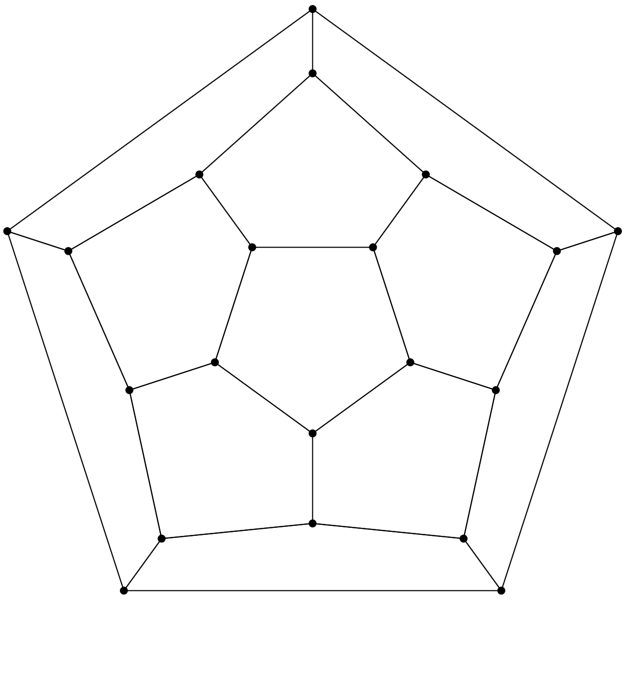

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards.

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Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation.

Fun with Pascal's triangle

Paul Jackson

Consider Pascal's triangle.

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & & & & & & \ldots & & & & & & &
\end{array}$$

Let the diagonal elements be called $t_n = t_{n,r}$; so the triangle numbers would be t_3 for example. Of course these each form infinite sequences. Then in general $t_{n,r} = \binom{n-1+r}{r}$. We shall show that

$$\sum_{r=0}^{\infty} \frac{1}{t_{n,r}} = \frac{n-1}{n-2} \quad \text{for } n \geq 3.$$

We also know that the sums of the reciprocals of t_1 and t_2 are both divergent. Using elementary methods we find that for $n = 3, 4, 5$, the sums are $2/1$, $3/2$, $4/3$ respectively, so we seek to generalize on these first results. To find these sums we invert the general formula for t_n and after factoring out the common numerator we get a general term that is the reciprocal of a product of factors in arithmetic progression.

Special case

So as an example, for $n = 5$ we have a general term

$$\frac{1}{t_5} = \frac{1}{\binom{4+r}{r}} = \frac{4!}{(r+1)(r+2)(r+3)(r+4)}$$

for $r = 0, 1, 2, \ldots$. Now just to make the structure a little clearer we replace $r+1$ by r for $r = 1, 2, 3, \ldots$. That is we have redefined r to begin at $r = 1$ instead of zero. Then we need to express $\dfrac{1}{r(r+1)(r+2)(r+3)}$ in partial fractions. So we have a general term:

$$\frac{1}{3!}\left(\frac{1}{r} - \frac{3}{r+1} + \frac{3}{r+2} - \frac{1}{r+3}\right).$$

Then when we sum to infinity we find all except the first few terms disappear; in fact we get the following pattern for the terms inside the brackets:

$$\begin{array}{ll} & \frac{1}{1} - \frac{3}{2} + \frac{3}{3} - \frac{1}{4} \\ + & \frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5} \\ + & \frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6} \\ + & \frac{1}{4} - \ldots. \end{array}$$

We see that the terms diagonally starting from the top right corner must equal zero, as they all have the same denominator and the numerators are just the value of $(1-x)^3$ when $x = 1$. So we see that we are just left with the first upper triangle of terms. Now, curiously, when we sum these we find the value is just $1/3$; hence we see that the sum we require is just $4/3$. For the general case we find the same pattern occurring; that is, the first upper triangle of terms has value $\dfrac{1}{n-2}$. We shall next prove this is the case.

Toward the general case

In general using the above example we would need to express

$$\frac{1}{r(r+1)(r+2)\ldots(r+k)}$$

in partial fractions; that is, we need

$$\frac{1}{r(r+1)(r+2)\ldots(r+k)} = \frac{A_0}{r} + \frac{A_1}{r+1} + \frac{A_2}{r+2} + \cdots + \frac{A_k}{r+k}. \qquad (1)$$

We have k terms and k A_i coefficients to evaluate. Say we wished to evaluate an arbitrary coefficient A_i. We simply multiply (1) throughout by $r+i$. In the left-hand side of (1) this term will cancel out, and on the right-hand side each term except the one with coefficient A_i will be multiplied by $r+i$. Now when we put $r = -i$ into both sides of our expression, we are left with the value of A_i alone, the other terms on the right-hand side being each zero. So for each $i = 0, 1, 2, \ldots, k$ we can evaluate (1) and nowhere on the left-hand side will we get a zero in the denominator.

Also in general the sign of A_i depends on the sign of the denominator

$$r(r+1)(r+2)\ldots(r+i-1)(r+i+1)(r+i+2)\ldots(r+k)$$

in the left-hand side of (1). We see that the product of terms after $r+i-1$ is positive, and as there are i terms before and including this term which will all be negative, when we substitute for $i=-1$, we see the sign of A_i is just $(-1)^i$.

So we find

$$A_i = \frac{(-1)^i}{i(i-1)(i-2)\cdot\cdots 3\cdot 2\cdot 1\cdot 1\cdot 2\cdot 3\cdot (k-i)} = \frac{(-1)^i}{i!(k-i)!} = \frac{1}{k!}\binom{k}{i}.$$

Hence if we let $B_i = \binom{k}{i}$, then $a_i = \dfrac{B_i}{k!}$. So we see that in our general expression the numerators in the array of fractions with denominators in arithmetic progression starting $1, 2, 3, \ldots$, as in the case for $n=5$ above, will be binomial coefficients. Hence each diagonal of n terms will sum to zero, as in the case illustrated above, and thus the sum of the series will depend on the sum of the upper left triangle of terms, which we need to show is always equal to $\dfrac{1}{n-2}$.

Now when we form upper right triangles of terms as we did above, for the cases $n=5$, $n=6$, and $n=7$, and in each case when we sum the diagonals of non-disappearing terms we find for $n=6$, that is $k=4$, we have $\frac{1}{1}-\frac{3}{2}+\frac{3}{3}-\frac{1}{4}$, being the first three non-disappearing terms for the triangle for $k=3$, plus the extra term. The pattern is the same for the next few values of k. It seems possible that we can generate the next sum of the upper right triangle from the previous one. This should not be too surprising as we are dealing with binomial coefficients, in the form of Pascal's triangle, that seems to be mapped in slightly strange ways. It would appear that as we manipulate this structure the same coefficients keep popping up.

Now consider the diagonal elements in the triangle for $k=4$. The first diagonal sum d_1 is 1, the second diagonal sum d_2 is $\frac{1}{2}-\frac{4}{2}=\frac{-3}{2}$, the third $d_3 = \dfrac{1-4+6}{3} = \dfrac{3}{3}$, and lastly $d_4 = \dfrac{1-4+6-4}{4} = \dfrac{-1}{4}$. But of course $4=1+3$, so we are undoing the formation of the next line of coefficients in Pascal's triangle. In fact we can reverse the procedure we have just outlined to generate the upper right triangle of terms for $k=4$ from $k=3$. So we have $\frac{1}{1}-\frac{3}{2}+\frac{3}{3}$ and we expand these to give $\frac{1}{1}$. The next diagonal has two terms. So from Pascal's triangle $1+3=4$, $-3=1-4$; so this is the next coefficient to be expanded: $\dfrac{-3}{2} = \dfrac{1-4}{2} = \dfrac{1}{2}-\dfrac{4}{2}$. The next term we need is the sum of $3+3=6$; so $3=-3+6$, using the last starting number. Thus

$\frac{3}{3} = \frac{-3+6}{3} = \frac{1-4+6}{3}$, as $-3 = 1-4$ from above, we then plug 3 into the next diagonal expansion as $-1 = 3-4$.

For the last diagonal, we know it has k terms, and we know that the sum of the terms $1-4+6-4+1$ is zero, so the sum of $1-4+6-4 = -1$; therefore the sum of the upper right diagonal for $k=4$ is just the sum of $\frac{1}{1} - \frac{3}{2} + \frac{3}{3}$ and $\frac{-1}{4}$, which is $\frac{1}{4}$, as required.

So in this particular case we will be able to find the value of the upper triangle for $k=4$ if we know the value of the first row of the upper triangle for $k=3$. This is because we can easily deduce the value of the last diagonal for the upper triangle for $k=4$, using the binomial theorem in the expansion of $(1-x)^4$ when $x=1$.

To generalize this we need some notation. Let the first row of the upper triangle for k be denoted by R_k, and the upper triangle and last diagonal for $k+1$ be denoted by U_{k+1} and d_{k+1} respectively. Then the value of d_{k+1} is just the value of $(1-x)^{k+1}$ when $x=1$ (which is zero of course), without the last term this being $\frac{\mp 1}{k+1}$.

Hence $U_{k+1} = R_k - d_{k+1}$. So we could solve this problem if we knew in general the value of R_k. This is easy to find for small values of k, which we tabulate below.

k	1	2	3	4	5	6	7	8
R_k	1	0	$\frac{1}{2}$	0	$\frac{1}{3}$	0	$\frac{1}{4}$	0

This again seems to imply interesting symmetry properties of the expressions for R_k and the binomial coefficients, but we leave that aside to solve the current problem. Now recall the example for $k=3$; at the start of the above discussion we had the expression

$$\frac{1}{r(r+1)(r+2)(r+3)} = \frac{1}{3!}\left(\frac{1}{r} - \frac{3}{r+1} + \frac{3}{r+2} - \frac{1}{r+3}\right)$$

expressed on the right-hand side in partial fractions. When we let $r=1$ and rearrange we get

$$\frac{3!}{4!} = \frac{1}{1} - \frac{3}{2} + \frac{3}{3} - \frac{1}{4}.$$

But this is just R_3 with the extra term! So we see that $R_3 = \frac{2}{4} = \frac{1}{2}$ as we expected. It is not hard to extend this example to the general case, from which we find that R_k has value zero if k is even, and value $\dfrac{1}{2k-1}$ if k is odd. Remember from above that $A_i = \dfrac{B_i}{k!}$; so the B_i are the binomial coefficients, or numerators in R_k. Therefore we see that the result $U_{k+1} = \dfrac{1}{k} = \dfrac{1}{n-2}$ follows, and so does what we set out to initially show; that is,

$$\sum_{r=0}^{\infty} \frac{1}{t_{n,r}} = \frac{n-1}{n-2} \qquad \text{for } n \geq 3.$$

Finally we note in passing the similarity of this expression to

$$\sum_{r=0}^{\infty} (-1)^r t_{n,r} x^r = \sum_{r=0}^{\infty} (-1)^r \binom{n-1+r}{r} x^r = \frac{1}{(1+x)^n},$$

which if valid in the context of infinite series, would give, when we put $x = 1$, the value of the reciprocal of the nth row in Pascal's triangle. It is almost as if we have a mapping of the diagonals of Pascal's triangle to the rows. We seem to have a function

$$f : \frac{n-1}{n-2} \mapsto \frac{1}{2^n} \qquad \text{for } n \geq 3.$$

The above expression is a generating function, so it would seem we can put

$$\left(\sum_{r=0}^{\infty} (-1)^r x^r \right)^n = \sum_{r=0}^{\infty} (-1)^r \binom{n-1+r}{r} x^r = \left(\frac{1}{1+x} \right)^n$$

as the last expression is just the nth power of the standard geometric series.

We also note that there are other manipulations of Pascal's triangle that lead to the Fibonacci sequence and to Lucas sequences in general, and thence to Pell's equation, second-order recurrence relations and beyond.

TF writes — If this article has got you interested in Pascal's triangle, here's something useful you can do.

Let $S(n,k) = \sum_{i=0}^{k} \binom{n}{k}$, the sum of the first $k+1$ elements of row n of the triangle. Find examples of n and k where $S(n,k)$ is a power of two. Apart from the easy cases, (i) $k = n$, (ii) $k = (n-1)/2$ for odd n and (iii) $n-1$ itself is a power of two, the only examples I know of are $S(23,3) = 2^{11}$ and $S(90,2) = 2^{12}$. Are there any more?

A note on the non-commutative $C_2 \times C_2$ algebras

Dennis Morris

The $C_2 \times C_2$ group has algebraic matrix form

$$\begin{bmatrix} a & b & c & d \\ \alpha b & a & \delta d & \varepsilon c \\ \eta c & \theta d & a & \kappa b \\ \mu d & \nu c & \pi b & a \end{bmatrix}.$$

By insisting upon multiplicative closure of this matrix form, we initially reduce this to

$$\begin{bmatrix} a & b & c & d \\ \alpha b & a & \dfrac{\alpha}{\varepsilon} d & \varepsilon c \\ \eta c & \theta d & a & \dfrac{\eta}{\theta} b \\ \dfrac{\theta\alpha}{\varepsilon} d & \dfrac{\eta}{\varepsilon} c & \dfrac{\alpha\theta}{\eta} b & a \end{bmatrix}.$$

Continuing to insist on multiplicative closure leads to $\theta^2 = \eta^2/\varepsilon^2$ with both positive and negative solutions. If we take the positive solution (and exponentiate the matrix), we get the $C_2 \times C_2$ algebraic fields. These are eight types of the sixteen types of 4-dimensional (commutative) complex numbers. These are covered elsewhere[1]. This article is concerned with the negative solution: $\theta = -\eta/\varepsilon$.

Since, choosing the positive solution gives commutative algebras; it is surprising that choosing the negative solution (and exponentiating the matrix) gives non-commutative division algebras[2].

The form of these algebras is

$$\exp\left(\begin{bmatrix} a & b & c & d \\ \alpha b & a & \dfrac{\alpha}{\varepsilon} d & \varepsilon c \\ \eta c & -\dfrac{\eta}{\varepsilon} d & a & -\varepsilon b \\ -\dfrac{\eta\alpha}{\varepsilon^2} d & \dfrac{\eta}{\varepsilon} c & -\dfrac{\alpha}{\varepsilon} b & a \end{bmatrix}\right).$$

[1] Dennis Morris, *Complex Numbers – The Higher Dimensional Forms*, ISBN 978-0-955600-30-2.

[2] These algebras satisfy all the algebraic field axioms except multiplicative commutativity.

The determinant of the enclosed matrix is

$$\frac{\left(a^2\varepsilon^2 - \varepsilon^2 b^2\alpha + \eta d^2\alpha - \eta c^2\varepsilon^2\right)^2}{\varepsilon^4}.$$

We are forced (in some but not all cases) to exponentiate the matrix because the multiplicative inverse will not exist if the matrix is singular, and exponentiation avoids this problem. In the case, $\alpha = \eta = -1$, we do not need to exponentiate the matrix to avoid singular matrices. The case $\alpha = \eta = \varepsilon = -1$ is

$$\begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{bmatrix},$$

which is the quaternions—a surprising connection between the group $C_2 \times C_2$ and the quaternions. The seven other algebras are given by the various combinations of assigning either $+1$ or -1 to the parameters $\alpha, \varepsilon, \eta$. Note that the polar part of the quaternions (the rotation matrix) is the Lie group SU(2).

Problem 237.1 – Three squares

Dick Boardman

Recently, in search of a problem, I got out my copy of Diophantus [Thomas Heath, *Diophantus of Alexandria*, Dover, New York, 1964] and discovered the following problem. **Find three numbers such that the product of any pair plus the square of the third is a square.**

This struck me as a typical Diophantus problem and I noted (i) if (a, b, c) were a solution then (ka, kb, kc) would also be a solution, (ii) if (a, b, c) were a solution then $(1, b/a, c/a)$ would be a solution, (iii) if there is a solution in fractions there would also be a solution in integers obtained by multiplying by the least common multiple of the denominators of a, b and c.

I tried all my favourite methods but got nowhere. Brute force found several solutions but with no hint of how to get them. So I gave in and went back to the book to find the answer. There wasn't one! I had misread the problem. So if anyone can find an elegant solution I should be most interested.

Mathematics student: “I hate logs!”
Tutor: “Try to avoid the timber industry.”

Solution 235.2 – Quartic roots

Let α, β, γ and δ be the roots of the quartic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e \;=\; 0.$$

Show that the equation

$$\sqrt[3]{\alpha\beta + \gamma\delta - x} + \sqrt[3]{\alpha\gamma + \beta\delta - x} + \sqrt[3]{\alpha\delta + \beta\gamma - x} \;=\; 0$$

has the solution

$$x \;=\; \frac{2\left(c^3 - 2ad^2 - 2b^2e + 3ace\right)}{a\,(3c^2 - 4bd + ae)}.$$

Tony Forbes

Define f, g and h by

$$f = \alpha\beta + \gamma\delta - x, \quad g = \alpha\gamma + \beta\delta - x, \quad h = \alpha\delta + \beta\gamma - x.$$

Then the equation we have to solve is

$$f^{1/3} + g^{1/3} + h^{1/3} \;=\; 0, \tag{1}$$

or $f^{1/3} + g^{1/3} = -h^{1/3}$, which on cubing becomes

$$f + g + h + 3f^{1/3}g^{1/3}\left(f^{1/3} + g^{1/3}\right) \;=\; 0.$$

But $f^{1/3} + g^{1/3} = -h^{1/3}$ and so $f + g + h = 3(fgh)^{1/3}$, which can be cubed to get

$$(f + g + h)^3 \;=\; 27fgh. \tag{2}$$

Now is a good time to recall that symmetric functions of the roots of a polynomial can be expressed as rational functions of the coefficients. Specifically, if we write the quartic as $a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$, expand and equate coefficients, we obtain these four equalities:

$$\alpha + \beta + \gamma + \delta \;=\; -\frac{4b}{a}, \tag{3}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta \;=\; \frac{6c}{a}, \tag{4}$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta \;=\; -\frac{4d}{a}, \tag{5}$$

$$\alpha\beta\gamma\delta \;=\; \frac{e}{a}. \tag{6}$$

Using (4), the sum $f+g+h$ in the left-hand side of (2) becomes

$$f+g+h \;=\; \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta-3x \;=\; \frac{6c}{a}-3x.$$

The product fgh is not so easy. When you multiply it out you get a monstrous cubic in x. But the coefficients of this cubic will be symmetric functions of α, β, γ and δ and therefore they must be expressible in terms of $4b/a$, $6c/a$, $4d/a$ and e/a, essentially the coefficients of the quartic. Well, I have done all the hard work and here is the result, which you can verify by using (3)–(6):

$$fgh \;=\; -x^3+\frac{6c}{a}x^2+\left(\frac{4e}{a}-\frac{16bd}{a^2}\right)x+\frac{16b^2e}{a^3}+\frac{16d^2-24ce}{a^2}.$$

Thus (2) becomes

$$\left(\frac{2c}{a}-x\right)^3 \;=\; -x^3+\frac{6c}{a}x^2+\left(\frac{4e}{a}-\frac{16bd}{a^2}\right)x+\frac{16b^2e}{a^3}+\frac{16d^2-24ce}{a^2}.$$

When we multiply out the left-hand side something wonderful happens. The x^2 and x^3 terms vanish to leave this linear equation

$$-2c^3+4b^2e-6ace+3ac^2x+a^2ex+4ad(d-bx) \;=\; 0,$$

which has the solution

$$x \;=\; \frac{2\big(c^3+3ace-2(ad^2+b^2e)\big)}{a(3c^2-4bd+ae)}. \tag{7}$$

The problem is solved. However, there is still a little something that is bothering me. When we substitute (7) into the original equation (1) we should get zero. And I expect we do—sometimes. But in general this can't always happen, surely. Each of the cube roots in (1) has three values; so there appear to be 27 possible values that the left-hand side of (1) can take, and there is no reason to suppose that they are all equal to zero on substituting for x.

If a, b, c, d, e and all four roots of the quartic are real, then the solution x in (7) is real and hence so are f, g and h, and it is clear (I think) that we must take the real cube roots in (1). On the other hand, if f, say, is not real, then it is not at all obvious (to me) which of the three values of $f^{1/3}$ to take. I am happy to leave it for others to resolve this difficulty.

Solution 233.6 – The quartic and the golden mean

A quartic polynomial has two points of inflection, with x coordinates p and q. Show that the straight line which passes through the points of inflection meets the quartic again at two points with x coordinates

$$\frac{\sqrt{5}+1}{2}\,p - \frac{\sqrt{5}-1}{2}\,q \quad \text{and} \quad \frac{\sqrt{5}+1}{2}\,q - \frac{\sqrt{5}-1}{2}\,p.$$

Stuart Walmsley

The quartic is modified so that x is zero for one of the points of inflection. The quartic has the general form $y = \sum_{j=0}^{4} a_j x^j$. This may be simplified without loss of generality as follows. First, a value of k may be found to transform $x \to x + k$ so that the term in one power of x is eliminated. Here the quadratic term is chosen. Secondly, $y \to y/a_4$ sets the coefficient of x^4 to 1. Thirdly, $y \to y - a_0$ eliminates the constant term.

The quartic then has the form

$$y = x^4 + a_3 x^3 + a_1 x.$$

The points of inflexion occur when the second derivative vanishes:

$$y'' = 12x^2 + 6a_3 x = 0;$$

that is when

$$x = 0, \quad y = 0 \qquad \text{or} \qquad x = -\frac{a_3}{2}, \quad y = -\frac{a_3^4}{16} - \frac{a_1 a_3}{2}.$$

The equation of the straight line joining the two points is

$$y = \left(\frac{a_3^3}{8} + a_1\right) x.$$

This meets $y = x^4 + a_3x^3 + a_1x$ when $x^4 + a_3x^3 + a_1x = (a_3^3/8 + a_1)x$, or

$$x^4 + a_3 x^3 - \frac{a_3^3}{8} x = 0.$$

Two roots of this quartic are known from the points of inflection so that $x^2 + (a_3/2)x$ is a factor. Dividing by this yields the quadratic

$$x^2 + \frac{a_3 x}{2} - \frac{a_3^2}{4} = 0.$$

Multiplying across by $4/a_3^2$ and setting $y = 2x/a_3$ gives $y^2 + y - 1 = 0$, which is recognized as the quadratic leading to the golden mean. This yields roots $y = 2x/a_3 = (-1 \pm \sqrt{5})/2$. Letting $r = -a_3/2$, the x-coordinates of the two points of inflection are 0 and r. Then the x coordinates of the two other points of intersection are

$$r\frac{\sqrt{5}+1}{2} \quad \text{and} \quad -r\frac{\sqrt{5}-1}{2},$$

which are consistent with the result required.

To generalize the result, let $x \to x + q$ and $r + q = p$. Then $0 \to q$, $r \to p$, so that

$$r\frac{\sqrt{5}+1}{2} \;\to\; (p-q)\frac{\sqrt{5}+1}{2} + q,$$

which simplifies to $\dfrac{\sqrt{5}+1}{2}p - \dfrac{\sqrt{5}-1}{2}q$. Similarly

$$-r\frac{\sqrt{5}-1}{2} \;\to\; (q-p)\frac{\sqrt{5}-1}{2} + q,$$

which simplifies to $\dfrac{\sqrt{5}+1}{2}q - \dfrac{\sqrt{5}-1}{2}p$, and the result is proved.

Solution 230.2 – Sum

> In how many ways can you sum n different non-negative integers which add to p? The same numbers in a different order count as different selections.

Sebastian Hayes

The formula is $(p + 1)(p + 2)\ldots(p + n - 1)/(n - 1)!$, $p = 0, 1, 2\ldots$, $n = 1, 2, 3\ldots$. Also known as polytopic numbers. The proof is via the multinomial theorem.

	p					
n	0	1	2	3	4	5
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	3	6	10	15	21
4	1	4	10	20	35	56
5	1	5	15	35	56	126

Solution 230.3 – Magic

> A number N is 'magic' if any number which ends with N (base 10) is divisible by N. (a) How many magic numbers are there $\leq 10^n$? (b) Give a sufficient and necessary condition for a positive integer to be magic.

Sebastian Hayes

The key point to grasp is that if a number N is to divide any number which ends $\ldots N$, it must divide the first power of $10 > N$. For if it does this, it will divide any multiple of this power, and this covers all numbers with more digits than N which terminate with $\ldots abcd = N$.

By trial we find that 1, 2 and 5 are the only magic numbers less than 10. And, excluding these, below 100 we have 10, 20, 25 and 50.

For $n \geq 2$ there seem to be five magic numbers with n digits between 10^n and 10^{n+1}, where we include the lower limit 10^n but not the higher. For example, if we consider magic numbers with four digits we are looking for numbers which divide 10^4 and which are $\geq 10^3$. Now, the only factors of powers of 10 are powers of 2 and 5; so the only possibilities are $10^4/2 = 5000$, $10^4/4 = 2500$, $10^4/5 = 2000$, $10^4/8 = 1250$ and $10^4/10 = 1000$. Any other numbers which divide 10^4 will have fewer than four digits.

More generally, for $n \geq 2$ over $[10^{n-1}, 10^n)$ we have the five numbers $10^n/2$, $10^n/2^2$, $10^n/5$, $10^n/2^3$ and 10^{n-1}, or

$$5^{n-1}2^n, \quad 5^n 2^{n-1}, \quad 5^n 2^{n-2}, \quad 5^n 2^{n-3}, \quad 5^{n-1}2^{n-1}.$$

(For $n = 1$, $n = 2$ we get less because we have to discount fractional divisors such as $10^2/2^3$ and $10/2^3$.)

Since there are three magic numbers between 1 and 10 and four between 10 and 100 (including 1 and 10), and five in all subsequent stretches between 10^{n-1} and 10^n, the answer to the first part of the problem is $5(n-2)+7$, and if we include 10^n itself, we obtain this result.

There are $5n-2$ magic numbers between 1 and 10^n inclusive.

Bearing the above in mind, it will be seen that the following is a necessary and sufficient condition:

N is magic if $N = 2^p 5^q$, where $0 \leq q - p + 1 \leq 4$.

Note: I came across this problem in Davis and Hersch, *The Mathematical Experience.*

Re: Problem 234.1 – Two ellipses

Ralph Hancock

In Tony's Problem 234.1 [see page 14], the volume enclosed by the surface between the two line segments is, of course, 568 cubic centimetres, and it's full of milk. This was the shape of the original Tetrapak pints of milk, which is why the container-making company is called Tetrapak when its products are now boring cuboids. Not, of course, that the carton is actually a tetrahedron, but it has a certain resemblance to one. The process heat-sealed a long flat strip of waxed paper into a cylindrical tube which was fed down vertically and filled with milk from above the joining point. Below this, it was crimped alternately north-south and east-west and cut across the crimped seams to form the cartons.

A consequence of this forming process was that—uniquely among liquid containers—the carton was completely full, with no head space of air above the milk. You opened it by putting it on a table and cutting off the uppermost corner with scissors. This inevitably caused a small amount of milk to dribble out on to the tablecloth. And then even the slightest touch to the flexible container, such as an attempt to pick it up, would produce a brisk spurt of milk from the open top. People hated them, and it is no surprise that the company quickly abandoned the design for a more conventional shape.

Returning to the problem more seriously, if you were to make the shape shown in the drawing from card, it would have to be made in three parts, with a cylindrical centre glued to conical ends.

Incidentally, I don't know why R. M. Boardman ('Where does mathematics come from?') feels the need to imagine a 'super creature' that can count and compare sizes. Real creatures have these abilities here and now; I think that even a dog can realise whether its opponent is bigger than it. Among the most intelligent animals, ravens have been found to be able to count up to seven, and Dr Irene Pepperberg's famous speaking African grey parrot Alex could sort objects by their colour, size and shape, and distinguish these abstract concepts in his answers to questions.

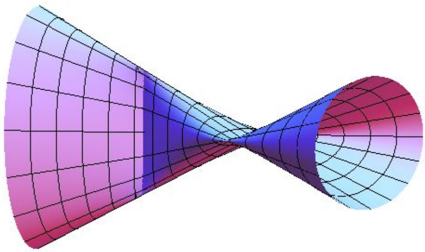
Problem 237.2 – Arctan sum

Show that

$$\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} \;=\; \frac{3\pi}{4}.$$

Solution 234.1 – Two ellipses

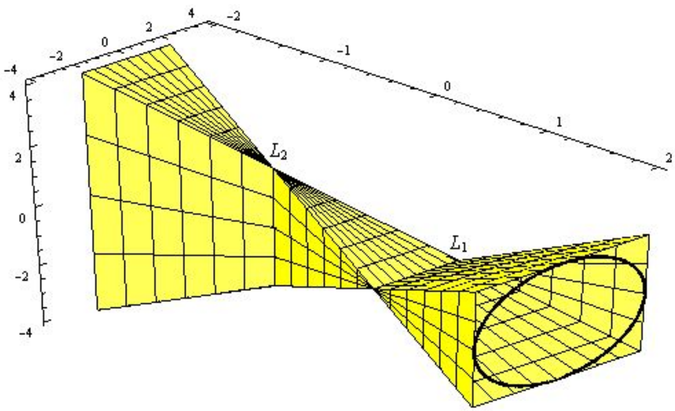
> Let a, b and c be positive numbers with $b \neq a$. Let E_1 be an ellipse with axes $2a$ and $2b$ and situated in the plane $x = -c$ with its centre at $(-c, 0, 0)$ and its $2b$-axis vertical. Let E_2 be a similar ellipse but centred at $(c, 0, 0)$ and with its $2a$-axis vertical. Now join each point of E_1 to the diametrically opposite point of E_2. The resulting surface has two singularities in the form of straight line segments L_1 and L_2, say. What is the volume enclosed by the surface between the two line segments?



Robin Marks

After thinking about this problem for a few days I decided to first try to solve another, easier, problem. I would replace each ellipse with a rectangle. Then I realized that solving the second problem would easily lead to the desired solution.

The diagram below shows the second problem with $a = 4$, $b = 2$, $c = 2$.



The central section is clearly a tetrahedron. The volume of the tetrahedron can be found by integration.

Let the nearest line segment to the viewer be L_1. Call the nearest rectangle R_1. Moving from R_1 to L_1 in the x-direction, the (y, z)-plane cuts the surface in progressively smaller rectangles. Between R_1 and L_1 the height of the rectangle has reduced from $2b$ to 0. It is fairly easy to see that at the same time the width has reduced from $2a$ to $2(a - b)$. Hence L_1 has length $2(a - b)$. Call this length L. Similarly, L_2 also has length L. The distance between R_1 and L_1 in the x-direction is $2cb/(a + b)$. Hence the distance between L_1 and L_2 is

$$2c - \frac{2 \cdot 2cb}{a+b} \;=\; \frac{2c(a-b)}{a+b}.$$

Call this length H. The volume is

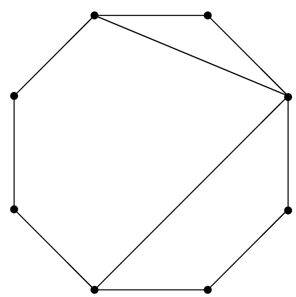
$$V_{\text{rect}} \;=\; \int_0^H \frac{Lx}{H}\left(L - \frac{Lx}{H}\right) dx \;=\; \frac{4c(a-b)^3}{3(a+b)}.$$

Finally, let's get back to the original problem. If we draw an ellipse to touch R_1 at four points as shown, the ratio of the area of the ellipse to the area of R_1 is the same as the ratio of the area of a circle to the area of a square. We can draw further touching ellipses at each (y, z)-plane which, when joined together, form the curved surface of the original problem, inside the angular surface shown on the diagram. Hence the volume required is

$$V_{\text{ell}} \;=\; \frac{\pi}{4} V_{\text{rect}} \;=\; \frac{\pi c(a-b)^3}{3(a+b)}.$$

Problem 237.2 – Cycles

Given an integer $n \geq 3$, what is the smallest number of edges that a graph with n points must have if it is to contain cycles of all possible lengths (i.e. 3, 4, …, n)?



The graph on the right illustrates the case $n = 8$. You can see that it contains 3-, 4-, 5-, 6-, 7- and 8-cycles, and therefore the requirements of the problem are satisfied with 10 edges. And with a little thinking you can deduce that you can't do it with nine edges. So 10 is the answer for $n = 8$.

Problem 237.3 – Rearrangements

Let N be a positive integer and make a list of all $N!$ permutations of the integers 1, 2, ..., N. Now, one by one, rearrange each sequence into the correct order by a finite number of *moves*. A move is to pick up a number and put it down somewhere else in the sequence. Thus, with $N = 5$,

$$54321 \;\to\; 15432 \;\to\; 12543 \;\to\; 12354 \;\to\; 12345$$

are four valid moves to get 54321 into the correct order.

What is the minimum number of moves required to order all $N!$ sequences?

Thanks to Robin Whitty for communicating this to me (TF).

Problem 237.4 – Continued fraction

Show that

$$\cfrac{1}{1 - \cfrac{1^4}{5 - \cfrac{2^4}{13 - \cfrac{3^4}{25 - \ldots}}}} \;=\; \frac{\pi^2}{6}.$$

The number underneath n^4 is $2n^2 + 2n + 1$.

Solution 234.3 – Fixed point

> Let α denote the (unique) unique real number that solves the equation $\cos \alpha = \alpha$. Prove that α is transcendental.

Tony Forbes

We take the easy option by starting from the **Hermite–Lindemann theorem**: *Let $n \geq 2$ be an integer. Let a_1, a_2, ..., a_n be arbitrary distinct algebraic numbers and let A_1, A_2, ..., A_n be arbitrary non-zero algebraic numbers. Then the equality*

$$A_1 e^{a_1} + A_2 e^{a_2} + \cdots + A_n e^{a_n} \;=\; 0$$

cannot hold.

Now rewrite $\cos \alpha = \alpha$ as $e^{i\alpha} + e^{-i\alpha} - 2\alpha e^0 = 0$, and we see that non-zero algebraic α would contradict the theorem.

Information-age mathematics

Eddie Kent

Once upon a time up in the north of Britain there lived an illiterate race of people called Picts. They were Celtish tribes who for centuries existed close to cultures such as the Romans, the Irish and the Anglo-Saxons, all of whom left extensive records about themselves to exhibit their literacy. The Picts left nothing, except some symbols on stone and jewelery, and these were impossible to interpret.

Archaeologists have long puzzled over these markings. Could they be religious in intent? Or perhaps they were tribal names, or some form of heraldry. There are not enough to be certain: only fewer than 500 inscriptions remain, and each is very short. However, now mathematics has been called on to help in the decipherment. Rob Lee of the University of Exeter has published a paper in the *Proceedings of the Royal Society* A (31 March 2010) using Shannon entropy to show that the marks have some characteristics of a written language. The Picts were not so illiterate after all.

From Wikipedia: 'The Shannon entropy equation provides a way to estimate the average minimum number of bits needed to encode a string of symbols, based on the frequency of the symbols.' The article uses *A Guide to Data Compression Methods* by David Solomon which is easily available at about \$20. The notion is that written languages are distinguishable from random sequences of symbols because they contain some statistical predictability. For instance, in English a q is nearly always followed by a u, or a w by an h if it is a consonant.

In his Summary Lee says 'The paper shows that the Pictish symbols are characters of a lexicographic written language, as opposed to the most general form of writing, which includes things like the [non-verbal] instructions on your Ikea flat packs.' Of course (what did you expect?) the statistical methods aren't enough to crack the code but, Lee says, there are some short inscriptions and a little Latin script which might help in the interpretation of the language, at least in part.

'I am somewhat hopeful that we may be able to do it.'

Problem 237.5 – Another sum

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} \;=\; \frac{\pi^2-9}{3}.$$

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Cover: Dodecahedron graph